Extra credit
November 27, 2013
Turn in the day of the exam.
(I am counting on everyone treating this as an individual effort not a group effort.)

NEEP 547
DLH

1. (3pts) Find the Fourier expansions of the periodic function whose definition on one period is
$$f(t) = \begin{cases} 0 & \text{for } -\pi \le t \le 0 \\ \sin(t) & \text{for } 0 \le t \le \pi. \end{cases}$$

2. (3pts) Find the solution of the following differential equation which satisfies the given initial conditions and where $f(t)$ is a periodic function:
$$y'' + 9y = f(t) \quad ; y(0) = y'(0) = 0 \text{ and } f(t) = |t| \quad \text{for } -\pi \le t \le \pi.$$

3. (3pts) Find the Fourier transform of the function $f(t) = \begin{cases} 0 & \text{for } -\infty < t \le 0 \\ \sin(t) & \text{for } 0 \le t \le \pi \\ 0 & \text{for } \pi \le t < \infty \end{cases}$ using the basic definition of Fourier transform.

4. (3pts) Find the inverse transform of
$$f(\omega) = \frac{\sin(\omega - 2)}{\omega - 2}.$$
(Use theorems discussed in class such as the shifting theorem to solve for the inverse. Do not just copy the inverse from a table of Fourier Transforms).

5. (3pts) Solve the integral equation:
$$\psi(x) = 1 + \lambda^2 \int_0^x (t - x)\,\psi(t)\,dt.$$

6. (3pts) Solve the integral equation:
$$\frac{dy(t)}{dt} = 1 - \sin(t) - \int_0^t y(\tau)\,d\tau, \text{ with I.C.: } y(0) = 0.$$

7. (2pts) Given the matrix equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$, show that the particular solution integral is given by $\mathbf{v(t)} = \mathbf{X(t)} \int \mathbf{X^{-1}(t) f(t)\, dt}$ where $\mathbf{X(t)}$ is the fundamental matrix of the homogeneous equation.

8. (3pts) Obtain the solution of the simultaneous equations
$$\begin{aligned} x' + y' + x &= -e^{-t}, \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}$$
which satisfies the initial conditions: $x(0) = -1$, and $y(0) = 1$.

9. (4pts) Solve the following differential equation using Fourier Transforms:
$3y'' + 10y' + 3y = 64e^{3t}(H(t) - H(t-5))$ for $t > 0$ with conditions $y(0) = -1$ and $y'(0) = 0$. ~~Invert the resulting transform expression in the complex plane to obtain the final result for $y(t)$.~~ (Recall that one can use Fourier Transforms on the half-space ($0 < t < \infty$). The definition of the transforms for $y'(t)$ and $y''(t)$ are modified and incorporate the initial conditions similar to Laplace transforms).

1. Find the Fourier expansion of the periodic function whose definition on one period is

$$f(t) = \begin{cases} 0 & \text{for } -\pi \leq t \leq 0 \\ \sin(t) & \text{for } 0 \leq t \leq \pi \end{cases}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}t\right) \qquad p = \pi$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} \sin(t)\,dt = \frac{1}{\pi}(-\cos(t))\Big|_0^{\pi} = \frac{1}{\pi}(-\cos(\pi)+1)$$
$$= \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\,d\left(\frac{1}{n}\sin(nt)\right)$$
$$= \frac{1}{\pi}\left[\left(\sin(t)\left(\frac{1}{n}\right)\sin(nt)\right)\Big|_0^{\pi} - \left(\frac{1}{n}\right)\int_0^{\pi} \sin(nt)\cos(t)\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left[\left[\left(\frac{1}{n}\right)\cancelto{0}{\sin(\pi)\sin(n\pi)} - \left(\frac{1}{n}\right)\cancelto{0}{\sin(0)\sin(0)} - \left(\frac{1}{n}\right)\int_0^{\pi} \cos(t)\,d\left(-\frac{1}{n}\cos(nt)\right)\right]\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n}\right)^2\left[\cos(t)\cos(nt)\Big|_0^{\pi} - \int_0^{\pi} \cos(nt)(-\sin(t))\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n}\right)^2\left[(\cos(\pi)\cos(n\pi) - 1) + \int_0^{\pi} \cos(nt)\sin(t)\,dt\right]$$
$$\therefore \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\left[(-1)(-1)^n - 1\right] + \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\int_0^{\pi} \cos(nt)\sin(t)\,dt$$
$$\left(\frac{1}{\pi}\right)\left(1-\frac{1}{n^2}\right)\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)(-1)\underbrace{\left((-1)^n+1\right)}$$

0 for n odd
2 for n even

$$\therefore a_n = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{-2}{n^2-1}\right) = \left(\frac{-2}{\pi}\right)\left(\frac{1}{n^2-1}\right) \quad \text{for } n \text{ even}$$

$$b_n = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\sin(nt)\,dt = \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\,d\left(-\frac{1}{n}\cos(nt)\right)$$
$$= \frac{1}{\pi}\left[\left(\sin(t)\left(-\frac{1}{n}\right)\cos(nt)\right)\Big|_0^{\pi} + \left(\frac{1}{n}\right)\int_0^{\pi} \cos(nt)\cos(t)\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left[\cancelto{0}{\left(-\frac{1}{n}\right)\sin(\pi)\cos(n\pi)} + \cancelto{0}{\left(\frac{1}{n}\right)\sin(0)\cos(0)} + \left(\frac{1}{n}\right)\int_0^{\pi} \cos(t)\,d\left(\frac{1}{n}\sin(nt)\right)\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\left[\cos(t)\sin(nt)\Big|_0^{\pi} + \int_0^{\pi} \sin(nt)\sin(t)\,dt\right]$$
$$= \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\left[0 - 0 + \int_0^{\pi} \sin(nt)\sin(t)\,dt\right]$$
$$\therefore \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin(t)\sin(nt)\,dt = \left(\frac{1}{\pi}\right)\left(\frac{1}{n^2}\right)\int_0^{\pi} \sin(nt)\sin(t)\,dt$$
$$\left(\frac{1}{\pi}\right)\left(1-\frac{1}{n^2}\right)\int_0^{\pi} \sin(t)\sin(nt)\,dt = 0 \qquad n \not\equiv 1$$

