Extra credit
October 18, 2013
Turn in the day of the exam

NEEP 547
DLH

I am counting on everyone treating this as an individual effort not a group effort. Not all differential equation methods we examined are included below.

1. (3pts) Find the solution to the differential equation that satisfies the initial condition
$(2\,xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$; where $y(1) = \ln(2)$.

2. (3pts) Solve the following differential equation
$2\,xy^3(y\,dx + x\,dy) = (y\,dx - x\,dy)\sin(x/y)$

3. (3pts) Find the solution which satisfies the given condition
$(x^4 + y^4)\,dx = 2\,x^3\,y\,dy$; where $y(1) = 0$

4. (3pts) Find the general solution for the equation, using the indicated solution to the homogeneous equation to reduce the order of the equation
$(2x + 1)y'' - 4(x + 1)y' + 4y = (2x + 1)^2/(x + 1); y_1 = e^{2x}$.

5. (3pts) Find the complete solution for the follow equation:
$(D^3 + D^2 + 3D - 5)\,y = e^x$.

6. (3pts) Find ~~a~~ the solution ~~for~~ to the following equation:
$3\,xy' + y + x^2y^4 = 0$

7. (3pts) Solve the following equation: $(y')^2\,y'' = 1 + (y')^2$.

8. (4pts) Obtain the solution of the simultaneous equations

$$\begin{aligned} x' + y' + x &= -e^{-t}, \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}$$

which satisfies the initial conditions: $x(0) = -1$, and $y(0) = 1$.

1. Find the solution to the differential equation that satisfies the initial condition

$$\underbrace{(2xy + e^y)}_{M} dx + \underbrace{(x^2 + xe^y)}_{N} dy = 0 \; ; \text{ where } y(1) = \ln(2)$$

let's check for exactness.

$$\frac{\partial M}{\partial y} = 2x + e^y \, , \quad \frac{\partial N}{\partial x} = 2x + e^y$$

Eq. is exact

$$\phi(x,y) = c \; ; \quad \underbrace{\frac{\partial \phi}{\partial x}}_{M} dx + \underbrace{\frac{\partial \phi}{\partial y}}_{N} dy = 0$$

$$\frac{\partial \phi}{\partial x} = 2xy + e^y \Rightarrow \int d\phi = \int (2xy + e^y) dx \Rightarrow \phi(x,y) = x^2 y + xe^y + g(y)$$

$$\frac{\partial \phi}{\partial y} = x^2 + xe^y + \frac{dg(y)}{dy}$$

$$N = \frac{\partial \phi}{\partial y} \Rightarrow \cancel{x^2} + xe^y = x^2 + xe^y + \frac{dg(y)}{dy}$$

$$\Rightarrow \frac{dg(y)}{dy} = 0 \Rightarrow g(y) = \text{Constant.}$$

$\phi(x,y) = x^2 y + xe^y + c_1 = c \Rightarrow x^2 y + xe^y = c_2$ now use $y(x=1) = \ln(2)$

$$(1)^2 \ln(2) + (1) e^{\ln(2)} = c_2 \Rightarrow \ln(2) + 2 = c_2$$

$$\therefore x^2 y + xe^y = 2 + \ln(2)$$

2. Solve the following differential equation

$$2xy^3(y\,dx + x\,dy) = (y\,dx - x\,dy)\sin(x/y)$$

$$2xy(y\,dx + x\,dy) = \frac{(y\,dx - x\,dy)}{y^2}\sin(x/y)$$

note the following $d\left(\frac{x}{y}\right) = \frac{1}{y}dx - \frac{x}{y^2}dy = \frac{y\,dx - x\,dy}{y^2}$

also $2xy(y\,dx + x\,dy) = 2xy\,d(xy) = d((xy)^2)$

$$\therefore\ 2xy(y\,dx + x\,dy) = \frac{(y\,dx - x\,dy)}{y^2}\sin\left(\frac{x}{y}\right)$$

$$\Rightarrow\ d((xy)^2) = d\left(\frac{x}{y}\right)\sin\left(\frac{x}{y}\right)$$ let $u = \frac{x}{y}$ and $z = xy$

$$d(z^2) = d(u)\sin(u)$$

$$d(z^2) = d(-\cos(u))$$

$$\Rightarrow\ z^2 = -\cos(u) + C$$ back substitution gives

$$(xy)^2 = -\cos\left(\frac{x}{y}\right) + C$$

3) Find the solution which satisfies the given condition

$(x^4+y^4)dx = 2x^3y\,dy$; where $y(1)=0$.

