Homework Set No. 7
Due November 1, 2013

NEEP 547
DLH

Solve the system with use of the Fundamental Matrix

1. (6pts) Solve the following system of equations with the initial conditions, $x(0) = 3$ and $y(0) = 1$:

$$\begin{aligned} x' &= 3x + y - 2\sin(t) \\ y' &= 4x + 3y + 6\cos(t). \end{aligned}$$

Solve the system with use of the variation of parameters

2. (6pts) Find the complete solution of the system with the initial conditions, $x(0) = -1$, $y(0) = 2$ and $z(0) = 8$.

$$\begin{aligned} x' &= 3x - z \\ y' &= -2x + 2y + z \\ z' &= 8x - 3z. \end{aligned}$$

Diagonalization

3. (6pts) page 339, prob. 6

Solve system with Diagonalization

4. (6pts) Find the general solution of the system:

$$\begin{aligned} x' &= -x + 3y \\ y' &= 3x - y \\ z' &= -2x - 2y + 6z \end{aligned}$$

5. (6pts) Find the general solution of the following system:

$$\begin{aligned} 2x' + x + y' + 2y &= e^t \\ 3x' - 7x + 3y' + y &= 0. \end{aligned}$$

Matrix Exponential

6. (6pts) Using the relation $\int e^{\mathbf{A}t}\, dt = e^{\mathbf{A}t} \times \mathbf{A}^{-1}$, determine the general solution of the following matrix equation;

$$\frac{d\overline{N}}{dt} = \mathbf{A}\,\overline{N}(t) + \overline{F}(t)$$

where $\overline{F}(\mathrm{t}) = \overline{B}t^2$ and $\overline{B}$ is a constant vector.

7. (6pts) Solve problem 5 using the Matrix Exponential method outlined in class.

1.) Solve the following system of equations with the initial conditions, $x(0) = 3$ and $y(0) = 1$

$D = \frac{d}{dt}$

$$x' = 3x + y - 2\sin(t) \quad \Rightarrow \quad (D-3)x - y = -2\sin(t)$$
$$y' = 4x + 3y + 6\cos(t) \qquad\qquad -4x + (D-3)y = 6\cos(t)$$

$$\begin{bmatrix} (D-3) & -1 \\ -4 & (D-3) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2\sin(t) \\ 6\cos(t) \end{bmatrix}$$

we will solve the homogeneous system first

we assume a solution of the form $\bar{a}e^{\lambda t}$. This gives

$$\begin{bmatrix} \lambda-3 & -1 \\ -4 & \lambda-3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we obtain a nontrivial solution only if the coef. matrix is equal to zero. Let's find the eigenvalues

$$\begin{vmatrix} \lambda-3 & -1 \\ -4 & \lambda-3 \end{vmatrix} = 0 \Rightarrow (\lambda-3)^2 - 4 = 0 \Rightarrow (\lambda-3)^2 = 4$$
$$(\lambda-3) = \pm 2 \Rightarrow \lambda = 3 \pm 2 \Rightarrow \lambda = 1, 5$$

for $\lambda = 1$ $\begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2a_1 - a_2 = 0 \\ -4a_1 - 2a_2 = 0 \end{matrix}$ or $-2a_1 = a_2$ let $a_1 = 1$ $a_2 = -2$

for $\lambda = 5$ $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2a_1 - a_2 = 0 \\ -4a_1 + 2a_2 = 0 \end{matrix}$ or $2a_1 = a_2$ let $a_1 = 1$, $a_2 = 2$

$$\bar{V}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} \qquad \bar{V}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t}$$

our fundamental matrix is $\bar{\bar{X}}(t) = \begin{array}{cc} \vec{V}_1 & \vec{V}_2 \\ \downarrow & \downarrow \end{array} \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix}$

$$\bar{X}_h(t) = C_1\vec{V}_1 + C_2\vec{V}_2 = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t}$$

The particular solution associated with the system is

$$\vec{U}_p = \bar{\bar{X}}(t)\int \bar{\bar{X}}^{-1}(t)\, f(t)\,dt \quad \text{where } f(t) = \begin{bmatrix} -2\sin(t) \\ 6\cos(t) \end{bmatrix}$$

$$\bar{\bar{X}}^{-1}(t) = \frac{\text{Adj } \bar{\bar{X}}(t)}{|\bar{\bar{X}}(t)|} \qquad |\bar{\bar{X}}(t)| = \begin{vmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{vmatrix} = 2e^{6t} + 2e^{6t} = 4e^{6t}$$

$\bar{\bar{C}}$ = Co-factor matrix

$$Adj\,\bar{\bar{X}}(t) = \bar{\bar{C}}^T \quad \bar{\bar{C}} = \begin{bmatrix} 2e^{5t} & 2e^{t} \\ -e^{5t} & e^{t} \end{bmatrix} \quad \bar{\bar{C}}^T = \begin{bmatrix} 2e^{5t} & -e^{5t} \\ 2e^{t} & e^{t} \end{bmatrix}$$

$$\therefore \bar{\bar{X}}^{-1}(t) = \frac{\bar{\bar{C}}^T}{|\bar{\bar{X}}(t)|} = \frac{1}{4}e^{-6t}\begin{bmatrix} 2e^{5t} & -e^{5t} \\ 2e^{t} & e^{t} \end{bmatrix} = \begin{bmatrix} \frac{e^{-t}}{2} & -\frac{e^{-t}}{4} \\ \frac{e^{-5t}}{2} & \frac{e^{-5t}}{4} \end{bmatrix}$$

$$\bar{U}_p(t) = \bar{\bar{X}}(t)\int \bar{\bar{X}}^{-1}(t)\,\bar{f}(t)\,dt = \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix}\int_0^t \begin{bmatrix} \frac{e^{-t}}{2} & -\frac{e^{-t}}{4} \\ \frac{e^{-5t}}{2} & \frac{e^{-5t}}{4} \end{bmatrix}\begin{bmatrix} -2\sin(t) \\ 6\cos(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix}\int_0^t \begin{pmatrix} -e^{-t}\sin(t) - \frac{3}{2}e^{-t}\cos(t) \\ -e^{-5t}\sin(t) + \frac{3}{2}e^{-5t}\cos(t) \end{pmatrix} dt$$

the integrals are of the form $\int_0^t e^{-at}\sin(t)\,dt$ and $\int_0^t e^{-at}\cos(t)\,dt$

$$\int_0^t e^{-at}\sin(t)\,dt = \int_0^t e^{-at}\,d(-\cos(t)) = (-e^{-at}\cos(t))\Big|_0^t - a\int_0^t e^{-at}\cos(t)\,dt$$
$$= (-e^{-at}\cos(t))\Big|_0^t - a\int_0^t e^{-at}\,d(\sin(t))$$
$$= (-e^{-at}\cos(t))\Big|_0^t - a\left((e^{-at}\sin(t))\Big|_0^t + a\int_0^t e^{-at}\sin(t)\,dt\right)$$
$$= (-e^{-at}\cos(t))\Big|_0^t - a(e^{-at}\sin(t))\Big|_0^t - a^2\int_0^t e^{-at}\sin(t)\,dt$$
$$(1+a^2)\int_0^t e^{-at}\sin(t)\,dt = (-e^{-at}\cos(t))\Big|_0^t - a(e^{-at}\sin(t))\Big|_0^t$$
$$\int_0^t e^{-at}\sin(t)\,dt = \left(\frac{1}{1+a^2}\right)\left(-e^{-at}\cos(t) + 1 - a\,e^{-at}\sin(t)\right)$$