$$\text{for } n=1 \quad \left(\frac{1}{\pi}\right)\int_0^{\pi} \sin^2(t)\,dt = \left(\frac{1}{\pi}\right)\left(t/2 - \frac{1}{2}\sin(t)\cos(t)\right)\Big|_0^{\pi} = \left(\frac{1}{\pi}\right)\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

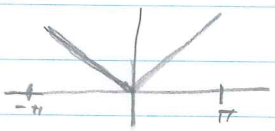
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt)$$
$$= \left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right) + \left(\frac{-2}{\pi}\right)\sum_{\substack{n=2 \\ n\text{-even}}}^{\infty}\left(\frac{1}{n^2-1}\right) + \left(\frac{1}{2}\right)\sin(t)$$
$$= \frac{1}{\pi} + \frac{1}{2}\sin(t) - \frac{2}{\pi}\sum_{\substack{n=2 \\ n\text{-even}}}^{\infty}\left(\frac{1}{n^2-1}\right)\cos(nt)$$

2. Find the solution of the following differential equation which satisfies the given initial conditions and where f(t) is a periodic function:

$y'' + 9y = f(t)$; $y(0) = y'(0) = 0$ and $f(t) = |t|$ for $-\pi \le t \le \pi$

note $y = y_h + y_p$ - we will find y_h by regular means and y_p by fourier series expansion

$y_h'' + 9y_h = 0 \Rightarrow y_h(x) = A\cos(3t) + B\sin(3t)$



f(t) is an even function

$p = \pi$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\underbrace{\int_{-\pi}^{0}(-t)dt}_{} + \frac{1}{\pi}\int_0^{\pi}(t)dt = \frac{1}{\pi}\int_0^{\pi} z\,dz + \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{2}{\pi}\int_0^{\pi} t\,dt$$

let $z = -t$, $dz = -dt$
$t = -\pi$, $z = \pi$
$t = 0$ $z = 0$

$$= \frac{2}{\pi}\left(\frac{t^2}{2}\right)\Big|_0^{\pi} = \left(\frac{2}{\pi}\right)\left(\frac{\pi^2}{2}\right) = \pi$$

$$\underline{a_0 = \pi}$$

let $z = -t$, $dz = -dt$
$t = -\pi$, $z = \pi$
$t = 0$, $z = 0$
$\cos(nz) = \cos(-nz)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\cos(nt)dt + \frac{1}{\pi}\int_0^{\pi}(t)\cos(nt)dt$$

$$= \frac{1}{\pi}\int_0^{\pi} z\cos(nz)dz + \frac{1}{\pi}\int_0^{\pi} t\cos(nt)dt$$

$$= \frac{2}{\pi}\int_0^{\pi} t\cos(nt)dt = \frac{2}{\pi}\int_0^{\pi} t\,d\left(\frac{1}{n}\sin(nt)\right) = \frac{2}{\pi}\left(\frac{t}{n}\sin(nt)\Big|_0^{\pi} - \int_0^{t}\frac{1}{n}\sin(nt)dt\right)$$

$$= \frac{2}{\pi}\left(0 - \frac{1}{n}\int_0^{t}\sin(nt)dt\right) \Rightarrow \frac{2}{\pi}\left(-\frac{1}{n}\right)\left(-\frac{1}{n}\cos(nt)\Big|_0^{\pi}\right) = \left(\frac{2}{\pi}\right)\left(\frac{1}{n^2}\right)(\cos(n\pi) - 1)$$

$$= \left(\frac{2}{\pi}\right)\left(\frac{1}{n^2}\right)((-1)^n - 1) = \left(\frac{-4}{\pi}\right)\left(\frac{1}{n^2}\right) \text{ for } n = \text{odd}$$

let $z = -t$ $dz = -dt$
$t = -\pi$, $z = \pi$
$t = 0$, $z = 0$
$\sin(nz) = -\sin(nz)$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\sin(nt)dt + \frac{1}{\pi}\int_0^{\pi}(t)\sin(nt)dt$$

$$= \frac{1}{\pi}\int_{\pi}^{0} z\sin(nz)dz + \frac{1}{\pi}\int_0^{\pi} t\sin(nt)dt$$

$$= -\frac{1}{\pi}\int_0^{\pi} t\sin(nt)dt + \frac{1}{\pi}\int_0^{\pi} t\sin(nt)dt = 0 \quad \text{as it should be}$$

we now assume $y_p(t) = \frac{c_0}{2} + \sum_{\substack{n=1\\ n\text{-odd}}}^{\infty} c_n\cos(nt)$, $y_p'(t) = \sum_{\substack{n=1\\ n\text{-odd}}}^{\infty} c_n(-n)\sin(nt)$

$$y_p''(t) = -\sum_{\substack{n=1\\ n\text{-odd}}}^{\infty} c_n(n^2)\cos(nt)$$

$$\therefore y_p''(t) + 9\cdot y_p(t) = \frac{a_0}{2} + \sum_{\substack{n=1\\ n\text{-odd}}} a_n\cos(nt)$$

$$-\sum_{\substack{n=1\\ n\text{-odd}}}^{\infty} c_n(n^2)\cos(nt) + 9\left(\frac{c_0}{2} + \sum_{\substack{n=1\\ n\text{-odd}}}^{\infty} c_n\cos(nt)\right) = \frac{\pi}{2} + \sum_{\substack{n=1\\ n\text{-odd}}}^{\infty}\left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2}\right)\cos(nt)$$

$$\frac{9c_0}{2} + \sum_{\substack{n=1\\n\text{-odd}}}^{\infty} c_n(9-n^2)\cos(nt) = \frac{\pi}{2} + \sum_{\substack{n=1\\n\text{-odd}}}^{\infty}\left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2}\right)\cos(nt)$$

