Homework Set No. 6
Due October 25, 2013

NEEP 547
DLH

Integral Equation (Homogeneous)

1. (6pts) Show that the only values of λ for which

$$f(x) = \lambda \int_0^1 xy\,(x+y)\,f(y)\,dy$$

has a non-trivial solution are the roots of the equation

$$\lambda^2 + 120\lambda - 240 = 0.$$

(Hint: This is an eigenvalue problem for the case of a homogeneous Fredholm Eq. It can be solved by the differential equation method.)

Integral Equation (Neumann series solution method)

2. (6pts) Solve the following integral equation using the Neumann series method:

$$\psi(x) = 1 + \frac{x^2}{2!} - \int_0^x (x-t)\,\psi(t)\,dt$$

3. (6pts) Determine $\psi(x)$ using the Neumann series method:

$$\psi(x) = x\cos x + \int_0^x t\,\psi(t)\,dt$$

Integral Equation (Differential equation solution method)

$$\begin{aligned}
4.\text{(6pts)}\quad 2\cosh(x) - \sinh(x) - (2-x) &= 1 + \int_0^x (2-x+t)\,\psi(t)\,dt \\
5.\text{(6pts)}\quad u(x) &= \cos(x) - x - 2 + \int_0^x (t-x)\,u(t)\,dt \\
6.\text{(6pts)}\quad u(x) &= x + \lambda \int_0^1 (1+x+t)\,u(t)\,dt \\
7.(6pts)\quad \frac{dy(t)}{dt} &= 2 - \frac{t^2}{2} - \frac{1}{4}\int_0^t y(\tau)\,d\tau, \text{ with I.C.: } y(0) = 0.
\end{aligned}$$

Integral Equation

8. (6pts) Solve the integral equation using any technique:

$$y(x) = f(x) - A\int_a^x xt\,e^{\lambda(x-t)}\,y(t)dt$$

1) Show that the only values of λ for which
$$f(x) = \lambda \int_0^1 xy(x+y)\, f(y)\,dy$$
has a non-trivial solution are the roots of the equation
$$\lambda^2 + 120\lambda - 240 = 0.$$
(Hint: this is an eigenvalue problem for the case of a homogeneous Fredholm Eq. It can be solved by the differential equation method).

$$f(x) = \lambda \int_0^1 xy(x+y)\, f(y)\,dy$$

$$\frac{df}{dx} = \lambda\left[\frac{d(1)}{dx}(1)y(1+y)f(1) - \frac{d(0)}{dx}(0)y(0+y)f(0)\right] + \lambda\int_0^1 \frac{d}{dx}\left(xy(x+y)f(y)\right)dy$$

$$= \lambda\int_0^1 (y(x+y) + xy)\, f(y)\,dy$$

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\lambda\int_0^1 (y(x+y)+xy) f(y)\,dy\right) = \lambda\int_0^1 (2y) f(y)\,dy$$

$\frac{d^3f}{dx^3} = 0$ so our functional form for $f(x)$ is $f(x) = A + Bx + Cx^2$

now to insert this into our integral equation

$$A + Bx + Cx^2 = \lambda\int_0^1 xy(x+y)(A + By + Cy^2)\,dy$$
$$= \lambda\int_0^1 \left(A(x^2y + xy^2) + B(x^2y^2 + xy^3) + C(x^2y^3 + xy^4)\right)dy$$
$$= \lambda\left[A\left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)\Big|_0^1 + B\left(\frac{x^2y^3}{3} + \frac{xy^4}{4}\right)\Big|_0^1 + C\left(\frac{x^2y^4}{4} + \frac{xy^5}{5}\right)\Big|_0^1\right]$$
$$= \lambda\left[A\left(\frac{x^2}{2} + \frac{x}{3}\right) + B\left(\frac{x^2}{3} + \frac{x}{4}\right) + C\left(\frac{x^2}{4} + \frac{x}{5}\right)\right]$$
$$A + Bx + Cx^2 = \lambda\left(\frac{A}{2} + \frac{B}{3} + \frac{C}{4}\right)x^2 + \lambda\left(\frac{A}{3} + \frac{B}{4} + \frac{C}{5}\right)x$$

Equating x, x^2 and constant terms on both sides gives

$A = 0$; $B = \left(\frac{B}{4} + \frac{C}{5}\right)\lambda$; $C = \left(\frac{B}{3} + \frac{C}{4}\right)\lambda$