$$\int_0^t e^{-at}\cos(t)\,dt = \int_0^t e^{-at}\,d(\sin(t)) = (e^{-at}\sin(t))\Big|_0^t + a\int_0^t e^{-at}\sin(t)\,dt$$
$$= (e^{-at}\sin(t))\Big|_0^t + a\int_0^t e^{-at}\,d(-\cos(t))$$
$$= (e^{-at}\sin(t))\Big|_0^t + a\left((-e^{-at}\cos(t))\Big|_0^t - a\int_0^t e^{-at}\cos(t)\,dt\right)$$
$$= (e^{-at}\sin(t))\Big|_0^t - a(e^{-at}\cos(t))\Big|_0^t - a^2\int_0^t e^{-at}\cos(t)\,dt$$
$$(1+a^2)\int_0^t e^{-at}\cos(t)\,dt = (e^{-at}\sin(t))\Big|_0^t - a(e^{-at}\cos(t))\Big|_0^t$$
$$\int_0^t e^{-at}\cos(t)\,dt = \left(\frac{1}{1+a^2}\right)\left(e^{-at}\sin(t) - a\,e^{-at}\cos(t) + a\right)$$

$$\therefore \int_0^t e^{-t}\sin(t)\,dt = \left(\frac{1}{2}\right)\left(1 - e^{-t}\cos(t) - e^{-t}\sin(t)\right)$$
$$\int_0^t e^{-t}\cos(t)\,dt = \left(\frac{1}{2}\right)\left(1 + e^{-t}\sin(t) - e^{-t}\cos(t)\right)$$
$$\int_0^t e^{-5t}\sin(t)\,dt = \left(\frac{1}{26}\right)\left(1 - e^{-5t}\cos(t) - 5e^{-5t}\sin(t)\right)$$
$$\int_0^t e^{-5t}\cos(t)\,dt = \left(\frac{1}{26}\right)\left(5 + e^{-5t}\sin(t) - 5e^{-5t}\cos(t)\right)$$

$$\overline{U}_p(t) = \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -(\frac{1}{2})(1-e^{-t}\cos(t) - e^{-t}\sin(t)) - (\frac{3}{4})(1+e^{-t}\sin(t) - e^{-t}\cos(t)) \\ -(\frac{1}{26})(1-e^{-5t}\cos(t) - 5e^{-5t}\sin(t)) + (\frac{3}{52})(5+e^{-5t}\sin(t) - 5e^{-5t}\cos(t)) \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -\frac{5}{4} + \frac{5}{4}e^{-t}\cos(t) - \frac{1}{4}e^{-t}\sin(t) \\ \frac{13}{52} - \frac{13}{52}e^{-5t}\cos(t) + \frac{13}{52}e^{-5t}\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{5t} \\ -2e^{t} & 2e^{5t} \end{bmatrix} \begin{bmatrix} -\frac{5}{4} + \frac{5}{4}e^{-t}\cos(t) - \frac{1}{4}e^{-t}\sin(t) \\ \frac{1}{4} - \frac{1}{4}e^{-5t}\cos(t) + \frac{1}{4}e^{-5t}\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{4}e^{t} + \frac{5}{4}\cos(t) - \frac{1}{4}\sin(t) + \frac{1}{4}e^{5t} - \frac{1}{4}\cos(t) + \frac{1}{4}\sin(t) \\ \frac{10}{4}e^{t} - \frac{10}{4}\cos(t) + \frac{2}{4}\sin(t) + \frac{1}{2}e^{5t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{4}e^{t} + \frac{1}{4}e^{5t} + \cos(t) \\ \frac{10}{4}e^{t} + \frac{1}{2}e^{5t} - \frac{12}{4}\cos(t) + \sin(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{4}e^{t} + \frac{1}{4}e^{5t} + \cos(t) \\ \frac{10}{4}e^{t} + \frac{1}{2}e^{5t} - 3\cos(t) + \sin(t) \end{bmatrix}$$

Our general solution is

$$\overline{X}(t) = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} + \begin{bmatrix} -\frac{5}{4}e^{t} + \frac{1}{4}e^{5t} + \cos(t) \\ \frac{10}{4}e^{t} + \frac{1}{2}e^{5t} - 3\cos(t) + \sin(t) \end{bmatrix}$$

now to the I.C.

$$\overline{X}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underbrace{\begin{bmatrix} -\frac{5}{4} + \frac{1}{4} + 1 \\ \frac{10}{4} + \frac{1}{2} - 3 \end{bmatrix}}_{=0}$$

$$\Rightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} C_1 + C_2 = 3 \\ -2C_1 + 2C_2 = 1 \end{matrix} \quad \begin{matrix} \Rightarrow C_2 = 3 - C_1 \\ \Rightarrow -2C_1 + 2(3-C_1) = 1 \\ -2C_1 + 6 - 2C_1 = 1 \\ -4C_1 = -5 \Rightarrow C_1 = \frac{5}{4} \\ C_2 = \frac{7}{4} \end{matrix}$$

$$\overline{X}(t) = \frac{5}{4}\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{t} + \frac{7}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} + \begin{bmatrix} -\frac{5}{4}e^{t} + \frac{1}{4}e^{5t} + \cos(t) \\ \frac{10}{4}e^{t} + \frac{1}{2}e^{5t} - 3\cos(t) + \sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{4} - \frac{5}{4} \\ -\frac{10}{4} + \frac{10}{4} \end{bmatrix}e^{t} + \begin{bmatrix} \frac{7}{4} + \frac{1}{4} \\ \frac{14}{4} + \frac{2}{4} \end{bmatrix}e^{5t} + \begin{bmatrix} \cos(t) \\ -3\cos(t) + \sin(t) \end{bmatrix}$$

$$= 0 + \begin{bmatrix} 2 \\ 4 \end{bmatrix}e^{5t} + \begin{bmatrix} \cos(t) \\ -3\cos(t) + \sin(t) \end{bmatrix}$$

$$\overline{X}(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}e^{5t} + \begin{bmatrix} \cos(t) \\ -3\cos(t) + \sin(t) \end{bmatrix}$$

2) Find the complete solution of the system with the initial conditions

$x(0) = -1$, $y(0) = 2$ and $z(0) = 8$:

$$\begin{aligned} x' &= 3x - z \\ y' &= -2x + 2y + z \\ z' &= 8x - 3z \end{aligned} \quad \Rightarrow \quad \begin{aligned} x' - 3x + z &= 0 \\ y' + 2x - 2y - z &= 0 \\ z' - 8x + 3z &= 0 \end{aligned}$$

$$\begin{bmatrix} (D-3) & 0 & 1 \\ 2 & D-2 & -1 \\ -8 & 0 & (D+3) \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

assume $\bar{X} = \bar{a} e^{\lambda t}$

$$\begin{bmatrix} (\lambda-3) & 0 & 1 \\ 2 & (\lambda-2) & -1 \\ -8 & 0 & (\lambda+3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{vmatrix} (\lambda-3) & 0 & 1 \\ 2 & (\lambda-2) & -1 \\ -8 & 0 & (\lambda+3) \end{vmatrix} = 0$$

$$(\lambda-3) \begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda+3 \end{vmatrix} + (1) \begin{vmatrix} 2 & (\lambda-2) \\ -8 & 0 \end{vmatrix} = 0 \Rightarrow (\lambda-3)(\lambda-2)(\lambda+3) - (\lambda-2)(-8) = 0$$

$$(\lambda-2)\left((\lambda-3)(\lambda+3) + 8\right) = 0 \Rightarrow (\lambda-2)\left((\lambda^2 - 9 + 8\right) = 0 \Rightarrow (\lambda-2)\left((\lambda^2 - 1)\right) = 0$$

$$(\lambda-2)(\lambda+1)(\lambda-1) = 0 \qquad \therefore \lambda = 2, 1, -1$$

for $\lambda = 1$

$$\begin{bmatrix} -2 & 0 & 1 \\ 2 & -1 & -1 \\ -8 & 0 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -2a_1 + a_3 &= 0 \\ 2a_1 - a_2 - a_3 &= 0 \\ -8a_1 + 4a_3 &= 0 \end{aligned}$$