$\cos(nt)$

$$\frac{9c_0}{2} + 8c_1\cos(t) + 0c_3\cos(3t) + \sum_{\substack{n=5\\n\text{-odd}}}^{\infty} c_n(9-n^2)\cos(nt) = \frac{\pi}{2} - \overbrace{\left(\frac{4}{\pi}\right)\left(\frac{1}{1}\right)}^{a_1} - \overbrace{\left(\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)}^{a_3} - \frac{4}{\pi}\sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(\frac{1}{n^2}\right)$$

$\frac{9c_0}{2} = \frac{\pi}{2} \therefore c_0 = \frac{1}{9}$; $8c_1 = -\left(\frac{4}{\pi}\right) = c_1 = -\left(\frac{1}{2\pi}\right)$; $c_n(9-n^2) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2}\right) \Rightarrow c_n = \left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2(9-n^2)}\right)$

note, we have a problem with the a_3 term. the reason is it is equal to one of the homogeneous solutions; the $\cos(3t)$. So we must find the particular solution to this term by another means. We will call this particular solution $y_{p2}(t)$. For this term our differential equation is $y_{p2}''(t) + 9y_{p2}(t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$. Calling on our experience with D.E. ~~we~~ assume $y_{p2}(t) = at\cos(3t) + bt\sin(3t)$.

$y_{p2}'(t) = a\cos(3t) - 3t^{a}\sin(3t) + b\sin(3t) + 3bt\cos(3t)$

$y_{p2}''(t) = -3a\sin(3t) - 3a\sin(3t) - 9at\cos(3t) + 3b\cos(3t) + 3b\cos(3t) - 9bt\sin(3t)$

$= -6a\sin(3t) - 9at\cos(3t) + 6b\cos(3t) - 9bt\sin(3t)$

$y_{p2}''(t) + 9y_p(t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$

$-6a\sin(3t) - 9at\cancel{\cos}(3t) + 6b\cos(3t) - 9bt\cancel{\sin}(3t) + 9at\cancel{\cos}(3t) + 9b\cancel{t}\sin(3t) =$
$\left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$

$-6a\sin(3t) + 6b\cos(3t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t)$ equating terms

$-6a\sin(3t) = 0\sin(3t) \Rightarrow -6a = 0 \Rightarrow a = 0$

$6b\cos(3t) = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right)\cos(3t) \Rightarrow 6b = \left(-\frac{4}{\pi}\right)\left(\frac{1}{9}\right) \Rightarrow b = \left(-\frac{2}{\pi}\right)\left(\frac{1}{27}\right)$

$\therefore y_{p2}(t) = \left(-\frac{2}{\pi}\right)\left(\frac{1}{27}\right)t\sin(3t)$

Thus $y(t) = y_h(t) + y_p(t) + y_{p2}(t)$

$\cos(nt)$

$$y(t) = A\cos(3t) + B\sin(3t) + \frac{1}{9} - \left(\frac{1}{2\pi}\right)\cos(t) - \left(\frac{2}{\pi}\right)\left(\frac{1}{27}\right)t\sin(3t) + \sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(-\frac{4}{\pi}\right)\left(\frac{1}{n^2(9-n^2)}\right)$$

$$y(0) = A\cos(0) + B\overset{0}{\cancel{\sin(0)}} + \frac{1}{9} - \left(\frac{1}{2\pi}\right)\cos(0) - \left(\frac{2}{\pi}\right)\left(\frac{1}{27}\right)\overset{0}{\cancel{(0)\sin(0)}} + \sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(\frac{4}{\pi}\right)\left(\frac{1}{n^2(n^2-9)}\right)^{\cos(0)} = 0$$

$$= A + \frac{1}{9} - \frac{1}{2\pi} + \sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(\frac{4}{\pi}\right)\left(\frac{1}{n^2(n^2-9)}\right) = 0$$

$$\therefore A = -\frac{1}{9} + \frac{1}{2\pi} - \frac{4}{\pi}\sum_{\substack{n=5\\n\text{-odd}}}^{\infty}\left(\frac{1}{n^2(n^2-9)}\right)$$

$$y'(0) = -3A\cancelto{0}{\sin(0)} + 3B\cos(0) + \frac{1}{2\pi}\cancelto{0}{\sin(0)} - \left(\frac{2}{\pi}\right)\left(\frac{1}{27}\right)\cancelto{0}{\sin(0)} - \left(\frac{2}{\pi}\right)\left(\frac{3}{27}\right)\cancelto{0}{(0)\cos(0)} + \sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}{}' \cancelto{0}{\left(\frac{4}{\pi}\right)\left(\frac{-n}{n^2(n^2-9)}\right)\sin(0)} = 0$$

$$= 3B = 0$$

$$\therefore\ y(t) = \left(-\frac{1}{9} + \frac{1}{2\pi} - \frac{4}{\pi}\sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}{}'\left(\frac{1}{n^2(n^2-9)}\right)\cos(3t) + \frac{1}{9} - \frac{1}{2\pi}\cos(t) - \left(\frac{2}{\pi}\right)\left(\frac{t}{27}\right)\sin(3t) + \left(\frac{4}{\pi}\right)\sum_{\substack{n=5\\ n\text{-odd}}}^{\infty}{}'\left(\frac{1}{n^2(n^2-9)}\right)\cos(nt)$$