$B - \frac{B\lambda}{4} = \frac{\lambda C}{5}$	$C = \frac{\lambda B}{3} + \frac{\lambda C}{4}$
$B\left(1 - \frac{\lambda}{4}\right) = C\frac{\lambda}{5}$	$C\left(1 - \frac{\lambda}{4}\right) = B\frac{\lambda}{3}$

we have two equations and two unknowns

$B(1-\frac{\lambda}{4}) = C\frac{\lambda}{5}$ and $C(1-\frac{\lambda}{4}) = B\frac{\lambda}{3}$

$\Rightarrow B = C\frac{\lambda}{5}\cdot\left(\frac{1}{1-\frac{\lambda}{4}}\right)$ insert into $C(1-\frac{\lambda}{4}) = \left(\frac{\lambda}{3}\right)\left(C\frac{\lambda}{5}\right)\left(\frac{1}{1-\frac{\lambda}{4}}\right)$

the C's cancel and we have $(1-\frac{\lambda}{4})^2 = \frac{\lambda^2}{15}$

$\Rightarrow \frac{1}{16}(4-\lambda)^2 = \frac{\lambda^2}{15} \Rightarrow 15(4-\lambda)^2 = 16\lambda^2$

$240 - 120\lambda^2 + 15\lambda^2 = 16\lambda^2 \Rightarrow \lambda^2 + 120\lambda - 240 = 0$ Eigenvalue Equation

$\lambda = \frac{-120 \pm \sqrt{(120)^2 + 4(240)}}{2} = -60 \pm \frac{1}{2}\sqrt{(64)(240)} = -60 \pm 8\sqrt{60}$

The Eigenvalue Eq. states the values for which integral Eq. has a solution.

Note that after equating both eqs. for B, the C dropped out.
This means that C can be anything - one typically chooses a normalization condition or choose 1. Here we choose 1. Thus $C=1$

We have two expressions for B: 1) $B = C\frac{\lambda}{5}\cdot\left(\frac{1}{1-\frac{\lambda}{4}}\right)$ and 2) $B = \frac{3}{\lambda}C(1-\frac{\lambda}{4})$

Both expressions reduce to the same value for a given eigenvalue

for $\lambda = -4(15+4\sqrt{15})$ $B = -\frac{\sqrt{15}}{5}$

for $\lambda = -4(15-4\sqrt{15})$ $B = \frac{\sqrt{15}}{5}$

Our general solution was $f(x) = Bx + Cx^2$

Our eigenvalues are $f_1(x) = -\frac{\sqrt{15}}{5}x + x^2$ for $\lambda_1 = -\frac{\sqrt{15}}{5}$

$f_2(x) = \frac{\sqrt{15}}{5}x + x^2$ for $\lambda_2 = \frac{\sqrt{15}}{5}$

recall $C=1$

and our condition

$\lambda^2 + 120\lambda - 240 = 0$

2) Solve the following integral equation using the Neuman series method:

$$\varphi(x) = 1 + \frac{x^2}{2!} - \int_0^x (x-t)\varphi(t)\,dt \qquad \varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x)$$

$$\varphi_0(x) = 1 + \frac{x^2}{2!}$$

$$\varphi_1(x) = -\int_0^x (x-t)\left(1+\frac{t^2}{2!}\right)dt = -\int_0^x (x)\left(1+\frac{t^2}{2!}\right)dt + \int_0^x (t)\left(1+\frac{t^2}{2!}\right)dt$$
$$= -x\int_0^x \left(1+\frac{t^2}{2!}\right)dt + \int_0^x \left(t+\frac{t^3}{2!}\right)dt$$
$$= -x\left(t+\frac{t^3}{3!}\right)\Big|_0^x + \left(\frac{t^2}{2}+\frac{t^4}{2\cdot 4}\right)\Big|_0^x$$
$$= -x\left(x+\frac{x^3}{2\cdot 3}\right) + \left(\frac{x^2}{2}+\frac{x^4}{2\cdot 4}\right) = -x^2 - \frac{x^4}{2\cdot 3} + \frac{x^2}{2} + \frac{x^4}{2\cdot 4}$$
$$= -\frac{x^2}{2} - \frac{x^4}{2\cdot 3\cdot 4} = -\frac{x^2}{2!} - \frac{x^4}{4!} = -\left(\frac{x^2}{2!}+\frac{x^4}{4!}\right)$$

