Homework Set No. 8
Due November 8, 2013

NEEP 547
DLH

Fourier expansions

1. (4pts) Find the Fourier expansions of the periodic function whose definition on one period is

$$f(t) = \begin{cases} t & \text{for } 0 < t < 2 \\ 4 - t & \text{for } 2 < t < 4. \end{cases}$$

2. (6pts) Find the complex exponential Fourier series of the periodic function whose definition on one period is $f(t) = \cosh(t) \quad -1 < t < 1$.

3. (8pts) Find the solution of the following differential equation which satisfies the given initial conditions:

$$y'' - 3y' + 2y = f(t) \quad ; y(0) = y'(0) = 0 \text{ and } f(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi. \end{cases}$$

(Hint: solve for the homogeneous Eqn. using O.D.E techniques and expand $f(t)$ in a Fourier series).

4. (8pts) A vibrating string, clamped at $x = 0$ and at $x = \ell$, is in a resisting medium which damps it motion. Its motion is described by the damped wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2} - k \frac{\partial u(x,t)}{\partial t}$$

with I.C.: $u(x,0) = f(x)$ and $\dfrac{\partial u(x,0)}{\partial t} = g(x)$ and B.C.: $u(0,t) = u(\ell,t) = 0$.

where v and k are constants and represent the propagation speed and damping coefficient, respectively. Find the displacement of the string (motion of the string) assuming the damping is large. Assume a Fourier expansion of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{\ell} x\right).$$

Why did we use the sine series and not the cosine series?

1) Find the Fourier expansion (sine, cosine) of the periodic function whose definition on one period is

$$f(t) = \begin{cases} t & 0<t<2 \\ 4-t & 2<t<4 \end{cases}$$

$p=2$

$$a_0 = \frac{1}{2}\int_0^4 f(t)\,dt = \frac{1}{2}\left(\int_0^2 t\,dt + \int_2^4 (4-t)\,dt\right)$$
$$= \frac{1}{2}\left(\left(\frac{t^2}{2}\right)\Big|_0^2 + \left(4t-\frac{t^2}{2}\right)\Big|_2^4\right)$$
$$= \frac{1}{2}\left(\frac{4}{2} + 16 - 8 - 8 + \frac{4}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{8}{2}\right) = 2$$

$$a_n = \frac{1}{2}\int_0^4 f(t)\cos\left(\frac{n\pi}{2}t\right)dt = \frac{1}{2}\left(\int_0^2 t\cos\left(\frac{n\pi}{2}t\right)dt + \int_2^4 4\cos\left(\frac{n\pi}{2}t\right)dt - \int_2^4 t\cos\left(\frac{n\pi}{2}t\right)dt\right)$$
$$= \frac{1}{2}\left(\int_0^2 t\,d\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}t\right)\right) + 4\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}t\right)\right)\Big|_2^4 - \int_2^4 t\,d\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}t\right)\right)\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)t\sin\left(\frac{n\pi}{2}t\right)\Big|_0^2 - \int_0^2\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}t\right)dt + 4\left(\frac{2}{n\pi}\right)(\sin(2n\pi)-\sin(n\pi)) - \left(\left(\frac{2}{n\pi}\right)\left(t\sin\left(\frac{n\pi}{2}t\right)\right)\Big|_2^4 - \int_2^4\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}t\right)dt\right)\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)(2\sin(n\pi)-0) + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}t\right)\Big|_0^2 + 4\left(\frac{2}{n\pi}\right)(0-0) - \left(\left(\frac{2}{n\pi}\right)(4\sin(2n\pi)-2\sin(n\pi))\right) - \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}t\right)\Big|_2^4\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)^2(\cos(n\pi)-1) - \left(\frac{2}{n\pi}\right)^2(\cos(2n\pi)-\cos(n\pi))\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)^2(2\cos(n\pi)-1-\cos(2n\pi))\right) = \frac{1}{2}\left(\left(\frac{2}{n\pi}\right)^2(2(-1)^n-1-1)\right)$$
$$= \left(\frac{2}{n\pi}\right)^2((-1)^n-1)$$ $=0$ for n-even; $=(-2)\left(\frac{2}{n\pi}\right)^2$ for n-odd