$-2a_1 + a_3 = 0$ $\quad a_3 = 2a_1$

$-8a_1 + 4a_3 = 0$ $\quad a_3 = 2a_1$

let $a_1 = 1$ $\quad a_3 = 2$

$2a_1 - a_2 - a_3 = 0 \Rightarrow 2 - a_2 - 2 = 0 \quad a_2 = 0$

for $\lambda = -1$

$$\begin{bmatrix} -4 & 0 & 1 \\ 2 & -6 & -1 \\ -8 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -4a_1 + a_3 &= 0 \\ 2a_1 - 6a_2 - a_3 &= 0 \\ -8a_1 + 2a_3 &= 0 \end{aligned}$$

$-4a_1 + a_3 = 0$ $\quad 4a_1 = a_3$

$-8a_1 + 2a_3 = 0$ $\quad 4a_1 = a_3$

let $a_1 = 1$ $\quad a_3 = 4$

$2a_1 - 6a_2 - a_3 = 0 \Rightarrow 2 - 6a_2 - 4 = 0 \Rightarrow -6a_2 = 2 \quad a_2 = -\frac{2}{3}$

for $\lambda = 2$

$$\begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ -8 & 0 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -a_1 + a_3 &= 0 \\ 2a_1 - a_3 &= 0 \\ -8a_1 - 5a_3 &= 0 \end{aligned}$$

the only values that satisfy these 3 conditions is $a_1 = a_3 = 0$. a_2 is arbitrary, we choose $a_2 = 1$.

$$\bar{Y}(t) = C_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}^{V_1} e^{t} + C_2 \underbrace{\begin{bmatrix} 1 \\ -\frac{2}{3} \\ 4 \end{bmatrix}}^{V_2} e^{-t} + C_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

$$= C_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix} e^{-t} + C_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

now to the I.C. at $t=0$

$$\bar{X}(0) = \begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} C_1 + 3C_2 &= -1 \\ -2C_2 + C_3 &= 2 \\ 2C_1 + 12C_2 &= 8 \end{aligned}$$

$$\begin{aligned} C_1 + 3C_2 &= -1 \\ 2C_1 + 12C_2 &= 8 \end{aligned}$$

$\Rightarrow C_1 = -1 - 3C_2$

$\Rightarrow 2(-1-3C_2) + 12C_2 = 8$

$-2 - 6C_2 + 12C_2 = 8$

$6C_2 = 10$

$C_2 = \frac{10}{6} = \frac{5}{3}$

$C_1 = -1 - 3(5/3) = -1 - 5 = -6$

$\Rightarrow (-2)C_2 + C_3 = 2 \Rightarrow (-2)(5/3) + C_3 = 2$

$-10/3 + C_3 = 2$

$C_3 = 2 + \frac{10}{3} = \frac{16}{3}$

$$\bar{X}(t) = -6 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{t} + 5/3 \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix} e^{-t} + \frac{16}{3} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

3) p. 339, problem 6 $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ $\bar{\bar{D}} = \bar{\bar{P}}^{-1}\bar{\bar{A}}\bar{\bar{P}}$ we need to solve the eigenvalue problem to diagonalize

$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\bar{X}$ let $\bar{X} = \bar{C}e^{\lambda t}$; $\frac{d\bar{X}}{dt} = \bar{C}\lambda e^{\lambda t}$

$\bar{C}\lambda e^{\lambda t} = \bar{\bar{A}}\bar{C}e^{\lambda t} \Rightarrow \lambda\bar{\bar{I}} = \bar{\bar{A}}$ our eigenvalue problem is defined as $(\lambda\bar{\bar{I}} - \bar{\bar{A}}) = 0$ ① or $(\bar{\bar{A}} - \lambda\bar{\bar{I}}) = 0$ ②

we will use ①

$$\begin{pmatrix} \lambda & 0 & 0 \\ -1 & \lambda & -2 \\ 0 & -1 & \lambda-3 \end{pmatrix} = \lambda \begin{vmatrix} \lambda & -2 \\ -1 & \lambda-3 \end{vmatrix} = \lambda((\lambda)(\lambda-3) - 2) = 0$$

$$\lambda(\lambda^2 - 3\lambda - 2) = 0$$

$\lambda_1 = 0$ $(\lambda^2 - 3\lambda - 2) = 0 \Rightarrow (\lambda^2 - 3\lambda + (\frac{9}{4})) = 2 + (\frac{9}{4})$

$(\lambda - \frac{3}{2})^2 = \frac{17}{4} \Rightarrow \lambda - \frac{3}{2} = \pm\frac{\sqrt{17}}{2} \Rightarrow \lambda_{2,3} = \frac{3}{2} \pm \frac{\sqrt{17}}{2}$

$\lambda_1 = 0$, $\lambda_2 = \frac{3}{2} - \frac{\sqrt{17}}{2}$, $\lambda_3 = \frac{3}{2} + \frac{\sqrt{17}}{2}$

for $\lambda_1 = 0$ $\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{matrix} -a - 2c = 0 \\ -b - 3c = 0 \end{matrix} \Rightarrow \begin{matrix} -a = 2c \\ -b = 3c \end{matrix}$ let $c = 1$ $a = -2$, $b = -3$

for $\lambda_2 = \frac{3-\sqrt{17}}{2}$ $\begin{pmatrix} \frac{3-\sqrt{17}}{2} & 0 & 0 \\ -1 & \frac{3-\sqrt{17}}{2} & -2 \\ 0 & -1 & \frac{3-\sqrt{17}}{2} - 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow$ $a\left(\frac{3-\sqrt{17}}{2}\right) = 0$ $\underline{a = 0}$

$\cancel{-a} + \left(\frac{3-\sqrt{17}}{2}\right)b - 2c = 0$

$-b + \left(\frac{3-\sqrt{17}}{2} - 3\right)c = 0$

$-b + \left(\frac{-3-\sqrt{17}}{2}\right)c = 0$

$a = 0$ $\left(\frac{3-\sqrt{17}}{2}\right)b = 2c$

$(3-\sqrt{17})b = 4c$ $c = 3-\sqrt{17}$, $b = 4$

for $\lambda_3 = \frac{3+\sqrt{17}}{2}$ $\begin{pmatrix} \frac{3+\sqrt{17}}{2} & 0 & 0 \\ -1 & \frac{3+\sqrt{17}}{2} & -2 \\ 0 & -1 & \frac{3+\sqrt{17}}{2} - 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow$ $a\left(\frac{3+\sqrt{17}}{2}\right) = 0$ $\underline{a = 0}$