3) Find the Fourier transform of the function $f(t) = \begin{cases} 0 & \text{for } -\infty < t \leq 0 \\ \sin(t) & \text{for } 0 \leq t \leq \pi \\ 0 & \text{for } \pi \leq t < \infty \end{cases}$ using the basic definition of Fourier transform.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\pi} \sin(t) e^{-i\omega t}\, dt = \int_0^{\pi} \left(\frac{e^{it} - e^{-it}}{2i}\right) e^{-i\omega t}\, dt$$

$$= \left(\frac{1}{2i}\right) \int_0^{\pi} \left(e^{-i(\omega-1)t} - e^{-i(\omega+1)t}\right) dt$$

$$= \left(\frac{1}{2i}\right)\left(-\frac{e^{-i(\omega-1)t}}{i(\omega-1)} + \frac{e^{-i(\omega+1)t}}{i(\omega+1)}\right)\Bigg|_0^{\pi}$$

$$= \left(\frac{1}{2i}\right)\left(-\frac{e^{-i(\omega-1)\pi}}{i(\omega-1)} + \frac{1}{i(\omega-1)} + \frac{e^{-i(\omega+1)\pi}}{i(\omega+1)} - \frac{1}{i(\omega+1)}\right)$$

$$= \left(\frac{1-e^{-i(\omega+1)\pi}}{2(\omega+1)} - \frac{1-e^{-i(\omega-1)\pi}}{2(\omega-1)}\right)$$

4/ Find the inverse transform of $f(\omega) = \frac{\sin(\omega-2)}{\omega-2}$.

$$f(t) = F^{-1}\{f(\omega)\} = F^{-1}\left\{\frac{\sin(\omega-2)}{\omega-2}\right\}$$

recall the frequency shift theorem $F(\omega-\omega_0) = F\{e^{i\omega_0 t} f(t)\}$

$$F^{-1}\left\{\frac{\sin(\omega-2)}{(\omega-2)}\right\} = e^{i2t} F^{-1}\left\{\frac{\sin(\omega)}{\omega}\right\} = e^{i2t} F^{-1}\left\{\frac{e^{i\omega}-e^{-i\omega}}{2i\omega}\right\}$$

$$= \frac{e^{i2t}}{2}\Bigg(\underbrace{F^{-1}\left\{\frac{e^{i\omega}}{i\omega}\right\} - F^{-1}\left\{\frac{e^{-i\omega}}{i\omega}\right\}}_{\text{these are step functions}}\Bigg)$$

note $f(t) = H(t-a)$; $F\{f(t)\} = F\{H(t-a)\}$

$$F\{H(t-a)\} = \int_{-\infty}^{\infty} H(t-a)\, e^{-i\omega t}\, dt = \int_a^{\infty} e^{-i\omega t}\, dt = \left(\frac{-1}{i\omega}\right) e^{-i\omega t}\Big|_a^{\infty} = \frac{e^{-i\omega a}}{i\omega}$$

$$\therefore F^{-1}\left\{\frac{e^{-i\omega a}}{i\omega}\right\} = H(t-a)$$

Hence $F^{-1}\left\{\frac{e^{i\omega}}{i\omega}\right\} = F^{-1}\left\{\frac{e^{-i\omega(-1)}}{i\omega}\right\} = H(t+1)$

and $F^{-1}\left\{\frac{e^{-i\omega}}{i\omega}\right\} = F^{-1}\left\{\frac{e^{-i\omega(1)}}{i\omega}\right\} = H(t-1)$

Thus $F^{-1}\{f(\omega)\} = F^{-1}\left\{\frac{\sin(\omega-2)}{\omega-2}\right\} = \frac{e^{i2t}}{2} F^{-1}\left\{\frac{\sin(\omega)}{\omega}\right\} = \frac{e^{i2t}}{2}\left(H(t+1) - H(t-1)\right)$

$$\Rightarrow f(t) = \frac{e^{i2t}}{2}\left(H(t+1) - H(t-1)\right)$$

5. $y(x) = 1 + \lambda^2 \int_0^x (t-x)\, y(t)\, dt$

$$\frac{dy(x)}{dx} = \lambda^2\left[\frac{dx}{dx}(x-x)\,y(x) - \frac{d(0)}{dx}(0-x)\,y(0) + \int_0^x \left(\frac{d}{dx}(t-x)\,y(t)\,dt\right)\right]$$

$$= \lambda^2\left[0 - 0 - \int_0^x y(t)\,dt\right] = -\lambda^2 \int_0^x y(t)\,dt$$

$$\frac{d^2y(x)}{dx^2} = -\lambda^2\left[\frac{dx}{dx}y(x) - \frac{d(0)}{dt} + \int_0^x \frac{dy(t)}{dx}\,dt\right] = -\lambda^2 y(x)$$

$$\frac{d^2y(x)}{dx^2} + \lambda^2 y(x) = 0 \Rightarrow y(x) = A\sin(\lambda x) + B\cos(\lambda x)$$

now to find the I.C.