$$b_n = \frac{1}{2}\int_0^4 f(t)\sin\left(\frac{n\pi}{2}t\right)dt = \frac{1}{2}\left(\int_0^2 t\sin\left(\frac{n\pi}{2}t\right)dt + \int_2^4 4\sin\left(\frac{n\pi}{2}t\right)dt - \int_2^4 t\sin\left(\frac{n\pi}{2}t\right)dt\right)$$
$$= \frac{1}{2}\left(\int_0^2 t\,d\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\right) + 4\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}t\right)\Big|_2^4 - \int_2^4 t\,d\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\right)\right)$$
$$= \frac{1}{2}\left(\left(-\frac{2}{n\pi}\right)t\cos\left(\frac{n\pi}{2}t\right)\Big|_0^2 - \int_0^2\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}t\right)dt + 4\left(-\frac{2}{n\pi}\right)(\cos(2n\pi)-\cos(n\pi)) - \left(\left(-\frac{2}{n\pi}\right)\left(t\cos\left(\frac{n\pi}{2}t\right)\right)\Big|_2^4 + \left(\frac{2}{n\pi}\right)\int_2^4\cos\left(\frac{n\pi}{2}t\right)dt\right)\right)$$
$$= \frac{1}{2}\left(\left(-\frac{2}{n\pi}\right)(2)\cos(n\pi) + \left(\frac{4}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}t\right)\Big|_0^2 + 4\left(-\frac{2}{n\pi}\right)(1-(-1)^n) - \left(-\frac{2}{n\pi}\right)(4\cos(2n\pi) - 2\cos(n\pi)) - \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}t\right)\Big|_2^4\right)$$
$$= \frac{1}{2}\left((2)\left(-\frac{2}{n\pi}\right)(-1)^n + 4\left(-\frac{2}{n\pi}\right)(1-(-1)^n) - \left(-\frac{2}{n\pi}\right)(4-2(-1)^n)\right)$$
$$= \left(\frac{1}{2}\right)\left(-\frac{2}{n\pi}\right)\left(2(-1)^n + 4(1-(-1)^n) - 2(2-(-1)^n)\right) = \left(\frac{1}{2}\right)\left(-\frac{2}{n\pi}\right)\left(\cancel{2(-1)^n} + \cancel{4} - \cancel{4(-1)^n} - \cancel{4} + \cancel{2(-1)^n}\right) = 0$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{2}t\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{2}t\right)$$

$$= \frac{2}{2} - 8\sum_{\substack{n=1\\ n\text{-odd}}}^{\infty}\left(\frac{1}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}t\right) = 1 - \frac{8}{\pi^2}\sum_{m=1}^{\infty}\left(\frac{1}{2m-1}\right)^2\cos\left(\frac{(2m-1)\pi}{2}t\right)$$

2. Find the complex exponential Fourier series of the periodic function whose definition over one period is

$$f(t) = \cosh(t) \qquad -1 < t < 1 \qquad p = 1$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{n i \pi t}{p}} \qquad \text{where } c_n = \tfrac{1}{2}\int_{-1}^{1} f(t) e^{-n i \pi t/p}$$

$$c_n = \tfrac{1}{2}\int_{-1}^{1} \cosh(t) e^{-in\pi t}\,dt = \tfrac{1}{2}\left(-\tfrac{1}{in\pi}\right)\cosh(t) e^{-in\pi t}\Big|_{-1}^{1} + \tfrac{1}{2}\left(\tfrac{1}{in\pi}\right)\int_{-1}^{1} \sinh(t) e^{-in\pi t}\,dt$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{-\cosh(1)}{in\pi}\right)\left(e^{-in\pi} - e^{in\pi}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)\left(\left(\tfrac{-1}{in\pi}\right)\sinh(t) e^{-in\pi t}\Big|_{-1}^{1}\right) + \left(\tfrac{1}{in\pi}\right)\int_{-1}^{1}\cosh(t) e^{-in\pi t}\,dt\Big)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{\cosh(1)}{in\pi}\right)\left(e^{in\pi} - e^{-in\pi}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)^2\left(\sinh(1)\left(e^{in\pi} + e^{-in\pi}\right)\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)^2\int_{-1}^{1}\cosh(t) e^{-in\pi t}\,dt\Big)$$

$$\therefore \left(1 - \left(\tfrac{1}{in\pi}\right)^2\right)\left(\tfrac{1}{2}\right)\int_{-1}^{1}\cosh(t) e^{-in\pi t}\,dt = \left(\tfrac{1}{2}\right)\left(\tfrac{\cosh(1)}{in\pi}\right)\left(e^{in\pi} - e^{-in\pi}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{in\pi}\right)^2\left(\sinh(1)\left(e^{in\pi} + e^{-in\pi}\right)\right)$$