$-a + \left(\frac{3+\sqrt{17}}{2}\right)b - 2c = 0$

$-b + \left(\frac{3+\sqrt{17}}{2} - 3\right)c = 0$

$-b + \left(\frac{-3+\sqrt{17}}{2}\right)c = 0$

$a = 0$ $\left(\frac{3+\sqrt{17}}{2}\right)b = 2c$

$(3+\sqrt{17})b = 4c$ $c = 3+\sqrt{17}$, $b = 4$

for λ_1, $\bar{V}_1 = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$, $\lambda_1 = 0$ for λ_2, $\bar{V}_2 = \begin{pmatrix} 0 \\ 4 \\ 3-\sqrt{17} \end{pmatrix}e^{\lambda_2 t}$, $\lambda_2 = \frac{3-\sqrt{17}}{2}$ for λ_3, $\bar{V}_3 = \begin{pmatrix} 0 \\ 4 \\ 3+\sqrt{17} \end{pmatrix}e^{\lambda_3 t}$, $\lambda_3 = \frac{3+\sqrt{17}}{2}$

$$\bar{\bar{P}} = \begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{pmatrix} \qquad \bar{\bar{P}}^{-1} = \frac{\text{Adj}(\bar{\bar{P}})}{|\bar{\bar{P}}|}, \quad |\bar{\bar{P}}| = \begin{vmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{vmatrix}$$

$$|\bar{\bar{P}}| = (-2)\begin{vmatrix} 4 & 4 \\ 3-\sqrt{17} & 3+\sqrt{17} \end{vmatrix} = (-2)\left((4)(3+\sqrt{17}) - (4)(3-\sqrt{17})\right)$$

$$= (-2)(4)(2\sqrt{17}) = -16\sqrt{17}$$

$\text{Adj}(\bar{\bar{P}}) = \bar{\bar{C}}^T$ where $\bar{\bar{C}}$ is the co-factor matrix

$$\bar{\bar{C}} = \begin{pmatrix} 8\sqrt{17} & 13+3\sqrt{17} & -13+3\sqrt{17} \\ 0 & -6-2\sqrt{17} & 6-2\sqrt{17} \\ 0 & 8 & -8 \end{pmatrix} \qquad \bar{\bar{C}}^T = \begin{pmatrix} 8\sqrt{17} & 0 & 0 \\ 13+3\sqrt{17} & -6-2\sqrt{17} & 8 \\ -13+3\sqrt{17} & 6-2\sqrt{17} & -8 \end{pmatrix}$$

$$\bar{\bar{P}}^{-1} = \left(\frac{-1}{16\sqrt{17}}\right)\bar{\bar{C}}^T = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix}$$

$\bar{\bar{P}}\,\bar{\bar{P}}^{-1}$ should be $\bar{\bar{I}}$

$$\begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ checks}$$

$$\bar{\bar{D}} = \bar{\bar{P}}^{-1}\,\bar{\bar{A}}\,\bar{\bar{P}}$$

$$\bar{\bar{D}} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3-\sqrt{17} & 3+\sqrt{17} \end{pmatrix}}$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \left(\frac{-13-3\sqrt{17}}{6\sqrt{17}}\right) & \frac{3+\sqrt{17}}{8\sqrt{17}} & -\frac{1}{2\sqrt{17}} \\ \left(\frac{13-3\sqrt{17}}{16\sqrt{17}}\right) & \frac{-3+\sqrt{17}}{8\sqrt{17}} & \frac{1}{2\sqrt{17}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 6-2\sqrt{17} & 6+2\sqrt{17} \\ 0 & 13-3\sqrt{17} & 13+3\sqrt{17} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3-\sqrt{17}}{2} & 0 \\ 0 & 0 & \frac{3+\sqrt{17}}{2} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{diagonalizable}$$

4). Find the general solution of the system: $x' = -x+3y$

$y' = 3x - y$

$z' = -2x - 2y + 6z$

$$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\bar{X} \quad \text{where } \bar{\bar{A}} = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{pmatrix}$$

let $\bar{X} = \bar{C}e^{\lambda t}$; $\frac{d\bar{X}}{dt} = \bar{C}\lambda e^{\lambda t}$

$\bar{C}\lambda e^{\lambda t} = \bar{\bar{A}}\bar{C}e^{\lambda t} \Rightarrow \lambda\bar{\bar{I}} = \bar{\bar{A}}$ thus $\lambda\bar{\bar{I}} - \bar{\bar{A}} = 0$

$$\begin{pmatrix} \lambda+1 & -3 & 0 \\ -3 & \lambda+1 & 0 \\ 2 & 2 & \lambda-6 \end{pmatrix} = (\lambda+1)\begin{pmatrix} \lambda+1 & 0 \\ 2 & \lambda-6 \end{pmatrix} + 3\begin{pmatrix} -3 & 0 \\ 2 & \lambda-6 \end{pmatrix}$$

$(\lambda+1)(\lambda+1)(\lambda-6) + 3(-3(\lambda-6)) \Rightarrow (\lambda+1)(\lambda+1)(\lambda-6) - 9(\lambda-6) = 0$

$(\lambda-6)((\lambda+1)(\lambda+1)-9) = 0 \quad (\lambda-6)((\lambda+1)^2-9) = 0$

we have $(\lambda+6) = 0$ and $(\lambda+1)^2 - 9 = 0$

$\lambda = 6 \qquad \lambda+1 = \pm 3 \Rightarrow \lambda = 2, -4$

for $\lambda = 6$ $\begin{pmatrix} 7 & -3 & 0 \\ -3 & 7 & 0 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $7a_1 - 3a_2 = 0$, $3a_1 - 7a_2 = 0$, $2a_1 + 2a_2 = 0$; $7a_1 = 3a_2$, $3a_1 = 7a_2$, $2a_1 = -2a_2$; $a_1 = 0$, $a_2 = 0$, $a_3 = 1$

for $\lambda = 2$ $\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $3a_1 - 3a_2 = 0$, $-3a_1 + 3a_2 = 0$, $2a_1 + 2a_2 - 4a_3 = 0$; $\Rightarrow a_1 = a_2$, $a_1 = a_2$, $a_1 + a_2 = 2a_3$; let $a_1 = 1$, $a_2 = 1$, $a_3 = 1$

for $\lambda = -4$ $\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 2 & 2 & -10 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $-3a_1 - 3a_2 = 0$, $-3a_1 - 3a_2 = 0$, $2a_1 + 2a_2 - 10a_3 = 0$; $a_1 = -a_2$; let $a_1 = 1$, $a_2 = -1$, $a_3 = 0$

$$\bar{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{6t} \quad \bar{V}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} \quad \bar{V}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-4t}$$

$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\bar{X}$ let $\bar{X} = \bar{\bar{P}}\bar{Z}$, $\frac{d\bar{X}}{dt} = \bar{\bar{P}}\frac{d\bar{Z}}{dt}$

$$\bar{\bar{P}}\frac{d\bar{Z}}{dt} = \bar{\bar{A}}\bar{\bar{P}}\bar{Z} \Rightarrow \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{P}}}_{\bar{\bar{I}}}\frac{d\bar{Z}}{dt} = \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{A}}\bar{\bar{P}}}_{\bar{\bar{D}}}\bar{Z}$$

$$\frac{d\bar{Z}}{dt} = \bar{D}\bar{Z} \Rightarrow \frac{d\bar{Z}}{dt} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix} \bar{Z}$$

$$\Rightarrow \begin{matrix} z_1'(t) = 6z_1 \\ z_2'(t) = 2z_2 \\ z_3'(t) = -4z_3 \end{matrix} \Rightarrow \begin{matrix} z_1(t) = c_1 e^{6t} \\ z_2(t) = c_2 e^{2t} \\ z_3(t) = c_3 e^{-4t} \end{matrix} \qquad \bar{Z} = \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{2t} \\ c_3 e^{-4t} \end{pmatrix}$$

$$\bar{X} = \bar{P}\bar{Z} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{2t} \\ c_3 e^{-4t} \end{pmatrix}$$

$$= \begin{pmatrix} c_2 e^{2t} - c_3 e^{-4t} \\ c_2 e^{2t} + c_3 e^{-4t} \\ c_1 e^{6t} + c_2 e^{2t} \end{pmatrix}$$