let's check to see if the eq. is homogeneous $\quad x \to tx,\ y \to ty$

$((xt)^4+(yt)^4)d(tx) = 2(xt)^3(yt)\,d(ty) \Rightarrow t^5(x^4+y^4)dx = t^5(2x^3y)dy$

$\Rightarrow (x^4+y^4)dx = (2x^3y)dy$

Eq. is a homogeneous Eq. $\quad$ let $y=ux$

$dy = x\,du + u\,dx$

$(x^4+y^4)dx = (2x^3y)dy$

$\Rightarrow (x^4+u^4x^4)dx = (2x^3ux)(x\,du+u\,dx)$

$x^4(1+u^4)dx = 2x^4u(x\,du+u\,dx) \Rightarrow (1+u^4)dx = 2u(x\,du+u\,dx)$

$(1+u^4)dx = 2ux\,du + 2u^2dx$

$(1+u^4-2u^2)dx = 2ux\,du$

$$\frac{dx}{x} = \frac{2u\,du}{(1+u^4-2u^2)} \Rightarrow \frac{dx}{x} = \frac{2u\,du}{(u^2-1)^2} = \frac{d(u^2)}{(u^2-1)^2} \qquad \text{let } u^2=z$$

$$\frac{dx}{x} = \frac{dz}{(z-1)^2} \Rightarrow \ln(x) = -\frac{1}{z-1} + c \qquad z=u^2$$

$$\ln(x) = -\frac{1}{u^2-1} + c \qquad \text{now } y=ux \Rightarrow u=\frac{y}{x}$$

$$\ln(x) = -\frac{1}{\left(\frac{y}{x}\right)^2-1} + c \Rightarrow \ln(x) = -\frac{x^2}{y^2-x^2} + c$$

let's find C from $y(1)=0$: $\ln(x=1) = -\frac{1}{0-1} + c \Rightarrow 0 = 1+c \Rightarrow c=-1$

$$\ln(x) = -1 - \frac{x^2}{y^2-x^2} \Rightarrow 1+\ln(x) = \frac{x^2}{x^2-y^2}$$

$(x^2-y^2)(1+\ln(x)) = x^2 \Rightarrow x^2(1+\ln(x)) - y^2(1+\ln(x)) = x^2$

$x^2+x^2\ln(x) - y^2(1+\ln(x)) = x^2 \Rightarrow y^2(1+\ln(x)) = x^2\ln(x)$

$$\therefore y^2 = \frac{x^2\ln(x)}{(1+\ln(x))}$$

4. Find the general solution for the equation using the indicated solution to the homogeneous equation to reduce the order of the equation.

$(2x+1)y'' - 4(x+1)y' + 4y = (2x+1)^2/(x+1); \quad y_1 = e^{2x}$

$y_2 = uy_1, \ y_2 = ue^{2x}, \ y_2' = u'e^{2x} + 2ue^{2x}, \ y_2'' = u''e^{2x} + 2u'e^{2x} + 2u'e^{2x} + 4ue^{2x}$

$= u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$

let's work on the homogeneous eq.