$$\text{thus } c_n = \tfrac{1}{2}\int_{-1}^{1}\cosh(t) e^{-in\pi t}\,dt = \left(\frac{(in\pi)^2}{(in\pi)^2 - 1}\right)\left(\frac{\cosh(1)}{n\pi}\underbrace{\left(\frac{e^{in\pi} - e^{-in\pi}}{2i}\right)}_{\sin(n\pi)} + \left(\tfrac{1}{n\pi}\right)^2\left(\sinh(1)\right)\underbrace{\left(\frac{e^{in\pi} + e^{-in\pi}}{2}\right)}_{\cos(n\pi)}\right)$$

$$c_n = \left(\frac{(in\pi)^2}{(in\pi)^2 - 1}\right)\left(\frac{\cosh(1)}{n\pi}\xrightarrow{} \cancel{\sin(n\pi)}^{\,0} + \left(\tfrac{1}{n\pi}\right)^2\left(\sinh(1)\underbrace{\cos(n\pi)}_{(-1)^n}\right)\right)$$

$$= \left(\frac{(in\pi)^2}{-(n\pi)^2 - 1}\right)\left(\left(\tfrac{1}{n\pi}\right)^2\sinh(1)\,(-1)^n\right) = \left(\frac{(n\pi)^2}{(n\pi)^2 + 1}\right)\left(\tfrac{1}{n\pi}\right)^2\sinh(1)\,(-1)^n$$

$$= \left(\frac{(-1)^n}{(n\pi)^2 + 1}\right)\sinh(1)$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{1}{(n\pi)^2 + 1}\right)\sinh(1)\, e^{in\pi t}$$

$$= \sinh(1) + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{(n\pi)^2 + 1}\right)\sinh(1)\left(\frac{e^{in\pi t} + e^{-in\pi t}}{2}\right)(2)$$

$$= \sinh(1) + \sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{(n\pi)^2 + 1}\right)\sinh(1)\cos(n\pi t)$$

3. $y'' - 3y' + 2y = f(t)$; $y(0) = y'(0) = 0$ and $f(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi \end{cases}$

Solution $y(t) = y_h(t) + y_p(t)$

solve $y'' - 3y' + 2y = 0$ for homogeneous solution, assume $y_h(t) = e^{mt}$

$m^2 - 3m + 2 = 0 \Rightarrow (m-2)(m-1) = 0$ ∴ $m = 1$ and 2

$y_h(t) = Ae^{t} + Be^{2t}$

$y_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}$, $y_p'(t) = \sum_{n=-\infty}^{\infty} (in) e^{int}$, $y_p''(t) = \sum_{n=-\infty}^{\infty} (in)^2 e^{int}$

$y_p''(t) - 3y_p' + 2y_p = f(t) \Rightarrow \sum_{n=-\infty}^{\infty} [(in)^2 - 3(in) + 2] C_n e^{int} = f(t) =$

$\Rightarrow \sum_{n=-\infty}^{\infty} [(in)^2 - 3(in) + 2] C_n e^{int} = \sum_{-\infty}^{\infty} b_n e^{int}$ where $b_n = \frac{1}{2\pi}\int_0^{\pi} (e^{-int'}) f(t')\,dt'$

b_n — need to find these

∴ $b_n = \frac{1}{2\pi}\int_0^{\pi} e^{-int'}\,dt' = \frac{1}{2\pi}\left(-\frac{1}{ni}\right) e^{(-int)}\Big|_0^{\pi}$

$= \left(\frac{1}{2\pi}\right)\left(\frac{1}{ni}\right)\left(1 - e^{-in\pi}\right) = \left(\frac{1}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(1 - e^{-in\pi}\right)$

$b_0 = \frac{1}{2\pi}\int_0^{\pi} dt' = \left(\frac{1}{2\pi}\right)(\pi) = \frac{1}{2}$

$$2C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n [(in)^2 - 3(in) + 2] e^{int} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\underbrace{\left(1 - e^{-in\pi}\right)}_{(1-(-1)^n)} e^{int}$$

$(1-(-1)^n)$ $= 0$ when n is even, $= 2$ when n is odd

$e^{-in\pi} = \cos(n\pi) - i\sin(n\pi) = (-1)^n$, ($\sin(n\pi) = 0$)

$$2C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n [(in)^2 - 3(in) + 2] e^{int} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0 \\ n\text{-odd}}}^{\infty} \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right) e^{int}$$