5) Find the general solution of the following system

$$2x' + x + y' + 2y = e^t$$
$$3x' - 7x + 3y' + y = 0 \Rightarrow \bar{\bar{B}}\bar{X}' = \bar{\bar{A}}\bar{X} + \bar{f}(t)$$

$$2x' + y' = -x - 2y + e^t$$
$$\Rightarrow 3x' + 3y' = 7x - y$$

$$\bar{\bar{B}} = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} \quad \bar{\bar{A}} = \begin{pmatrix} -1 & -2 \\ 7 & -1 \end{pmatrix} \quad \bar{f}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

$$\bar{\bar{B}}\bar{X}' = \bar{\bar{A}}\bar{X} + \bar{f}(t) \Rightarrow \bar{X}' = \bar{\bar{B}}^{-1}\bar{\bar{A}}\bar{X} + \bar{\bar{B}}^{-1}\bar{f}(t)$$
$$= \bar{\bar{C}}\bar{X} + \bar{g}(t) \qquad \text{where } \bar{\bar{C}} = \bar{\bar{B}}^{-1}\bar{\bar{A}} \text{ and } \bar{g}(t) = \bar{\bar{B}}^{-1}\bar{f}(t)$$

$$\begin{array}{cc} \bar{\bar{B}} & \bar{\bar{I}} \end{array}$$
$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}\cdot r_1} \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 3 & 0 & 1 \end{array}\right) \xrightarrow{(-3)r_1 + r_2} \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right)$$
$$\xrightarrow{-\frac{1}{3}r_2 + r_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{3} \\ 0 & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right) \xrightarrow{\frac{2}{3}r_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 1 & -1 & \frac{2}{3} \end{array}\right)$$

$$\bar{\bar{B}}^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix} \quad \text{lets check} \quad \begin{pmatrix} 1 & -\frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{checks}$$

$$\bar{\bar{C}}_1 = \bar{\bar{B}}^{-1}\bar{\bar{A}} = \begin{pmatrix} 1 & -\frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 7 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{10}{3} & -\frac{5}{3} \\ \frac{17}{3} & \frac{4}{3} \end{pmatrix} \qquad \bar{g}(t) = \bar{\bar{B}}^{-1}\bar{f}(t) = \begin{pmatrix} 1 & -\frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} e^t \\ 0 \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

our new Eq. is $\bar{X}' = \bar{\bar{C}}_1\bar{X} + \bar{g}(t) \Rightarrow \bar{X}' = \begin{pmatrix} -\frac{10}{3} & -\frac{5}{3} \\ \frac{17}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$ $\qquad D = \frac{d}{dt}$

$$\Rightarrow \begin{array}{l} x' + \frac{10}{3}x + \frac{5}{3}y = e^t \\ y' - \frac{4}{3}y - \frac{17}{3}x = -e^t \end{array} \Rightarrow \begin{bmatrix} (D + \frac{10}{3}) & \frac{5}{3} \\ -\frac{17}{3} & (D - \frac{4}{3}) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

we will solve the homogeneous solution first.

we assume a solution of the form $\bar{a}e^{\lambda t}$

$$\begin{bmatrix} (\lambda + \frac{10}{3}) & \frac{5}{3} \\ -\frac{17}{3} & (\lambda - \frac{4}{3}) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we obtain a nontrivial solution if and only if the coef. matrix is equal to zero.

$$(\lambda + \tfrac{10}{3})(\lambda - \tfrac{4}{3}) + (\tfrac{5}{3})(\tfrac{17}{3}) = 0 \Rightarrow \lambda^2 + \tfrac{6}{3}\lambda - \tfrac{40}{9} + \tfrac{85}{9} = 0$$
$$\lambda^2 + 2\lambda + \tfrac{45}{9} = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = 1 - \tfrac{45}{9} \Rightarrow (\lambda+1)^2 = \tfrac{9}{9} - \tfrac{45}{9}$$
$$(\lambda+1)^2 = -\tfrac{36}{9} \Rightarrow (\lambda+1)^2 = -4 \Rightarrow \lambda + 1 = \pm 2i \Rightarrow \lambda = -1 \pm 2i$$

for $\lambda_1 = -1 + 2i$

$$\begin{bmatrix} -1 + 2i + \frac{10}{3} & \frac{5}{3} \\ -\frac{17}{3} & -1 + 2i - \frac{4}{3} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} (\frac{10}{3} - \frac{3}{3} + 2i)a_1 + \frac{5}{3}a_2 = 0 \\ (-\frac{17}{3})a_1 + (-\frac{3}{3} - \frac{4}{3} + 2i)a_2 = 0 \end{array}$$

$$\Rightarrow \begin{array}{l} (\frac{7}{3} + 2i)a_1 + \frac{5}{3}a_2 = 0 \\ (-\frac{17}{3})a_1 + (-\frac{7}{3} + 2i)a_2 = 0 \end{array} \qquad \text{let } a_1 = -5 \quad a_2 = 7 + 6i$$

for $\lambda_2 = -1-2i$ $\begin{bmatrix} -1-2i+\frac{10}{3} & \frac{5}{3} \\ -\frac{17}{3} & -1-2i-\frac{4}{3} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} \left(\frac{10}{3}-\frac{3}{3}-2i\right)a_1 + \frac{5}{3}a_2 = 0 \\ \left(-\frac{17}{3}\right)a_1 + \left(-\frac{3}{3}-\frac{4}{3}-2i\right)a_2 = 0 \end{array}$

$\Rightarrow \begin{array}{l} \left(\frac{7}{3}-2i\right)a_1 + \frac{5}{3}a_2 = 0 \\ \left(-\frac{17}{3}\right)a_1 + \left(-\frac{7}{3}-2i\right)a_2 = 0 \end{array}$ let $a_1 = -5 \quad a_2 = 7-6i$

$\vec{v}_1 = \begin{pmatrix} -5 \\ 7+6i \end{pmatrix} e^{(-1+2i)t} \qquad \vec{v}_2 = \begin{pmatrix} -5 \\ 7-6i \end{pmatrix} e^{(-1-2i)t}$

$\bar{\bar{P}} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix} \qquad \bar{\bar{P}}^{-1} = \frac{Adj(\bar{\bar{P}})}{|\bar{\bar{P}}|} \qquad |\bar{\bar{P}}| = \begin{vmatrix} -5 & -5 \\ 7+6i & 7-6i \end{vmatrix} = \begin{array}{l}(-5)(7+6i) \\ -(-5)(7+6i)\end{array}$

$= -35+30i+35+30i = 60i$

cofactor matrix $\bar{\bar{C}} = \begin{pmatrix} 7-6i & -(7+6i) \\ 5 & -5 \end{pmatrix}, \quad \bar{\bar{C}}^T = \begin{pmatrix} 7-6i & 5 \\ -(7+6i) & -5 \end{pmatrix} \quad Adj(\bar{\bar{P}}) = \bar{\bar{C}}^T$