$$\varphi_2(x) = -\int_0(x-t)\left(-\frac{t^2}{2!}-\frac{t^4}{4!}\right)dt = \int_0^x (x)\left(\frac{t^2}{2!}+\frac{t^4}{4!}\right)dt - \int_0^x (t)\left(\frac{t^2}{2!}+\frac{t^4}{4!}\right)dt$$
$$= x\int_0^x \left(\frac{t^2}{2!}+\frac{t^4}{4!}\right)dt - \int_0^x \left(\frac{t^3}{2!}+\frac{t^5}{4!}\right)dt$$
$$= x\left(\frac{t^3}{3!}+\frac{t^5}{5!}\right)\Big|_0^x - \left(\frac{t^4}{2\cdot 4}+\frac{t^6}{6\cdot 4!}\right)\Big|_0^x$$
$$= x\left(\frac{x^3}{3!}+\frac{x^5}{5!}\right) - \left(\frac{x^4}{2\cdot 4}+\frac{x^6}{6\cdot 4!}\right) = \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^4}{2\cdot 4} - \frac{x^6}{6\cdot 4!}$$
$$= \left(\frac{4}{2\cdot 3\cdot 4} - \frac{3}{2\cdot 3\cdot 4}\right)x^4 + \left(\frac{6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\right)x^6 = \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$\varphi_3(x) = -\int_0^x (x-t)\left(\frac{t^4}{4!}+\frac{x^6}{6!}\right)dt = -\int_0^x (x)\left(\frac{t^4}{4!}+\frac{t^6}{6!}\right)dt + \int_0^x (t)\left(\frac{t^4}{4!}+\frac{t^6}{6!}\right)dt$$
$$= -x\int_0^x \left(\frac{t^4}{4!}+\frac{t^6}{6!}\right)dt + \int_0^x \left(\frac{t^5}{4!}+\frac{t^7}{6!}\right)dt$$
$$= -x\left(\frac{t^5}{5!}+\frac{t^7}{7!}\right)\Big|_0^x + \left(\frac{t^6}{6\cdot 4!}+\frac{t^8}{8\cdot 6!}\right)\Big|_0^x = -\frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^6}{6\cdot 4!} + \frac{x^8}{8\cdot 6!}$$
$$= -\left(\frac{6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\right)x^6 - \left(\frac{8}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} - \frac{7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\right)x^8$$
$$= -\frac{x^6}{6!} - \frac{x^8}{8!}$$

$$\therefore \varphi(x) = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3, \ \cdots$$
$$= 1 + \frac{x^2}{2!} - \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^6}{6!} - \frac{x^8}{8!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} - \cdots$$
$$= 1$$

3). Determine $\varphi(x)$ using the neumann series method.

$$\varphi(x) = x\cos(x) + \int_0^x t\,\varphi(t)\,dt \qquad \varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x)$$

$$\varphi_0(x) = x\cos(x)$$

$$\varphi_1(x) = \int_0^x (t)(t\cos(t))dt = \int_0^x t^2\cos(t)dt = \int_0^x t^2 d(\sin(t))$$

$$= \left(t^2\sin(t)\Big|_0^x - \int_0^x \sin(t)(2t)dt\right)$$

$$= x^2\sin(x) - 2\left(\int_0^x t\,d(-\cos(t))\right)$$

$$= x^2\sin(x) - 2\left(-t\cos(t)\Big|_0^x + \int_0^x \cos(t)dt\right)$$

$$= x^2\sin(x) - 2\left(-x\cos(x) + \sin(t)\Big|_0^x\right)$$

$$= x^2\sin(x) + 2x\cos(x) - 2\sin(x)$$

$$\varphi_2(x) = \int_0^x (t)(t^2\sin(t) + 2t\cos(t) - 2\sin(t))dt$$

$$= \int_0^x (t^3\sin(t) + 2t^2\cos(t) - 2t\sin(t))dt$$

$$= \int_0^x t^3\sin(t)dt + 2\int_0^x t^2\cos(t)dt - 2\int_0^x t\sin(t)dt$$

$$= \int_0^x t^3 d(-\cos(t)) + 2\int_0^x t^2\cos(t)dt - 2\int_0^x t\sin(t)dt$$

$$= \left(-3t^2\cos(t)\Big|_0^x + \int_0^x 3t^2\cos(t)dt\right) + 2\int_0^x t^2\cos(t)dt - 2\int_0^x t\sin(t)dt$$

$$= -3x^2\cos(x) + 5\int_0^x t^2\cos(t)dt - 2\int_0^x t\sin(t)dt$$

$$= -3x^2\cos(x) + 5\int_0^x t^2\cos(t)dt - 2\int_0^x t\,d(-\cos(t))$$

$$= -3x^2\cos(x) + 5\int_0^x t^2\cos(t)dt - 2\left(-t\cos(t)\Big|_0^x + \int_0^x \cos(t)dt\right)$$

$$= -3x^2\cos(x) + 5\int_0^x t^2\cos(t)dt - 2\left(-x\cos(x) + \sin(x)\right)$$

$$= -3x^2\cos(x) + 5\int_0^x t^2\cos(t)dt + 2x\cos(x) - 2\sin(x)$$

was compute in $\varphi_1(x)$.