from the original eq. $y(0) = 1$ and $y'(0) = 0$

$$y(0) = A\sin(0) + B\cos(0) = 1 \Rightarrow \therefore B = 1$$

$$y'(0) = A\lambda\cos(0) - B\lambda\sin(0) = 0 \Rightarrow A = 0$$

$\therefore y(x) = \cos(\lambda x)$ let's check as original solution

$$y(x) = 1 + \lambda^2 \int_0^x (t-x)\, y(t)\, dt$$

$$\cos(\lambda x) = 1 + \lambda^2 \int_0^x (t-x)\cos(\lambda t)\,dt = 1 + \lambda^2 \int_0^x t\cos(\lambda t)\,dt - x\lambda^2 \int_0^x \cos(\lambda t)\,dt$$

$$= 1 + \lambda^2 \int_0^x t\, d\left(\tfrac{1}{\lambda}\sin(\lambda t)\right) - x\lambda^2\left(\tfrac{1}{\lambda}\sin(\lambda t)\right)\Big|_0^x$$

$$= 1 + \lambda^2\left(\tfrac{1}{\lambda}\, t\sin(\lambda t)\Big|_0^x - \tfrac{1}{\lambda}\int_0^x \sin(\lambda t)\,dt\right) - x\lambda\sin(\lambda x)$$

$$= 1 + \lambda^2\left(\tfrac{x}{\lambda}\sin(\lambda x) + \tfrac{1}{\lambda^2}\cos(\lambda t)\Big|_0^x\right) - x\lambda\sin(\lambda x)$$

$$= \cancel{1} + \cancel{\lambda x\sin(\lambda x)} + \cos(\lambda x) - \cancel{1} - \cancel{\lambda x\sin(\lambda x)}$$

$\cos(\lambda x) = \cos(\lambda x)$ checks

5. Solve the integral equation:

$$\frac{dy}{dt} = 1 - \sin(t) - \int_0^t y(t)\,dt \quad , \text{ with I.C.: } y(0) = 0$$

$$\frac{d^2y}{dt^2} = -\cos(t) - \left[\frac{dt}{dt}y(t) - \cancel{\frac{d(0)}{dt}}\overset{??}{(y(0))} + \overset{?.}{\cancel{\int_0^t \frac{dy(\tau)}{dt}d\tau}}\right]$$

$$= -\cos(t) - y(t)$$

$$\frac{d^2y}{dt^2} + y(t) = -\cos(t) \quad \Rightarrow \quad \frac{d^2y_h}{dt^2} + y(t) = 0 \qquad y_h(t) = A\cos(t) + B\sin(t)$$

now to find $y_p(t)$. assume $y_p(t) = bt\cos(t) + ct\sin(t)$

$$y_p'(t) = b\cos(t) - bt\sin(t) + c\sin(t) + tc\cos(t)$$

$$y_p''(t) = -b\sin(t) - b\sin(t) - bt\cos(t) + c\cos(t) + c\cos(t) - tc\sin(t).$$

$$= -2b\sin(t) - bt\cos(t) + 2c\cos(t) - tc\sin(t)$$ Insert in D.E.

$$-2b\sin(t) - bt\cancel{\cos(t)} + 2c\cos(t) - tc\cancel{\sin(t)} + bt\cancel{\cos(t)} + ct\cancel{\sin(t)} = -\cos(t)$$

$-2b\sin(t) + 2c\cos(t) = -\cos(t)$ comparing sides $-2b = 0 \therefore b = 0$

$2c = -1 \therefore c = -\frac{1}{2}$

$$y(t) = y_h(t) + y_p(t)$$

$$= A\cos(t) + B\sin(t) - \tfrac{1}{2}t\sin(t)$$ now to find A and B using I.C.

$$y(0) = 0 = A\cos(0) \Rightarrow A = 0$$

$y(t) = B\sin(t) - \frac{1}{2}t\sin(t)$ Still need to find B. will use $y'(0)$ and find $y'(0)$ from the above eq. $y'(0) = 1$.

$$y'(t) = B\cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}t\cos(t)$$

$$y'(0) = B\cos(0) - \tfrac{1}{2}\sin(0) - \tfrac{1}{2}(0)\cos(0) = 1$$

$$y'(0) = B = 1$$

$\therefore y(t) = \sin(t) - \frac{1}{2}t\sin(t)$ now to stick into integral eq.

$$\frac{dy}{dt} = \cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}t\cos(t) = 1 - \sin(t) - \int_0^t (\sin(t) - \tfrac{1}{2}t\sin(t))\,dt$$

$$\cos(t) - \tfrac{1}{2}\sin(t) - \tfrac{1}{2}t\cos(t) = 1 - \sin(t) - \int_0^t \sin(t)\,dt + \tfrac{1}{2}\int_0^t t\sin(t)\,dt$$

$$= 1 - \sin(t) + \cos(t)\Big|_0^t + \tfrac{1}{2}\int_0^t t\,d(-\cos(t))$$

$$= \cancel{1} - \sin(t) + \cos(t) - \cancel{1} + \tfrac{1}{2}\left(-t\cos(t)\Big|_0^t + \int_0^t \cos(t)\,dt\right)$$

$$= -\sin(t) + \cos(t) - \tfrac{1}{2}t\cos(t) + \tfrac{1}{2}\sin(t)\Big|_0^t$$

$$\cancel{\cos(t)} - \tfrac{1}{2}\cancel{\sin(t)} - \tfrac{1}{2}t\cancel{\cos(t)} = -\sin(t) + \cancel{\cos(t)} - \tfrac{1}{2}t\cancel{\cos(t)} + \tfrac{1}{2}\cancel{\sin(t)}$$

$0 = 0$ checks

7- Given the matrix equation $\bar{x}' = \bar{\bar{A}}\bar{x} + \bar{f}$, show that the particular solution integral is given by $\bar{v}(t) = \bar{\bar{X}}(t) \int \bar{\bar{X}}^{-1}(t)\,\bar{f}(t)\,dt$ where $\bar{\bar{X}}(t)$ is the fundamental matrix of the homogeneous equation.