Equating terms $2C_0 = \frac{1}{2} \Rightarrow C_0 = \frac{1}{4}$

$C_n [(in)^2 - 3(in) + 2] = \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)$

$$C_n = \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{1}{(in)^2 - 3(in) + 2}\right)$$

$$= \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{1}{(2-n^2) - 3(in)}\right)\left(\frac{(2-n^2) + 3(in)}{(2-n^2) + 3(in)}\right)$$

$$= \left(\frac{2}{\pi}\right)\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{(2-n^2) + 3(in)}{(2-n^2)^2 + 9n^2}\right)$$

$$y_p(t) = \frac{1}{4} + \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0 \\ n\text{-odd}}}^{\infty} \left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{(2-n^2) + 3(in)}{(2-n^2)^2 + 9n^2}\right) e^{int}$$

$$y_p(t) = \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0 \\ n\text{-odd}}}^{\infty}{}' \left(\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2} e^{int}\right) + \left(\frac{3}{2}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) e^{int} \right)$$

$$= \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left[\left(\frac{1}{2i}\right)\left(\frac{1}{n}\right)\left(\frac{i(2-n^2)}{(2-n^2)^2+9n^2}\right)\left(e^{int} - e^{-int}\right) + \left(\frac{3}{2}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\left(e^{int} + e^{-int}\right) \right]$$

$$= \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left[\left(\frac{1}{n}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) \sin(nt) + (3)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) \cos(nt) \right]$$

Thus

$$y(t) = Ae^{t} + Be^{2t} + \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left[\left(\frac{1}{n}\right)\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) \sin(nt) + 3\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) \cos(nt) \right]$$

$$y(0) = 0 \Rightarrow A + B + \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' 3\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0 \qquad \text{I}$$

$$y'(0) = 0 \Rightarrow A + 2B + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0 \qquad \text{II}$$

$$\text{II} - \text{I} \Rightarrow B + \left(\frac{2}{\pi}\right) \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) - \frac{1}{4} - \left(\frac{2}{\pi}\right)(3) \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0$$

$$B = \frac{1}{4} + \frac{4}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)$$

using I

$$A + \frac{1}{4} + \frac{4}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) + \frac{1}{4} + \frac{6}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right) = 0$$

$$A = -\frac{1}{2} - \frac{10}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)$$

Thus

$$y(t) = \left(-\frac{1}{2} - \frac{10}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\right) e^{t} + \left(\frac{1}{4} + \frac{4}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\right) e^{2t}$$

$$+ \frac{1}{4} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\text{-odd}}}^{\infty}{}' \left[\left(\frac{(2-n^2)}{(2-n^2)^2+9n^2}\right)\left(\frac{1}{n} \sin(nt) + 3\cos(nt)\right) \right]$$

4. A vibrating string, clamped at $x=0$ and at $x=\ell$, is in a resisting medium which damps its motion. Its motion is described by the damped wave Eq.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial u(x,t)}{\partial t}$$

with I.C. $u(x,0)=f(x)$ and $\frac{\partial u(x,0)}{\partial t} = g(x)$ and B.C.: $u(0,t)=u(\ell,t)=0$ and where v and k are constants and represent the propagation speed and damping coefficient, respectively. Find the displacement of the string (motion of the string) assuming the damping is large. Assume a fourier expansion of the form $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi}{\ell}x\right)$

Why did we use the sine series and not the cosine series?

$$\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2} - k\frac{\partial u}{\partial t}$$ insert series solution form into the PDE.

$$\sum_{n=1}^{\infty}\left(\frac{d^2(b_n(t))}{dt^2}\sin\left(\frac{n\pi}{\ell}x\right) = v^2(-1)\left(\frac{n\pi}{\ell}\right)^2 b_n(t)\sin\left(\frac{n\pi}{\ell}x\right) - k\frac{d(b_n(t))}{dt}\sin\left(\frac{n\pi}{\ell}x\right)\right)$$

$$\sum_{n=1}^{\infty}\frac{d^2(b_n(t))}{dt^2} + k\frac{d(b_n(t))}{dt} + v^2\left(\frac{n\pi}{\ell}\right)^2 b_n(t) = 0$$ we have a 2^{nd} order D.E. for the b_n's. Assume $b_n(t) = e^{\lambda_n t}$

$$\lambda_n^2 + k\lambda_n + v^2\left(\frac{n\pi}{\ell}\right)^2 = 0$$

$$\lambda_n^2 + k\lambda_n + \left(\frac{k}{2}\right)^2 = \left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{\ell}\right)^2$$

$$\left(\lambda_n + \frac{k}{2}\right)^2 = \left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{\ell}\right)^2 \Rightarrow \lambda_n = \frac{-k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{\ell}\right)^2}$$

for large damping $\left(\frac{k}{2}\right)^2 > v^2\left(\frac{n\pi}{\ell}\right)^2$ we have real roots