$$= -3x^2\cos(x) + 5\left(x^2\sin(x) + 2x\cos(x) - 2\sin(x)\right) + 2x\cos(x) - 2\sin(x)$$

$$= -3x^2\cos(x) + 5x^2\sin(x) + 10x\cos(x) - 10\sin(x) + 2x\cos(x) - 2\sin(x)$$

$$= -3x^2\cos(x) + 5x^2\sin(x) + 12x\cos(x) - 12\sin(x)$$

$$\psi_3(x) = \int_0^x (t)(-3t^2\cos(t) + 5t^2\sin(t) + 12t\cos(t) - 12\sin(t))\,dt$$

$$= -3\int_0^x t^3\cos(t)\,dt + 5\int_0^x t^3\sin(t)\,dt + 12\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$

$$= -3\int_0^x t^3\,d(\sin(t)) + 5\int_0^x t^3\,d(-\cos(t)) + 12\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$

$$= -3\left(t^3\sin(t)\Big|_0^x - \int_0^x 3t^2\sin(t)\,dt\right) + 5\left(-t^3\cos(t)\Big|_0^x + \int_0^x 3t^2\cos(t)\,dt\right.$$

$$+12\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$

$$= -3x^3\sin(x) + 9\int_0^x t^2\sin(t)\,dt - 5x^3\cos(x) + 15\int_0^x t^2\cos(t)\,dt + 12\int_0^x t^2\cos(t)\,dt$$

$$-12\int_0^x t\sin(t)\,dt$$

$$= -3x^3\sin(x) + 9\int_0^x t^2\sin(t)\,dt - 5x^3\cos(x) + 27\int_0^x t^2\cos(t)\,dt - 12\int_0^x t\sin(t)\,dt$$

This seems to be getting quite messy. Let's try a different tactic, Let's expand the cosine term right from the start.

$$\psi(x) = x\cos(x) + \int_0^x t\,\psi(t)\,dt \quad = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\psi_0(x) = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} \cdots$$

$$\psi_1(x) = \int_0^x t\left(t - \frac{t^3}{2!} + \frac{t^5}{4!} - \frac{t^7}{6!} + \cdots\right)dt = \int_0^x \left(t^2 - \frac{t^4}{2!} + \frac{t^6}{4!} - \frac{t^8}{6!} + \cdots\right)dt$$

$$= \left(\frac{t^3}{3} - \frac{t^5}{5\cdot 2!} + \frac{t^7}{7\cdot 4!} - \frac{t^9}{9\cdot 6!}\right)\Big|_0^x = \frac{x^3}{3} - \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 4!} - \frac{x^9}{9\cdot 6!}$$

$$\psi_2(x) = \int_0^x t\left(\frac{t^3}{3} - \frac{t^5}{5\cdot 2!} + \frac{t^7}{7\cdot 4!} - \frac{t^9}{9\cdot 6!} + \cdots\right)dt = \int_0^x\left(\frac{t^4}{3} - \frac{t^6}{5\cdot 2!} + \frac{t^8}{7\cdot 4!} - \frac{t^{10}}{9\cdot 6!} + \cdots\right)dt$$

$$= \left(\frac{t^5}{3\cdot 5} - \frac{t^7}{5\cdot 7\cdot 2!} + \frac{t^9}{7\cdot 9\cdot 4!} - \frac{t^{11}}{9\cdot 11\cdot 6!} + \cdots\right)\Big|_0^x = \frac{x^5}{3\cdot 5} - \frac{x^7}{5\cdot 7\cdot 2!} + \frac{x^9}{7\cdot 9\cdot 4!} - \frac{x^{11}}{9\cdot 11\cdot 6!} + \cdots$$

$$\psi_3(x) = \int_0^x t\left(\frac{t^5}{3\cdot 5} - \frac{t^7}{5\cdot 7\cdot 2!} + \frac{t^9}{7\cdot 9\cdot 4!} - \frac{t^{11}}{9\cdot 11\cdot 6!} + \cdots\right)dt$$