$(2x+1)(u''e^{2x} + 4u'e^{2x} + 4ue^{2x}) - 4(x+1)(u'e^{2x} + 2ue^{2x}) + 4ue^{2x} = 0$

$(2x+1)(u'' + 4u' + 4u) - 4(x+1)(u' + 2u) + 4u = 0$

$(2x+1)u'' + 4(2x+1)u' + 4(2x+1)u - 4(x+1)u' - 8(x+1)u + 4u = 0$

$(2x+1)u'' + 8xu' + 4u' + \cancel{8xu} + \cancel{4u} - 4xu' - \cancel{4u'} - \cancel{8xu} - \cancel{8u} + \cancel{4u} = 0$

$(2x+1)u'' + 4xu' = 0$

$u'' + \frac{4x}{2x+1}u' = 0 \qquad \Rightarrow u'' + \left(\frac{4x+2-2}{2x+1}\right)u' = 0$

$u'' + \left(\frac{2(2x+1)}{2x+1} - \frac{2}{2x+1}\right)u' = 0$

$u'' + \left(2 - \frac{2}{2x+1}\right)u' = 0 \qquad$ let $v = u' \quad v' = u''$

$v' + \left(2 - \frac{2}{2x+1}\right)v = 0$

$\frac{dv}{v} = \left(-2 + \frac{2}{2x+1}\right)dx \Rightarrow \ln(v) = -2x + \ln(2x+1) + c$

$v(x) = e^{(-2x + \ln(2x+1) + c)} \Rightarrow v(x) = Ce^{-2x}(2x+1)$

now $v = \frac{du}{dx} \Rightarrow \frac{du}{dx} = Ce^{-2x}(2x+1)$

$\int du = \int Ce^{-2x}(2x+1)\,dx \Rightarrow u(x) = c\left(\int e^{-2x}2x\,dx + \int e^{-2x}dx\right)$

$u(x) = c\left(-\tfrac{1}{2}e^{-2x} + \int x\,d(-e^{-2x})\right)$

$= c\left(-\tfrac{1}{2}e^{-2x} - xe^{-2x} + \int e^{-2x}dx\right)$

$= c\left(-\tfrac{1}{2}e^{-2x} - xe^{-2x} - \tfrac{1}{2}e^{-2x}\right) = c\left(-e^{-2x} - xe^{-2x}\right)$

$= -ce^{-2x}(1+x)$

$y_2 = uy_1 = \left(-ce^{-2x}(1+x)\right)\left(e^{2x}\right) = -c(1+x) = c_2(1+x)$

$$y_h = C_1 y_1 + y_2 = C_1 e^{2x} + C_2(1+x)$$

now to find the particular solution using variation of parameters.

$$(2x+1)y'' - 4(x+1)y' + 4y = \frac{(2x+1)^2}{(x+1)}, \quad y_1 = e^{2x} \quad y_2 = 1+x, \; y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{-y_2 f(x)}{a(x) W(x)}\,dx \quad \text{and } u_2 = \int \frac{y_1 f(x)}{a(x) W(x)}\,dx$$

where $a(x) = (2x+1)$ and $W(x) \equiv$ Wronskian.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & 1+x \\ 2e^{2x} & 1 \end{vmatrix} = e^{2x} - 2e^{2x}(1+x) = e^{2x} - 2e^{2x} - 2xe^{2x}$$

$$= -e^{2x} - 2xe^{2x} = -e^{2x}(1+2x)$$

$$u_1 = \int \frac{(1+x)\left(\frac{(2x+1)^2}{(1+x)}\right)(-1)}{(2x+1)(-e^{2x})(1+2x)}\,dx = \int \frac{(2x+1)^2 e^{-2x}}{(2x+1)^2}\,dx = \int e^{-2x}\,dx = -\tfrac{1}{2}e^{-2x}$$

$$u_2 = \int \frac{(e^{2x})\left(\frac{(2x+1)^2}{x+1}\right)}{(2x+1)(-e^{2x})(1+2x)}\,dx = -\int \frac{1}{1+x}\,dx = -\ln(1+x)$$

$$y_p = u_1 y_1 + u_2 y_2 = (e^{2x})(-\tfrac{1}{2}e^{-2x}) + (1+x)(-\ln(1+x))$$

$$= -\tfrac{1}{2} - (1+x)\ln(1+x)$$

$$y = y_h + y_p = C_1 e^{2x} + C_2(1+x) - \tfrac{1}{2} - (1+x)\ln(1+x)$$

5. Find the complete solution for the following equation.

$(D^3 + D^2 + 3D - 5)y = e^x$

assume $y = e^{mx}$

characteristic Eq for the homogeneous Eq.