We consider $\bar{x}' = \bar{\bar{A}}\bar{x} + \bar{f}$

Let's first find the homogeneous solution $\bar{x}' = \bar{\bar{A}}\bar{x} \Rightarrow \bar{x}' - \bar{\bar{A}}\bar{x} = 0$

If the system has a full set of eigenvalues and eigenvectors, we can create the fundamental matrix where the columns are the individual eigenvectors.

$\therefore \bar{x}_h = \bar{\bar{X}}\bar{c}$ where $\bar{\bar{X}}$ is the fundamental matrix where is the solution to the homogeneous system. $\bar{\bar{X}} = \bar{\bar{X}}(t)$

Particular Solution to $\bar{x}' = \bar{\bar{A}}\bar{x} + \bar{f}$

Let's look for the particular solution of the form

$\bar{v}_p(t) = \bar{\bar{X}}(t)\,\bar{u}(t)$ where $\bar{\bar{X}}$ is the fundamental matrix and $\bar{u}$ and unknown vector.

$\bar{v}_p'(t) = \bar{\bar{X}}'(t)\,\bar{u}(t) + \bar{\bar{X}}(t)\,\bar{u}'(t)$ let's substitute into the matrix D.E. $\bar{x}' = \bar{\bar{A}}\bar{x} + \bar{f}$

$$\bar{v}_p' = \bar{\bar{A}}\bar{v}_p + \bar{f} \Rightarrow \bar{\bar{X}}'(t)\bar{u}(t) + \bar{\bar{X}}(t)\bar{u}'(t) = \bar{\bar{A}}\,\bar{\bar{X}}\,\bar{u} + \bar{f} \qquad \text{now } \bar{\bar{X}}' = \bar{\bar{A}}\bar{\bar{X}}$$

$$\Rightarrow \cancel{\bar{\bar{A}}\bar{\bar{X}}\,\bar{u}(t)} + \bar{\bar{X}}(t)\,\bar{u}'(t) = \cancel{\bar{\bar{A}}\,\bar{\bar{X}}\bar{u}} + \bar{f}$$

$$\Rightarrow \bar{\bar{X}}(t)\,\bar{u}'(t) = \bar{f} \Rightarrow \bar{u}'(t) = \bar{\bar{X}}^{-1}(t)\,\bar{f}(t)$$

$$\Rightarrow \bar{u}(t) = \int \bar{\bar{X}}^{-1}(t)\,\bar{f}(t)\,dt$$

Recall $\bar{v}_p(t) = \bar{\bar{X}}(t)\,\bar{u}(t)$

$$\therefore \bar{v}_p(t) = \bar{\bar{X}}(t) \int \bar{\bar{X}}^{-1}\,\bar{f}(t)\,dt$$

8. Obtain the solution to the simulltaneous equations

$$x' + y' + x = -e^{-t}$$

$$x' + 2y' + 2x + 2y = 0$$

which satisfies the initial conditions: $x(0) = -1$ and $y(0) = 1$

$$\begin{matrix} x' + y' + x = -e^{-t} \\ x' + 2y' + 2x + 2y = 0 \end{matrix} \Rightarrow \begin{matrix} (D+1)x + Dy = -e^{-t} \\ (D+2)x + (2D+2)y = 0 \end{matrix} \quad \text{where } D \equiv \frac{d}{dt}$$

let $\bar{x} \equiv e^{\lambda t}$

$\bar{y} \equiv e^{\lambda t}$

$$\begin{pmatrix} (D+1) & D \\ (D+2) & 2(D+1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda+1 & \lambda \\ \lambda+2 & 2(\lambda+1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}$$

Homogeneous Solution $\begin{vmatrix} \lambda+1 & \lambda \\ \lambda+2 & 2(\lambda+1) \end{vmatrix} = 0 \Rightarrow$

$(\lambda+1)(2)(\lambda+1) - (\lambda)(\lambda+2) = 0$

$2(\lambda^2 + 2\lambda + 1) - \lambda^2 - 2\lambda = 0$

$2\lambda^2 + 4\lambda + 2 - \lambda^2 - 2\lambda = 0$

$\lambda^2 + 2\lambda + 2 = 0 \Rightarrow (\lambda+1)^2 = -1 \quad \lambda = -1 \pm i$

for $\lambda = -1+i$ $\begin{pmatrix} -\cancel{1}+i\cancel{+1} & -1+i \\ -1+i+2 & 2(-1+i+1) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} i & i-1 \\ i+1 & 2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$ai + b(i-1) = 0 \quad a = i-1 \quad b = -i$

$a(i+1) + 2ib = 0$

for $\lambda = -1-i$ $\begin{pmatrix} -1-i+1 & -1-i \\ -1-i+2 & 2(-1-i+1) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -i & -i-1 \\ 1-i & -2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$-ai - b(i+1) = 0 \quad a = i+1 \quad b = -i$

$a(1-i) - 2ib = 0$

$$v_1 = \begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{(-1+i)t} \quad v_2 = \begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-(1+i)t} \Rightarrow \bar{\bar{X}} = \begin{pmatrix} (i-1)e^{(-1+i)t} & (i+1)e^{-(1+i)t} \\ (-i)e^{(-1+i)t} & (-i)e^{-(1+i)t} \end{pmatrix}$$

$$\bar{X}_h(t) = C_1 \begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{(-1+i)t} + C_2 \begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-(1+i)t}$$

now to find the particular solution vector. try $\bar{v}_p(t) = \bar{b}\, e^{-t} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-t}$

insert into the matrix system.