$\bar{\bar{P}}^{-1} = \left(\frac{1}{60i}\right)\begin{pmatrix} 7-6i & 5 \\ -(7+6i) & -5 \end{pmatrix}$

checks

Let's check $\bar{\bar{P}}\,\bar{\bar{P}}^{-1} = \bar{\bar{I}} \Rightarrow \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\bar{\bar{D}} = \bar{\bar{P}}^{-1}\bar{\bar{C}}_1\bar{\bar{P}} = \begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} -\frac{10}{3} & -\frac{5}{3} \\ \frac{17}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}$

$= \begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} 5-10i & 5+10i \\ -19+8i & -19-8i \end{pmatrix} = \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix}$

we have $\bar{X}' = \bar{\bar{C}}_1\bar{X} + \bar{g}(t)$ let $\bar{X} = \bar{\bar{P}}\bar{Z}$ and $\bar{X}' = \bar{\bar{P}}\bar{Z}'$

$\bar{\bar{P}}\bar{Z}' = \bar{\bar{C}}_1\bar{\bar{P}}\bar{Z} + \bar{g}(t) \Rightarrow \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{P}}}_{\bar{\bar{I}}}\bar{Z}' = \underbrace{\bar{\bar{P}}^{-1}\bar{\bar{C}}_1\bar{\bar{P}}}_{\bar{\bar{D}}}\bar{Z} + \bar{\bar{P}}^{-1}\bar{g}(t)$

$\therefore \bar{Z}' = \bar{\bar{D}}\bar{Z} + \bar{\bar{P}}^{-1}\bar{g}(t) \Rightarrow \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} e^t \\ -e^t \end{pmatrix}$

$\frac{dz_1}{dt} = (-1+2i)z_1 + \left(\frac{2-6i}{60i}\right)e^t \Rightarrow \frac{dz_1}{dt} + (1-2i)z_1 = \left(\frac{1-3i}{30i}\right)e^t \qquad I_f = e^{\int(1-2i)dt} = e^{(1-2i)t}$

$\int_0^t d\left(z_1 e^{(1-2i)t}\right) = \left(\frac{1-3i}{30i}\right)\int_0^t e^t e^{(1-2i)t}dt = \left(\frac{1-3i}{30i}\right)\int_0^t e^{(2-2i)t}dt$

$z_1(t)e^{(1-2i)t} - z_1(0) = \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{(2-2i)t}\right)\Big|_0^t$

$z_1(t) = z_1(0)e^{-(1-2i)t} + \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{-(1-2i)t}\right)\left(e^{(2-2i)t}-1\right)$

$$z_1(t) = z_1(0)\, e^{-(1-2i)t} + \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right)$$

$$\frac{dz_2}{dt} = (-1-2i)z_2 + \left(\frac{-2-6i}{60i}\right)e^{t} \Rightarrow \frac{dz_2}{dt} + (1+2i)\, z_2 = -\left(\frac{1+3i}{30i}\right)e^{t} \qquad I_f = e^{\int (1+2i)dt} = e^{(1+2i)t}$$

$$\int_0^t d\left(z_2 e^{(1+2i)t}\right) = -\left(\frac{1+3i}{30i}\right)\int_0^t e^{t} e^{(1+2i)t}\, dt = -\left(\frac{1+3i}{30i}\right)\int_0^t e^{(2+2i)t}\, dt$$

$$z_2(t)\, e^{(1+2i)t} - z_2(0) = -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{(2+2i)t}\right)\Big|_0^t$$

$$z_2(t) = z_2(0)\, e^{-(1+2i)t} - \left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{-(1+2i)t}\right)\left(e^{(2+2i)t} - 1\right)$$

$$z_2(t) = z_2(0)\, e^{-(1+2i)t} - \left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right)$$

$$\bar{Z}(t) = \begin{pmatrix} z_1(0)\, e^{-(1-2i)t} + \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ z_2(0)\, e^{-(1+2i)t} - \left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}$$

Recall $\bar{X} = \bar{\bar{P}}\bar{Z}$ also note $\bar{X}(0) = \bar{\bar{P}} Z(0) \Rightarrow \bar{\bar{P}}^{-1}\bar{X}(0) = \bar{Z}(0)$

$$\bar{X} = \bar{\bar{P}}\bar{Z} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} z_1(0)\, e^{-(1-2i)t} \\ z_2(0)\, e^{-(1+2i)t} \end{pmatrix} + \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} e^{-(1-2i)t} & 0 \\ 0 & e^{-(1+2i)} \end{pmatrix}\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} + \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}$$

Recall $\bar{Z}(0) = \bar{\bar{P}}^{-1}\bar{X}(0)$

$$= \underbrace{\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} e^{-(1-2i)t} & 0 \\ 0 & e^{-(1+2i)} \end{pmatrix}\begin{pmatrix} \left(\frac{7-6i}{60i}\right) & \frac{5}{60i} \\ -\left(\frac{7+6i}{60i}\right) & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}}_{\mathrm{I}} + \underbrace{\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t} - e^{-(1-2i)t}\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t} - e^{-(1+2i)t}\right) \end{pmatrix}}_{\mathrm{II}}$$

$$\mathrm{I} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{7-6i}{60i}\right)e^{-(1-2i)t} & \left(\frac{5}{60i}\right)e^{-(1-2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{-(1+2i)t} & \left(-\frac{5}{60i}\right)e^{-(1+2i)t} \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} \frac{-7}{2i}\left(e^{2it} - e^{-2it}\right) + \frac{6i}{2i}\left(e^{2it} + e^{-2it}\right) & -\frac{5}{2i}\left(e^{2it} - e^{-2it}\right) \\ \frac{17}{2i}\left(e^{2it} - e^{-2it}\right) & \frac{7}{2i}\left(e^{2it} - e^{-2it}\right) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t) + 6\cos(2t) & -5\sin(2t) \\ 17\sin(2t) & 7\sin(2t) + 6\cos(2t) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$\text{II} = \begin{pmatrix} (-5)\left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t}-e^{-(1-2i)t}\right) + (5)\left(\frac{1+3i}{60i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t}-e^{-(1+2i)t}\right) \\ (7+6i)\left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{t}-e^{-(1-2i)t}\right) - (7-6i)\left(\frac{1+3i}{60i}\right)\left(\frac{1}{2+2i}\right)\left(e^{t}-e^{-(1+2i)t}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\left(\frac{1-3i}{6i}\right)\left(\frac{2+2i}{8}\right)\left(e^{t}-e^{-(1-2i)t}\right) + \left(\frac{1+3i}{6i}\right)\left(\frac{2-2i}{8}\right)\left(e^{t}-e^{-(1+2i)t}\right) \\ \left(\frac{25-15i}{30i}\right)\left(\frac{2+2i}{8}\right)\left(e^{t}-e^{-(1-2i)t}\right) + \left(\frac{25+15i}{30i}\right)\left(\frac{2-2i}{8}\right)\left(e^{t}-e^{-(1+2i)t}\right) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} -\left(\frac{2-i}{12i}\right)\left(e^{2t}-e^{2it}\right) + \left(\frac{2+i}{12i}\right)\left(e^{2t}-e^{-2it}\right) \\ \left(\frac{4+i}{12i}\right)\left(e^{2t}-e^{2it}\right) - \left(\frac{4-i}{12i}\right)\left(e^{2t}-e^{-2it}\right) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} -\left(\frac{1}{12i}\right)\left(\cancel{2e^{2t}} - 2e^{2it} - ie^{2t} + ie^{2it} - \cancel{2e^{2t}} + 2e^{-2it} - ie^{2t} + ie^{-2it}\right) \\ \left(\frac{1}{12i}\right)\left(\cancel{4e^{2t}} - 4e^{2it} + ie^{2t} - ie^{2it} - \cancel{4e^{2t}} + 4e^{-2it} + ie^{2t} - ie^{-2it}\right) \end{pmatrix}$$