$(m^3 + m^2 + 3m - 5) = 0$

$(m-1)(\underbrace{m^2+2m+5}) = 0$

$(m^2+2m+1) = -5+1 \Rightarrow (m+1)^2 = -4 \Rightarrow m+1 = \pm 2i$

$m = -1 \pm 2i$

roots are $m = 1, -1+2i, -1-2i$

$y_h(x) = C_1 e^x + C_2 e^{(-1+2i)x} + C_3 e^{(-1-2i)x}$

$= C_1 e^x + C_2 e^{-x} e^{2ix} + C_3 e^{-x} e^{-2ix}$

$= C_1 e^x + e^{-x}(C_4 \sin(2x) + C_5 \cos(2x))$

need to find the particular solution. note the inhomogeneous term is also one of the homogeneous solutions.

hence we will try $y_p(x) = ae^x + bxe^x$

$y_p'(x) = ae^x + be^x + bxe^x$

$y_p''(x) = ae^x + be^x + be^x + bxe^x = ae^x + 2be^x + bxe^x$

$y_p'''(x) = ae^x + 2be^x + be^x + bxe^x = ae^x + 3be^x + bxe^x$

substitute into the D.E.

$ae^x + 3be^x + bxe^x + ae^x + 2be^x + bxe^x + 3ae^x + 3be^x + 3bxe^x - 5ae^x - 5bxe^x = e^x$

$5ae^x + 8be^x + 5bxe^x - 5ae^x - 5bxe^x = e^x$

$8be^x = e^x \Rightarrow b = 1/8 \quad \therefore y_p = \frac{1}{8}xe^x$

$y = y_h + y_p = C_1 e^x + e^{-x}(C_4 \sin(2x) + C_5 \cos(2x)) + \frac{1}{8}xe^x$

6. Find the solution to the following Eq.

$$3xy' + y + x^2y^4 = 0$$

$$3xy' + y = -x^2y^4 \Rightarrow y' + \frac{1}{3x}y = -\frac{x}{3}y^4 \quad \text{Bernolli Eq. where } n=4$$

$$y' + \frac{1}{3x}y = -\frac{x}{3}y^4$$

let $u = y^{1-n} = y^{-3}$, $u = y^{-3}$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$du = -3y^{-4}dy$

$\frac{dy}{du} = -\frac{1}{3}y^4$

$$\frac{dy}{dx} = -\frac{1}{3}y^4\frac{du}{dx}$$

$$y' + \frac{1}{3x}y = -\frac{x}{3}y^4 \Rightarrow -\frac{1}{3}y^4\frac{du}{dx} + \frac{1}{3x}y = -\frac{x}{3}y^4$$

$u = y^{-3} \Rightarrow y = uy^4$

$$-\frac{1}{3}y^4\frac{du}{dx} + \frac{1}{3x}uy^4 = -\frac{x}{3}y^4$$

$$-\frac{1}{3}\frac{du}{dx} + \frac{1}{3x}u = -\frac{x}{3} \Rightarrow \frac{du}{dx} - \frac{1}{x}u = x \qquad I_f = e^{-\int\frac{1}{x}dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\Rightarrow d\left(\frac{1}{x}u\right) = x\cdot\left(\frac{1}{x}\right)dx$$

$$\frac{1}{x}u = x + c \Rightarrow u = x^2 + cx$$

recall $u = y^{-3}$

$$y^{-3} = x^2 + cx \Rightarrow y^3 = \frac{1}{x^2+cx} \Rightarrow y = \frac{1}{(x^2+cx)^{1/3}}$$