$\bar{v}_p'(t) = -\bar{b}\, e^{-t}$

$$\begin{pmatrix} (-1+1) & (-1) \\ (-1+2) & 2(-1+1) \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-t} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-t} \Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$(0)b_1 - b_2 = -1 \quad b_2 = 1$

$b_1 + (0)b_2 = 0 \quad b_1 = 0$

$$\therefore \vec{v}_p(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

$$\overline{X}(t) = \overline{X}_h(t) + \vec{v}_p(t) = C_1\begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{(-1+i)t} + C_2\begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-(1+i)t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

now to find C_1 and C_2 using the I.C. $\overline{X}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\overline{X}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} i-1 \\ -i \end{pmatrix} + C_2\begin{pmatrix} i+1 \\ -i \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad -1 = C_1(i-1) + C_2(i+1)$$
$$1 = C_1(-i) + C_2(-i) + 1 \quad \rightarrow \quad C_1(-i) + C_2(-i) = 0$$

Add $\quad 0 = -C_1 + C_2 + 1 \quad \Rightarrow \quad C_1 = 1 + C_2$

$$(1+C_2)(-i) + C_2 i = 0$$
$$-i - C_2 i - C_2 i = 0 \Rightarrow -2C_2 i = i$$
$$C_2 = -\tfrac{1}{2}$$
$$C_1 = \tfrac{1}{2}$$

$$\overline{X}(t) = \frac{1}{2}\begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{(-1+i)t} - \frac{1}{2}\begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-(1+i)t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

$$= \frac{e^{-t}}{2}\begin{pmatrix} i-1 \\ -i \end{pmatrix} e^{it} - \frac{e^{-t}}{2}\begin{pmatrix} i+1 \\ -i \end{pmatrix} e^{-it} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

$$\Rightarrow x(t) = e^{-t}(i-1)\frac{e^{it}}{2} - e^{-t}(i+1)\frac{e^{-it}}{2}$$
$$y(t) = e^{-t}(-i)\frac{e^{it}}{2} - e^{-t}(-i)\frac{e^{-it}}{2} + e^{-t}$$

$$\Rightarrow x(t) = e^{-t}\left(\left(\frac{i}{2}e^{it} - \frac{i}{2}e^{-it}\right) - \left(\frac{e^{it}}{2} - \frac{e^{-it}}{2}\right)\right) = e^{-t}\left(-\left(\frac{e^{it} - e^{-it}}{2i}\right) - \left(\frac{e^{it} + e^{-it}}{2}\right)\right)$$
$$y(t) = e^{-t}\left(-\frac{i}{2}e^{it} + \frac{i}{2}e^{-it}\right) + e^{-t} = e^{-t}\left(\frac{e^{it} - e^{-it}}{2i}\right) + e^{-t}$$

$$x(t) = -e^{-t}(\sin(t) + \cos(t))$$
$$y(t) = e^{-t}(1 + \sin(t))$$

$$\overline{X}(t) = e^{-t}\begin{pmatrix} -\sin(t) - \cos(t) \\ \sin(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

9. Solve the following differential equation using Fourier Transforms:

$3y'' + 10y' + 3y = 64e^{3t}(H(t) - H(t-5))$ for $t > 0$ with conditions

$y(0) = -1$ and $y'(0) = 0.$ half-space problem : $(0 \le t < \infty)$

On the half-space $F\{f'(t)\} = i\omega F\{f(t)\} - f(0) = i\omega F(\omega) - f(0)$

$F\{f''(t)\} = i\omega F\{f'(t)\} - f'(0) = (i\omega)^2 F(\omega) - i\omega f(0) - f'(0)$

$$3[(i\omega)^2 \bar{y}(\omega) - i\omega \overset{-1}{y(0)} - \overset{0}{\cancel{y'(0)}}] + 10[(i\omega)\bar{y}(\omega) - \overset{-1}{y(0)}] + 3\bar{y}(\omega) = F\{64e^{3t}(H(t) - H(t-5))\}$$

$$3[(i\omega)^2 \bar{y}(\omega) + i\omega] + 10[(i\omega)\bar{y}(\omega) + 1] + 3\bar{y}(\omega) = F\{64e^{3t}(H(t) - H(t-5))\}$$

$$3(i\omega)^2\bar{y}(\omega) + 3(i\omega) + 10(i\omega)\bar{y}(\omega) + 10 + 3\bar{y}(\omega) = F\{\quad\}$$

$$[3(i\omega)^2 + 10(i\omega) + 3]\,\bar{y}(\omega) + 3(i\omega) + 10 = F\{\quad\}$$

$$((i\omega)^2 + \tfrac{10}{3}(i\omega) + 1)\,\bar{y}(\omega) + i\omega + \tfrac{10}{3} = \tfrac{1}{3}F\{\quad\}$$

$$((i\omega)^2 + \tfrac{10}{3}(i\omega) + 1)\,\bar{y}(\omega) = -i\omega - \tfrac{10}{3} + \tfrac{1}{3}F\{\quad\}$$

$$\bar{y}(\omega) = \frac{-i\omega}{(i\omega)^2 + \frac{10}{3}(i\omega) + 1} - \left(\frac{10}{3}\right)\left(\frac{1}{(i\omega)^2 + \frac{10}{3}(i\omega) + 1}\right) + \frac{1}{3}\left(\frac{1}{(i\omega)^2 + \frac{10}{3}(i\omega) + 1}\right)F\{\quad\}$$