$$= \int_0^x\left(\frac{t^6}{3\cdot 5} - \frac{t^8}{5\cdot 7\cdot 2!} + \frac{t^{10}}{7\cdot 9\cdot 4!} - \frac{t^{12}}{9\cdot 11\cdot 6!} + \cdots\right)dt$$

$$= \left(\frac{t^7}{3\cdot 5\cdot 7} - \frac{t^9}{5\cdot 7\cdot 9\cdot 2!} + \frac{t^{11}}{7\cdot 9\cdot 11\cdot 4!} - \frac{t^{13}}{9\cdot 11\cdot 13\cdot 6!} + \cdots\right)\Big|_0^x$$

$$= \left(\frac{x^7}{3\cdot 5\cdot 7} - \frac{x^9}{5\cdot 7\cdot 9\cdot 2!} + \frac{x^{11}}{7\cdot 9\cdot 11\cdot 4!} - \frac{x^{13}}{9\cdot 11\cdot 13\cdot 6!} + \cdots\right)$$

now to sum up the first 4 terms

$$\psi(x) = \psi_0 + \psi_1 + \psi_2 + \psi_3$$

$$\psi(x) = \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}\right) + \left(\frac{x^3}{3} - \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 4!} - \frac{x^9}{9\cdot 6!}\right)$$
$$+ \left(\frac{x^5}{3\cdot 5} - \frac{x^7}{5\cdot 7\cdot 2!} + \frac{x^9}{7\cdot 9\cdot 4!} - \frac{x^{11}}{9\cdot 11\cdot 6!}\right) + \left(\frac{x^7}{3\cdot 5\cdot 7} - \frac{x^9}{5\cdot 7\cdot 9\cdot 2!} + \frac{x^{11}}{7\cdot 9\cdot 11\cdot 4!} - \frac{x^{13}}{9\cdot 11\cdot 13\cdot 6!}\right)$$

let only keep terms through the 7th power

$$\psi(x) = \left(x - \left(\frac{1}{2!} - \frac{1}{3}\right)x^3 + \left(\frac{1}{4!} - \frac{1}{5\cdot 2!} + \frac{1}{3\cdot 5}\right)x^5 - \left(\frac{1}{6!} - \frac{1}{7\cdot 4!} + \frac{1}{5\cdot 7\cdot 2!} - \frac{1}{3\cdot 5\cdot 7}\right)x^7 \cdots\right) + \cdots$$
$$= x - \left(\frac{3}{6} - \frac{2}{6}\right)x^3 + \left(\frac{5}{2\cdot 3\cdot 4\cdot 5} - \frac{3\cdot 4}{2\cdot 3\cdot 4\cdot 5} + \frac{2\cdot 4}{2\cdot 3\cdot 4\cdot 5}\right)x^5 - \left(\frac{7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} - \frac{5\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{3\cdot 4\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} - \frac{2\cdot 4\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\right)x^7$$
$$= x - \frac{1}{6}x^3 + \left(\frac{5-12+8}{2\cdot 3\cdot 4\cdot 5}\right)x^5 - \left(\frac{7-30+72-48}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\right) + \cdots$$
$$= x - \frac{x^3}{3!} + \left(\frac{13-12}{5!}\right)x^5 - \left(\frac{79-78}{7!}\right)x^7 + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{Series for } \sin(x)$$

$$\therefore \psi(x) = \sin(x)$$

let's check

$$\psi(x) = x\cos(x) + \int_0^x t\,\psi(t)\,dt \qquad \text{insert } \psi(x) = \sin(x)$$

$$\sin(x) = x\cos(x) + \int_0^x t\sin(t)\,dt$$
$$= x\cos(x) + \int_0^x t\,d(-\cos(t))$$
$$= x\cos(x) + \left(-t\cos(t)\Big|_0^x + \int_0^x \cos(t)\,dt\right)$$
$$= x\cos(x) - x\cos(x) + \sin(t)\Big|_0^x$$
$$= \sin(x)$$
$$\sin(x) = \sin(x) \checkmark$$

4. Solve using the Differential equation solution method

$$2\cosh(x) - \sinh(x) - (2-x) = \int_0^x (2-x+t)\,\psi(t)\,dt$$

take derivative of eq.