7. Solve the following equation $(y'^2)\, y'' = 1 + (y')^2$

This is an inhomogeneous eq. missing x and y explicitly

$(y')^2 y'' = 1 + (y')^2$ let $v(x) = y'$ $v' = y''$

$v^2 v' = 1 + v^2$

$v^2 \frac{dv}{dx} = 1 + v^2 \Rightarrow \frac{v^2 dv}{1+v^2} = dx \Rightarrow \left(\frac{1+v^2}{1+v^2} - \frac{1}{1+v^2}\right) dv = dx$

$\Rightarrow \int \left(1 - \frac{1}{1+v^2}\right) dv = \int dx \Rightarrow v - \tan^{-1}(v) = x + c$

$\Rightarrow x = v - \tan^{-1}(v) + c_1$

back to the original eq. $(y')^2 y'' = 1 + (y')^2$ let $\frac{dy}{dx} = v(y)$

$\frac{d^2y}{dx^2} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$

$(v^2)\left(v\frac{dv}{dy}\right) = 1 + (v)^2 \Rightarrow v^3 \frac{dv}{dy} = 1 + v^2$

$\left(\frac{v^3}{1+v^2}\right) dv = dy \Rightarrow \left(\frac{v^3+v}{1+v^2} - \frac{v}{1+v^2}\right) dv = dy = \left(\frac{v(v^2+1)}{v^2+1} - \frac{v}{v^2+1}\right) dv = dy$

$\Rightarrow \int \left(v - \frac{v}{v^2+1}\right) dv = \int dy \Rightarrow \frac{v^2}{2} - \frac{1}{2}\ln(1+v^2) = y + c$

$\Rightarrow y = \frac{v^2}{2} - \frac{1}{2}\ln(1+v^2) + c_2$

This is a parametric solution in terms of the parameter v

$x = v - \tan^{-1}(v) + c_1$

$y = \frac{v^2}{2} - \frac{1}{2}\ln(1+v^2) + c_2$ where $v = \frac{dy}{dx}$

8. Obtain the solution of the simultaneous equations

$$x' + y' + x = -e^{-t}$$

$$x' + 2y' + 2x + 2y = 0$$

which satisfies the initial conditions: $x(0) = -1$ and $y(0) = 1$.

$$s\bar{X}(s) - x(0) + s\bar{Y}(s) - y(0) + \bar{X}(s) = -\frac{1}{s+1}$$

$$s\bar{X}(s) - x(0) + 2(s\bar{Y}(s) - y(0)) + 2\bar{X}(s) + 2\bar{Y}(s) = 0$$

$$s\bar{X}(s) + 1 + s\bar{Y}(s) - 1 + \bar{X}(s) = -\frac{1}{s+1} \Rightarrow (s+1)\bar{X}(s) + s\bar{Y}(s) = -\frac{1}{s+1} \quad ①$$

$$s\bar{X}(s) + 1 + 2s\bar{Y}(s) - 2 + 2\bar{X}(s) + 2\bar{Y}(s) = 0 \Rightarrow (s+2)\bar{X}(s) + 2(s+1)\bar{Y}(s) = 1 \quad ②$$

we have two equation, and two unknowns

$$① \quad (s+1)\bar{X}(s) + s\bar{Y}(s) = -\frac{1}{s+1} \Rightarrow \bar{X}(s) = -\left(\frac{1}{s+1}\right)^2 - \frac{s}{(s+1)}\bar{Y}(s)$$

$$② \quad (s+2)\left(-\left(\frac{1}{s+1}\right)^2 - \left(\frac{s}{s+1}\right)\bar{Y}(s)\right) + 2(s+1)\bar{Y}(s) = 1$$

$$-\frac{s(s+2)}{(s+1)}\bar{Y}(s) + 2(s+1)\bar{Y}(s) = 1 + \frac{(s+2)}{(s+1)^2}$$