Thus $$b_n(t) = A_n e^{-\left(\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{\ell}\right)^2}\right)t} + B_n e^{\left(-\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{\ell}\right)^2}\right)t}$$

$$= A_n e^{\left(-\frac{k}{2} - \omega_n\right)t} + B_n e^{\left(-\frac{k}{2} + \omega_n\right)t}$$ where $\omega_n = \sqrt{\left(\frac{k}{2}\right)^2 - v^2\left(\frac{n\pi}{\ell}\right)^2}$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi}{\ell}x\right)$$

$$= e^{-\frac{k}{2}t}\sum_{n=1}^{\infty}\left(A_n e^{-\omega_n t} + B_n e^{\omega_n t}\right)\sin\left(\frac{n\pi}{\ell}x\right)$$

now to find A_n and B_n from the initial conditions

$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{\ell}x)$ multiply both sides by $\sin(\frac{m\pi}{\ell}x)$ and integrate

$$\int_0^\ell f(x)\sin(\frac{m\pi}{\ell}x)\,dx = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^\ell \sin(\frac{n\pi}{\ell}x)\sin(\frac{m\pi}{\ell}x)\,dx}$$

$$= \begin{cases} 0 & n \neq m \\ \frac{\ell}{2} & n = m \end{cases}$$

$$\int_0^\ell f(x)\sin(\frac{m\pi}{\ell}x)\,dx = C_m(\frac{\ell}{2}) \Rightarrow C_m = \frac{2}{\ell}\int_0^\ell f(x)\sin(\frac{m\pi}{\ell}x)\,dx \quad \text{now } m = n$$

$$C_n = \frac{2}{\ell}\int_0^\ell f(x)\sin(\frac{n\pi}{\ell}x)\,dx$$

$$\therefore u(x,0) = \sum_{n=1}^{\infty}(A_n + B_n)\sin(\frac{n\pi}{\ell}x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{\ell}x)$$

$$\Rightarrow A_n + B_n = C_n$$

$$\frac{\partial u(x,0)}{\partial t} = g(x) = \sum_{n=1}^{\infty}\left(-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n\right)\sin(\frac{n\pi}{\ell}x)$$

$$\int_0^\ell g(x)\sin(\frac{m\pi}{\ell}x)\,dx = \sum_{n=1}^{\infty}\left(-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n\right)\underbrace{\int_0^\ell \sin(\frac{m\pi}{\ell}x)\sin(\frac{n\pi}{\ell}x)\,dx}$$

$$= \begin{cases} \frac{\ell}{2} & n = m \\ 0 & n \neq m \end{cases}$$

$$\int_0^\ell g(x)\sin(\frac{n\pi}{\ell}x)\,dx = \left(-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n\right)(\frac{\ell}{2})$$

$$-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n = \frac{2}{\ell}\int_0^\ell g(x)\sin(\frac{n\pi}{\ell}x)\,dx = D_n$$

thus we have

$$A_n + B_n = C_n \qquad \text{need to find } A_n \,\&\, B_n$$
$$-(\frac{k}{2}+\omega_n)A_n - (\frac{k}{2}-\omega_n)B_n = D_n$$

Thus $u(x,t) = e^{-\frac{k}{2}t}\sum_{n=1}^{\infty}\left(A_n e^{-\omega_n t} + B_n e^{\omega_n t}\right)\sin(\frac{n\pi}{\ell}x)$

where $A_n = -\dfrac{D_n + C_n(\frac{k}{2}-\omega_n)}{2\omega_n}$, $B_n = \dfrac{D_n + C_n(\frac{k}{2}+\omega_n)}{2\omega_n}$

$$C_n = \frac{2}{\ell}\int_0^\ell f(x)\sin(\frac{n\pi}{\ell}x)\,dx, \quad D_n = \frac{2}{\ell}\int_0^\ell g(x)\sin(\frac{n\pi}{\ell}x)\,dx$$

The sine series was used because it satisfies the B.C. $u(0,t) = u(\ell,t) = 0$