$$2\sinh(x) - \cosh(x) + 1 = \left[\frac{dx}{dx}(2-x+x)\psi(x) - \frac{d0}{dx}(\ ) + \int_0^x (-1)\,\psi(t)\,dt\right]$$

$$= 2\psi(x) - \int_0^x \psi(t)\,dt$$

take another derivative

$$2\cosh(x) - \sinh(x) = 2\frac{d\psi(x)}{dx} - \left[\frac{dx}{dx}\psi(x) - \frac{d(0)}{dx}\psi(0) + \int_0^x \frac{d\psi(t)}{dx}\,dt\right]$$

$$= 2\frac{d\psi(x)}{dx} - \psi(x)$$

we have the D.E.

$$2\frac{d\psi}{dx} - \psi(x) = 2\cosh(x) - \sinh(x)$$

$$\frac{d\psi}{dx} - \tfrac{1}{2}\psi(x) = \cosh(x) - \tfrac{1}{2}\sinh(x) \qquad I_f \text{ is } e^{-\frac{1}{2}\int dx} = e^{-x/2}$$

$$\underset{①}{\int_0^x d(e^{-x/2}\psi(x))} = \underset{②}{\int_0^x e^{-x/2}\cosh(x)\,dx} - \underset{③}{\tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx}$$

① $\int_0^x d(e^{-x/2}\psi(x)) = e^{-x/2}\psi(x) - \psi(0)$

② $\int_0^x e^{-x/2}\cosh(x)\,dx = \int_0^x e^{-x/2}\,d(\sinh(x)) = \left(e^{-x/2}\sinh(x)\right)\Big|_0^x + \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$

$$= e^{-x/2}\sinh(x) - 0 + \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$$

② + ③ $= \int_0^x e^{-x/2}\cosh(x)\,dx - \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$

$$= e^{-x/2}\sinh(x) + \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx - \tfrac{1}{2}\int_0^x e^{-x/2}\sinh(x)\,dx$$

$$= e^{-x/2}\sinh(x)$$

now to combine everything: ① = ② + ③

$$e^{-x/2}\psi(x) - \psi(0) = e^{-x/2}\sinh(x)$$

$$\psi(x) = \sinh(x) + \psi(0)e^{x/2} \qquad \text{need to find } \psi(0)$$

we find $\psi(0)$ from the equation above: $2\sinh(x) - \cosh(x) + 1 = 2\psi(x) - \int_0^x \psi(t)\,dt$

$$2\sinh(x) - \cosh(x) + 1 = 2(\sinh(x) + \psi(0)e^{x/2}) - \int_0^x \sinh(x)\,dx - \int_0^x \psi(0)e^{x/2}$$

$$2\sinh(x) - \cosh(x) + 1 = 2\sinh(x) + 2\psi(0)e^{x/2} - \cosh(x) + 1 - 2\psi(0)e^{x/2} + 2\psi(0)$$

$$0 = 2\psi(0) \qquad \text{we see that for any } x \ \ \psi(0) = 0$$

$\therefore \psi(x) = \sinh(x)$

5) Solve using the differential Eq. method

$$u(x) = \cos(x) - x - 2 + \int_0^x (t-x)\,u(t)\,dt$$

$$\frac{du}{dx} = -\sin(x) - 1 + \left[\frac{dx}{dx}(x-x)u(x) - \frac{d(0)}{dx}(\;) + \int_0^x (-1)\,u(t)\,dt\right]$$

$$= -\sin(x) - 1 - \int_0^x u(t)\,dt$$

$$\frac{d^2u}{dx^2} = -\cos(x) - \left[\frac{dx}{dx}u(x) - \frac{d(0)}{dx}u(0) + \int_0^x \frac{du(t)}{dx}\,dt\right]$$

$$\frac{d^2u}{dx^2} = -\cos(x) - u(x)$$

$$\frac{d^2u}{dx^2} + u(x) = -\cos(x)$$

2^{nd} order Eq. with conditions: $u(0) = -1$, $u'(0) = -1$

$$u(x) = u_h(x) + u_p(x)$$

$$u_h(x) = A\cos(x) + B\sin(x)$$

$$u_p = c\,x\cos(x) + d\,x\sin(x)$$

$$u_p' = c(\cos(x) - x\sin(x)) + d(\sin(x) + x\cos(x))$$

$$u_p'' = c(-\sin(x) - \sin(x) - x\cos(x)) + d(\cos(x) + \cos(x) - x\sin(x))$$

substitute in D.E.