$$= \frac{e^{-t}}{12i}\begin{pmatrix} -2\left(e^{2it}-e^{-2it}\right) + 2ie^{2t} - i\left(e^{2it}+e^{-2it}\right) \\ -4\left(e^{2it}-e^{-2it}\right) + 2ie^{2t} - i\left(e^{2it}+e^{-2it}\right) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \frac{1}{3}\sin(2t) + \frac{1}{6}e^{2t} - \frac{1}{6}\cos(2t) \\ -\frac{2}{3}\sin(2t) + \frac{1}{6}e^{2t} - \frac{1}{6}\cos(2t) \end{pmatrix} = \frac{e^{-t}}{6}\begin{pmatrix} 2\sin(2t) + e^{2t} - \cos(2t) \\ -4\sin(2t) + e^{2t} - \cos(2t) \end{pmatrix}$$

$\overline{X}(t) = \text{I} + \text{II}$:

$$= \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t) + 6\cos(2t) & -5\sin(2t) \\ 17\sin(2t) & 7\sin(2t) + 6\cos(2t) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} + \frac{e^{-t}}{6}\begin{pmatrix} 2\sin(2t) + e^{2t} - \cos(2t) \\ -4\sin(2t) + e^{2t} - \cos(2t) \end{pmatrix}$$

6) Using the relation $\int e^{\bar{\bar{A}}t}\,dt = e^{\bar{\bar{A}}t} \times \bar{\bar{A}}^{-1}$, determine the general solution of the following matrix equation;

$$\frac{d\bar{N}}{dt} = \bar{\bar{A}}\,\bar{N}(t) + \bar{F}(t) \quad \text{where } \bar{F}(t) = \bar{B}t^2 \text{ and } \bar{B} \text{ is a constant vector.}$$

$$\frac{d\bar{N}}{dt} = \bar{\bar{A}}\,\bar{N}(t) + \bar{F}(t)$$

$$\frac{d\bar{N}}{dt} - \bar{\bar{A}}\,\bar{N}(t) = \bar{F}(t) \Rightarrow \int_0^t d\left(e^{-\bar{\bar{A}}t}\,\bar{N}(t)\right) = \int_0^t e^{-\bar{\bar{A}}t'}\,F(t')\,dt'$$

$$e^{-\bar{\bar{A}}t}\,\bar{N}(t) - \bar{N}(0) = \int_0^t e^{-\bar{\bar{A}}t'}\,F(t')\,dt'$$

$$\bar{N}(t) = e^{\bar{\bar{A}}t}\,\bar{N}(0) + e^{\bar{\bar{A}}t}\underbrace{\int_0^t e^{-\bar{\bar{A}}t'}\,\bar{F}(t')\,dt'}_{I_{nt}} \qquad \text{now } \bar{F}(t) = \bar{B}t^2$$

$$I_{nt} = \int_0^t e^{-\bar{\bar{A}}t'}\,\bar{B}\,t'^2\,dt' = \int_0^t e^{-\bar{\bar{A}}t'}\,t'^2\,dt'\;\bar{B}$$

$$= \left[\left(-t'^2 e^{-\bar{\bar{A}}t'}\,\bar{\bar{A}}^{-1}\right)\Big|_0^t + \int_0^t e^{-\bar{\bar{A}}t'}\,\bar{\bar{A}}^{-1}\,d(t'^2)\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}^{-1} + 2\int_0^t e^{-\bar{\bar{A}}t'}\,\bar{\bar{A}}^{-1}\,t'\,dt'\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}^{-1} + 2\left(\left(-t' e^{-\bar{\bar{A}}t'}\,\bar{\bar{A}}^{-1}\,\bar{\bar{A}}^{-1}\right)\Big|_0^t + \int_0^t e^{-\bar{\bar{A}}t'}\,\bar{\bar{A}}^{-1}\,\bar{\bar{A}}^{-1}\,dt'\right)\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}^{-1} - 2t\,e^{-\bar{\bar{A}}t}\left(\bar{\bar{A}}^{-1}\right)^2 + 2\,(-1)\left(e^{-\bar{\bar{A}}t'}\,\bar{\bar{A}}^{-1}\,\bar{\bar{A}}^{-1}\,\bar{\bar{A}}^{-1}\right)\Big|_0^t\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}^{-1} - 2t\,e^{-\bar{\bar{A}}t}\left(\bar{\bar{A}}^{-1}\right)^2 - 2\left(e^{-\bar{\bar{A}}t}\left(\bar{\bar{A}}^{-1}\right)^3 - \left(\bar{\bar{A}}^{-1}\right)^3\right)\right]\bar{B}$$

$$= \left[-t^2 e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}^{-1} - 2t\,e^{-\bar{\bar{A}}t}\left(\bar{\bar{A}}^{-1}\right)^2 - 2\left(e^{-\bar{\bar{A}}t} - \bar{\bar{I}}\right)\left(\bar{\bar{A}}^{-1}\right)^3\right]\bar{B}$$

$$\therefore \bar{N}(t) = e^{\bar{\bar{A}}t}\,\bar{N}(0) - \left[t^2 e^{-\bar{\bar{A}}t} + 2e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}^{-1} + 2\left(e^{-\bar{\bar{A}}t} - \bar{\bar{I}}\right)\left(\bar{\bar{A}}^{-1}\right)^2\right]\bar{\bar{A}}^{-1}\,\bar{B}$$

7| Solve problem 5 using the Matrix Exponential method outlined in class

$$\frac{d\bar{X}}{dt} = \bar{\bar{A}}\bar{X} + \bar{g}(t) \Rightarrow \bar{X}(t) = e^{\bar{\bar{A}}t}\bar{X}(0) + e^{\bar{\bar{A}}t}\int_0^t e^{-\bar{\bar{A}}t'}\bar{g}(t')\,dt'$$

where $e^{\bar{\bar{A}}t} = \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}$ $\quad \bar{\bar{P}} \equiv$ eigenvector matrix of $\bar{\bar{A}}$; $\bar{\bar{P}}^{-1}$ inverse of $\bar{\bar{P}}$

$\bar{\bar{D}} \equiv$ diagonal eigenvalue matrix

$$\bar{X}(t) = \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{X}(0) + \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\int_0^t \bar{\bar{P}}e^{\bar{\bar{D}}t'}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt'$$
$$= \bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{X}(0) + \bar{\bar{P}}e^{\bar{\bar{D}}t}\int_0^t e^{-\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt'$$

now to the problem

$$2x' + x + y' + 2y = e^t \qquad \Rightarrow \bar{\bar{B}}\bar{X}' = \bar{\bar{A}}x + \bar{f}(t) \Rightarrow \bar{X}' = \bar{\bar{B}}^{-1}\bar{\bar{A}}\bar{X} + \bar{\bar{B}}^{-1}\bar{f}(t)$$
$$3x' - 7x + 3y' + y = 0 \qquad\qquad = \bar{\bar{C}}\bar{X} + \bar{g}(t)$$

where $\bar{\bar{C}} = \bar{\bar{B}}^{-1}\bar{\bar{A}}$ and $\bar{g}(t) = \bar{\bar{B}}^{-1}\bar{f}(t)$