$$-s(s+2)\bar{Y}(s) + 2(s+1)^2\bar{Y}(s) = (s+1) + \frac{s+2}{s+1}$$

$$(-s^2 - 2s + 2s^2 + 4s + 2)\bar{Y}(s) = (s+1) + \frac{s+2}{s+1} = \frac{(s+1)^2 + s + 2}{(s+1)} = \frac{s^2 + 3s + 3}{(s+1)}$$

$$(s^2 + 2s + 2)\bar{Y}(s) = \frac{s^2 + 3s + 3}{(s+1)}$$

$$\bar{Y}(s) = \frac{s^2 + 3s + 3}{(s+1)(s^2 + 2s + 2)}$$

$$\bar{X}(s) = -\frac{1}{(s+1)^2} - \left(\frac{s}{s+1}\right)\left(\frac{s^2 + 3s + 3}{(s+1)(s^2 + 2s + 2)}\right) = -\frac{(s^2 + 2s + 2) - (s)((s+1)^2 + s + 2)}{(s+1)^2(s^2 + 2s + 2)}$$

$$= \frac{-((s+1)^2 + 1) - s(s+1)^2 - (s)(s+2)}{(s+1)^2(s^2 + 2s + 2)}$$

$$= \frac{-(s+1)^2}{(s+1)^2(s^2 + 2s + 2)} - \frac{1}{(s+1)^2(s^2 + 2s + 2)} - \frac{s(s+1)^2}{(s+1)^2(s^2 + 2s + 2)} - \frac{s^2 - 2s}{(s+1)^2(s^2 + 2s + 2)}$$

$$= -\frac{1}{s^2 + 2s + 2} - \frac{s}{s^2 + 2s + 2} - \frac{(s^2 + 2s + 1)}{(s+1)^2(s^2 + 2s + 2)}$$

$$= -\frac{1}{s^2 + 2s + 2} - \frac{s}{s^2 + 2s + 2} - \frac{1}{(s^2 + 2s + 2)} = -\frac{(s+2)}{(s^2 + 2s + 2)}$$

$\therefore \bar{X}(s) = \dfrac{-(s+2)}{(s^2+2s+2)}$ and $\bar{Y}(s) = \dfrac{s^2+3s+3}{(s+1)(s^2+2s+2)}$

let's work on $\bar{X}(t)$: $\bar{X}(s) = -\dfrac{s+2}{(s^2+2s+2)} = -\dfrac{(s+1)}{(s+1)^2+1} - \dfrac{1}{(s+1)^2+1}$

$$x(t) = -\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$ let's use the s-shift theorem.

$$= -e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\cos(t)} - e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{\sin(t)}$$

$$x(t) = -e^{-t}\cos(t) - e^{-t}\sin(t) = -e^{-t}(\cos(t)+\sin(t))$$

now to work on $\bar{Y}(s)$: $\bar{Y}(t) = \dfrac{s^2+3s+3}{(s+1)(s^2+2s+2)} = \dfrac{(s+1)^2+(s+2)}{(s+1)((s+1)^2+1)}$

$$\bar{Y}(s) = \frac{(s+1)^2}{(s+1)((s+1)^2+1)} + \frac{(s+2)}{(s+1)((s+1)^2+1)} = \frac{(s+1)}{((s+1)^2+1)} + \frac{(s+1)}{(s+1)((s+1)^2+1)} + \frac{1}{(s+1)((s+1)^2+1)}$$

$$= \cancel{\frac{(s+1)}{((s+1)^2+1)}} + \frac{1}{((s+1)^2+1)} + \frac{1}{(s+1)} - \cancel{\frac{(s+1)}{(s+1)^2+1}}$$

$$= \frac{1}{((s+1)^2+1)} + \frac{1}{(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\}$$

$$= e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{\sin(t)} + e^{-t}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{1}$$

$$y(t) = e^{-t}\sin(t) + e^{-t} = e^{-t}(1+\sin(t))$$

Let's summarize

$$x(t) = -e^{-t}(\cos(t)+\sin(t))$$

$$y(t) = e^{-t}(1+\sin(t))$$