$$c(-2\sin(x) - x\cos(x)) + d(2\cos(x) - x\sin(x)) + c\,x\cos(x) + d\,x\sin(x) = -\cos(x)$$

$$c(-2\sin(x) - x\cos(x) + x\cos(x)) + d(2\cos(x) - x\sin(x) + x\sin(x)) = -\cos(x)$$

$$-2c\sin(x) + 2d\cos(x) = -\cos(x)$$

$\therefore c = 0 \qquad 2d = -1 \Rightarrow d = -\tfrac{1}{2}$

$$u_p = -\tfrac{1}{2}x\sin(x)$$

$$u(x) = A\cos(x) + B\sin(x) - \tfrac{1}{2}x\sin(x)$$

$$u'(x) = -A\sin(x) + B\cos(x) - \tfrac{1}{2}\sin(x) - \tfrac{1}{2}x\cos(x)$$

now to find A & B from the I.C. $u(0) = -1$, $u'(0) = -1$

$u(0) = A + 0 - 0 = -1 \quad \therefore A = -1$

$u'(0) = -0 + B - 0 - 0 = -1 \quad \therefore B = -1$

$$u(x) = -\cos(x) - \sin(x) - \tfrac{1}{2}x\sin(x)$$

6). Solve using the differential Eq. method

$$u(x) = x + \lambda \int_0^1 (1+x+t)u(t)\,dt$$

$$\frac{du}{dx} = 1 + \lambda\left[\frac{d(1)}{dx}(\ ) - \frac{d(0)}{dx} + \int_0^1 u(t)\,dt\right]$$

$$\frac{d^2u}{dx^2} = 0 \qquad \therefore u(x) = Ax + B$$

$$Ax + B = x + \lambda\int_0^1 (1+x+t)(At+B)\,dt$$
$$= x + \lambda\left[\int_0^1 (1+x)(At+B)\,dt + \int_0^1 (At^2+Bt)\,dt\right]$$
$$= x + \lambda\left[(1+x)\int_0^1 (At+B)\,dt + \int_0^1 (At^2+Bt)\,dt\right]$$
$$= x + \lambda\left[(1+x)\left(A\frac{t^2}{2}+Bt\right)\Big|_0^1 + \left(A\frac{t^3}{3}+B\frac{t^2}{2}\right)\Big|_0^1\right]$$

$$Ax + B = x + \lambda\left[(1+x)\left(\frac{A}{2}+B\right) + \left(\frac{A}{3}+\frac{B}{2}\right)\right]$$
$$= x + \lambda\left[\left(\frac{A}{2}+B\right) + \left(\frac{A}{2}+B\right)x + \left(\frac{A}{3}+\frac{B}{2}\right)\right]$$
$$= x + \lambda\left(\frac{A}{2}+B\right)x + \lambda\left(\frac{A}{2}+B+\frac{A}{3}+\frac{B}{2}\right)$$
$$= \left(1+\lambda\left(\frac{A}{2}+B\right)\right)x + \lambda\left(\frac{5}{6}A+\frac{3}{2}B\right)$$

comparing coefficients of x and constant terms, we have the conditions

$$A = \left(1+\lambda\left(\frac{A}{2}+B\right)\right) \quad \text{and} \quad B = \lambda\left(\frac{5}{6}A+\frac{3}{2}B\right)$$

two Eqs. and two unknowns (A & B are unknown)

$$B = \lambda\left(\frac{5}{6}A+\frac{3}{2}B\right) \Rightarrow B = \frac{5}{6}A\lambda + \frac{3}{2}\lambda B \Rightarrow B\left(1-\frac{3}{2}\lambda\right) = \frac{5}{6}A\lambda \quad (1)$$

$$A = \left(1+\lambda\left(\frac{A}{2}+B\right)\right) \Rightarrow A = 1 + \frac{A}{2}\lambda + B\lambda \Rightarrow A\left(1-\frac{\lambda}{2}\right) = 1 + B\lambda \quad (2)$$

(1) $A = \frac{6}{5\lambda}\left(1-\frac{3}{2}\lambda\right)B$ sub in (2) $\frac{6}{5\lambda}\left(1-\frac{3}{2}\lambda\right)\left(1-\frac{\lambda}{2}\right)B = 1 + B\lambda \Rightarrow \left[\frac{6}{5\lambda}\left(1-\frac{3}{2}\lambda\right)\left(1-\frac{\lambda}{2}\right)-\lambda\right]B = 1$

$$\therefore B = \frac{1}{\frac{6}{5\lambda}\left(1-\frac{3}{2}\lambda\right)\left(1-\frac{\lambda}{2}\right)-\lambda} = \frac{5\lambda}{6(1-3/2\,\lambda)(1-\lambda/2)-5\lambda^2} = \frac{20\lambda}{6(2-3\lambda)(2-\lambda)-20\lambda^2}$$