$$= \frac{-i\omega}{(i\omega+3)(i\omega+\frac{1}{3})} - \left(\frac{10}{3}\right)\left(\frac{1}{(i\omega+3)(i\omega+\frac{1}{3})}\right) + \frac{1}{3}\left(\frac{1}{(i\omega+3)(i\omega+\frac{1}{3})}\right)F\{\quad\}$$

$$= -\left(\left(\frac{9}{8}\right)\left(\frac{1}{i\omega+3}\right) - \left(\frac{1}{8}\right)\left(\frac{1}{i\omega+\frac{1}{3}}\right)\right) - \left(\frac{10}{3}\right)\left(\left(-\frac{3}{8}\right)\left(\frac{1}{i\omega+3}\right) + \frac{3}{8}\left(\frac{1}{i\omega+\frac{1}{3}}\right)\right)$$
$$+ \frac{1}{3}\left(\left(-\frac{3}{8}\right)\left(\frac{1}{i\omega+3}\right) + \left(\frac{3}{8}\right)\left(\frac{1}{i\omega+\frac{1}{3}}\right)\right)F\{\quad\}$$

$$= -\frac{1}{8}\left(\frac{9}{i\omega+3} - \frac{1}{i\omega+\frac{1}{3}}\right) + \frac{5}{4}\left(\frac{1}{i\omega+3} - \frac{1}{i\omega+\frac{1}{3}}\right) - \frac{1}{8}\left(\frac{1}{i\omega+3} - \frac{1}{i\omega+\frac{1}{3}}\right)F\{\quad\}$$

$$F^{-1}\{\bar{y}(\omega)\} = y(t) = F^{-1}\left\{\frac{1}{8}\left(\frac{1}{i\omega+3} - \frac{9}{i\omega+\frac{1}{3}}\right) - 8\underbrace{\left(\frac{1}{i\omega+3} - \frac{1}{i\omega+\frac{1}{3}}\right)F\{\quad\}}_{\text{use convolution}}\right\}$$

$$y(t) = \frac{1}{8}\left(e^{-3t} - 9e^{-\frac{t}{3}}\right)\overset{H(t)}{\downarrow} - 8\int_{-\infty}^{\infty}\left(e^{-3\lambda} - e^{-\frac{\lambda}{3}}\right)H(\lambda)\,64e^{3(t-\lambda)}\left(H(t-\lambda) - H(t-\lambda-5)\right)d\lambda$$

$$= \underset{H(t)}{\frac{1}{8}\left(e^{-3t} - 9e^{-\frac{t}{3}}\right)} - 8\int_{0}^{\infty}\left(e^{-3\lambda} - e^{-\lambda/3}\right)e^{3(t-\lambda)}\left(H(t-\lambda) - H(t-\lambda-5)\right)d\lambda$$

$$y(t) = \frac{1}{8}\left(e^{-3t} - 9e^{-t/3}\right)\overset{H(t)}{\swarrow} - 8\underbrace{\int_0^\infty \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right)\left(H(t-\lambda) - H(t-\lambda-5)\right)d\lambda}_{①}$$

$$① = \int_0^\infty \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) H(t-\lambda)\,d\lambda - \int_0^\infty \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) H(t-\lambda-5)\,d\lambda$$

$t - \lambda - 5 = 0$

$\lambda = t-5$

$$= \int_0^t \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) d\lambda\, H(t) + \overset{\to 0}{\int_t^\infty \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) d\lambda\, H(t-\infty)}$$

$$- \int_0^{t-5} \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) d\lambda\, H(t-5) + \overset{\to 0}{\cancel{\int_{t-5}^\infty \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) d\lambda\, H(t-5-\infty)}}$$

$$= \int_0^t \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) d\lambda\, H(t) - \int_0^{t-5} \left(e^{3(t-2\lambda)} - e^{3(t-\frac{10}{9}\lambda)}\right) d\lambda\, H(t-5)$$

$$= \left(e^{3t}\int_0^t e^{-6\lambda} d\lambda - e^{3t}\int_0^t e^{-\frac{10}{3}\lambda} d\lambda\right) H(t) - \left(e^{3t}\int_0^{t-5} e^{-6\lambda} d\lambda - e^{3t}\int_0^{t-5} e^{-\frac{10}{3}\lambda} d\lambda\right) H(t-5)$$

$$= \left(e^{3t}\left(-\frac{1}{6}e^{-6\lambda}\right)\Big|_0^t - e^{3t}\left(-\frac{3}{10}e^{-\frac{10}{3}\lambda}\right)\Big|_0^t\right) H(t) - \left(e^{3t}\left(-\frac{1}{6}e^{-6\lambda}\right)\Big|_0^{t-5} - e^{3t}\left(-\frac{3}{10}e^{-10/3\,\lambda}\right)\Big|_0^{t-5}\right) H(t-5)$$

$$= \left[e^{3t}\left(\frac{1-e^{-6t}}{6}\right) - 3e^{3t}\left(\frac{1-e^{-10/3\,t}}{10}\right)\right] H(t) - \left[e^{3t}\left(\frac{1-e^{-6(t-5)}}{6}\right) - 3e^{3t}\left(\frac{1-e^{-10/3\,(t-5)}}{10}\right)\right] H(t-5)$$

$$\therefore y(t) = \frac{1}{8}\left(e^{-3t} - 9e^{-t/3}\right) H(t) - 8\left[e^{3t}\left(\frac{1-e^{-6t}}{6}\right) - 3e^{3t}\left(\frac{1-e^{-10/3\,t}}{10}\right)\right] H(t)$$

$$+ 8\left[e^{3t}\left(\frac{1-e^{-6(t-5)}}{6}\right) - 3e^{3t}\left(\frac{1-e^{-10/3\,(t-5)}}{10}\right)\right] H(t-5)$$