From problem 5 we have

$$\lambda_1 = -1+2i, \quad \lambda_2 = -1-2i \qquad \bar{V}_1 = \begin{pmatrix} -5 \\ 7+6i \end{pmatrix} e^{(-1+2i)t} \qquad \bar{V}_2 = \begin{pmatrix} -5 \\ 7-6i \end{pmatrix} e^{(-1-2i)t}$$

$$\bar{\bar{P}} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix} \qquad \bar{\bar{P}}^{-1} = \left(\frac{1}{60i}\right)\begin{pmatrix} 7-6i & 5 \\ -(7+6i) & -5 \end{pmatrix} \qquad \bar{\bar{D}} = \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix}$$

$$\bar{g}(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$\bar{\bar{P}}e^{\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{X}(0) = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{-(1+2i)t} \end{pmatrix}\begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ \frac{-(7+6i)}{60i} & \frac{-5}{60i} \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} \left(\frac{7-6i}{60i}\right)e^{(-1+2i)t} & \left(\frac{5}{60i}\right)e^{(-1+2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{-(1+2i)t} & \left(-\frac{5}{60i}\right)e^{-(1+2i)t} \end{pmatrix}}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} -\frac{7}{2i}\left(e^{2it} - e^{-2it}\right) + \frac{6i}{2i}\left(e^{2it} + e^{-2it}\right) & -\frac{5}{2i}\left(e^{2it} - e^{-2it}\right) \\ \frac{17}{2i}\left(e^{2it} - e^{-2it}\right) & \frac{7}{2i}\left(e^{2it} - e^{-2it}\right) + \frac{6i}{2i}\left(e^{2it} + e^{-2it}\right) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$= \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t) + 6\cos(2t) & -5\sin(2t) \\ 17\sin(2t) & 7\sin(2t) + 6\cos(2t) \end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

$$\bar{\bar{P}} e^{\bar{\bar{D}} t} = \begin{pmatrix} -5 & -5 \\ 7+6i & 7-6i \end{pmatrix}\begin{pmatrix} e^{(-1+2i)t} & 0 \\ 0 & e^{-(1+2i)t}\end{pmatrix} = \begin{pmatrix} -5e^{(-1+2i)t} & -5e^{-(1+2i)t} \\ (7+6i)e^{(-1+2i)t} & (7-6i)e^{-(1+2i)t}\end{pmatrix}$$

$$e^{-\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{g}(t') = \begin{pmatrix} e^{(1-2i)t} & 0 \\ 0 & e^{(1+2i)t}\end{pmatrix}\begin{pmatrix} \frac{7-6i}{60i} & \frac{5}{60i} \\ -\frac{(7+6i)}{60i} & -\frac{5}{60i}\end{pmatrix}\begin{pmatrix} e^t \\ -e^t\end{pmatrix} = \begin{pmatrix} \left(\frac{7-6i}{60i}\right)e^{(1-2i)t} & \left(\frac{5}{60i}\right)e^{(1-2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{(1+2i)t} & \left(\frac{-5}{60i}\right)e^{(1+2i)t}\end{pmatrix}\begin{pmatrix} e^t \\ -e^t\end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{7-6i}{60i}\right)e^{(2-2i)t} - \left(\frac{5}{60i}\right)e^{(2-2i)t} \\ -\left(\frac{7+6i}{60i}\right)e^{(2+2i)t} + \left(\frac{5}{60i}\right)e^{(2+2i)t}\end{pmatrix} = \begin{pmatrix} \left(\frac{1-3i}{30i}\right)e^{(2-2i)t} \\ -\left(\frac{1+3i}{30i}\right)e^{(2+2i)t}\end{pmatrix}$$

$$\int_0^t e^{-\bar{\bar{D}}t'}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt' = \int_0^t \begin{pmatrix} \left(\frac{1-3i}{30i}\right)e^{(2-2i)t} \\ -\left(\frac{1+3i}{30i}\right)e^{(2+2i)t}\end{pmatrix} dt = \begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{1}{2-2i}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{1}{2+2i}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1-3i}{30i}\right)\left(\frac{2+2i}{8}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{1+3i}{30i}\right)\left(\frac{2-2i}{8}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix} = \begin{pmatrix} \left(\frac{2-i}{60i}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{2+i}{60i}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix}$$

$$\bar{\bar{P}}e^{\bar{\bar{D}}t}\int_0^t e^{-\bar{\bar{D}}t}\bar{\bar{P}}^{-1}\bar{g}(t')\,dt' = \begin{pmatrix} -5e^{(-1+2i)t} & -5e^{-(1+2i)t} \\ (7+6i)e^{(-1+2i)t} & (7-6i)e^{-(1+2i)t}\end{pmatrix}\begin{pmatrix} \left(\frac{2-i}{60i}\right)\left(e^{(2-2i)t}-1\right) \\ -\left(\frac{2+i}{60i}\right)\left(e^{(2+2i)t}-1\right)\end{pmatrix}$$

$$= -e^{-t}\begin{pmatrix} \left(\frac{2-i}{12i}\right)(e^{2t}-e^{2it}) - \left(\frac{2+i}{12i}\right)(e^{2t}-e^{-2it}) \\ -\left(\frac{4+i}{12i}\right)(e^{2t}-e^{2it}) + \left(\frac{4-i}{12i}\right)(e^{2t}-e^{-2it})\end{pmatrix}$$

$$= \frac{-e^{-t}}{12i}\begin{pmatrix} 2(\cancel{e^{2t}}-e^{2it}) - i(e^{2t}-e^{2it}) - 2(\cancel{e^{2t}}-e^{-2it}) - i(e^{2t}-e^{-2it}) \\ -4(\cancel{e^{2t}}-e^{2it}) - i(e^{2t}-e^{2it}) + 4(\cancel{e^{2t}}-e^{-2it}) - i(e^{2t}-e^{-2it})\end{pmatrix}$$

$$= \frac{-e^{-t}}{12i}\begin{pmatrix} -2e^{2it} - ie^{2t} + ie^{2it} + 2e^{-2it} - ie^{2t} + ie^{-2it} \\ 4e^{2it} - ie^{2t} + ie^{2it} - 4e^{-2it} + ie^{2t} + ie^{-2it}\end{pmatrix}$$

$$= \frac{-e^{-t}}{12i}\begin{pmatrix} -2(e^{2it}-e^{2it}) - 2ie^{2t} + i(e^{2it}+e^{-2it}) \\ 4(e^{2it}-e^{-2it}) - 2ie^{2t} + i(e^{2it}+e^{-2it})\end{pmatrix}$$

$$= -e^{t}\begin{pmatrix} -\frac{1}{3}\sin(2t) - \frac{1}{6}e^{2t} + \frac{1}{6}\cos(2t) \\ \frac{2}{3}\sin(2t) - \frac{1}{6}e^{2t} + \frac{1}{6}\cos(2t)\end{pmatrix}$$

$$\bar{X}(t) = \frac{e^{-t}}{6}\begin{pmatrix} -7\sin(2t)+6\cos(2t) & -5\sin(2t) \\ 17\sin(2t) & 7\sin(2t)+6\cos(2t)\end{pmatrix}\begin{pmatrix} X_1(0) \\ X_2(0)\end{pmatrix} + \frac{e^{-t}}{6}\begin{pmatrix} 2\sin(2t)+e^{2t}-\cos(2t) \\ -4\sin(2t)+e^{2t}-\cos(2t)\end{pmatrix}$$