(2) $A\left(1-\frac{\lambda}{2}\right) = 1 + B\lambda \Rightarrow A = \frac{1}{\left(1-\frac{\lambda}{2}\right)} + \frac{B\lambda}{\left(1-\frac{\lambda}{2}\right)} = \frac{2}{(2-\lambda)} + \frac{2\lambda B}{(2-\lambda)}$

$$= \left(\frac{2}{2-\lambda}\right) + \left(\frac{2\lambda}{(2-\lambda)}\right)\left(\frac{20\lambda}{6(2-3\lambda)(2-\lambda)-20\lambda^2}\right)$$
$$= \left(\frac{2}{2-\lambda}\right) + \left(\frac{40\lambda^2}{6(2-3\lambda)(2-\lambda)^2-(2-\lambda)(20\lambda^2)}\right)$$

$$\therefore u(x) = Ax + B = \left[\left(\frac{2}{2-\lambda}\right) + \left(\frac{40\lambda^2}{6(2-3\lambda)(2-\lambda)^2-(2-\lambda)(20\lambda^2)}\right)\right]x + \frac{20\lambda}{6(2-3\lambda)(2-\lambda)-20\lambda^2}$$

7) Solving using the differential equation method

$$\frac{dy}{dt} = 2 - \frac{t^2}{2} - \frac{1}{4}\int_0^t y(\tau)\,d\tau \quad \text{with I.C.: } y(0)=0$$

$$\frac{d^2y}{dt^2} = -t - \frac{1}{4}\left[\frac{dt}{dt}y(t) - \frac{d(0)}{dt} + \int_0^t \frac{dy(\tau)}{dt}d\tau\right]$$

$$= -t - \frac{1}{4}y(t)$$

$$\frac{d^2y}{dt^2} + \frac{1}{4}y(t) = -t \qquad y_h(t) = A\cos(t/2) + B\sin(t/2)$$

assume $y_p = at \qquad y_p'' = 0$

$$0 + \frac{1}{4}at = -t \Rightarrow \frac{a}{4}t = -t \Rightarrow \frac{a}{4} = -1 \Rightarrow a = -4 \Rightarrow y_p(t) = -4t$$

$$y(t) = y_h + y_p = A\cos(t/2) + B\sin(t/2) - 4t \qquad y(0)=0,\ y'(0)=2$$

$$y(0) = A\cos(0) + B\sin(0) - 4(0) = 0 \Rightarrow A = 0$$

$$y'(t) = -\frac{A}{2}\sin(t/2) + \frac{B}{2}\cos(t/2) - 4$$

$$y'(0) = -\frac{A}{2}(0) + \frac{B}{2}\cos(0) - 4 = 2 \Rightarrow \frac{B}{2} = 6 \Rightarrow B = 12$$

$$y(t) = 12\sin(t/2) - 4t$$

8.) Solve the integral equation using any technique

$$y(x) = f(x) - A\int_a^x x\,t\,e^{\lambda(x-t)}\,y(t)\,dt$$

$$y(x) = f(x) - A\,x\,e^{\lambda x}\int_a^x t\,e^{-\lambda t}\,y(t)\,dt \qquad y(a) = f(a)$$

$$\frac{y(x)}{x e^{\lambda x}} = \frac{f(x)}{x e^{\lambda x}} - A\int_a^x t^2\,\frac{y(t)}{t e^{\lambda t}}\,dt \qquad \text{let } \phi(x) = \frac{y(x)}{x e^{\lambda x}} \text{ and } F(x) = \frac{f(x)}{x e^{\lambda x}}$$

$$\phi(x) = F(x) - A\int_a^x t^2\,\phi(t)\,dt \qquad \phi(a) = F(a)$$

$$\frac{d\phi(x)}{dx} = \frac{dF(x)}{dx} - A\left[\frac{dx}{dx}x^2\phi(x) - \frac{da}{dx}a^2\phi(a) + \int_a^x \frac{\partial}{\partial x}\left(t^2\phi(t)\right)dt\right]$$

$$\frac{d\phi}{dx} = \frac{dF(x)}{dx} - A x^2\phi(x) \Rightarrow \frac{d\phi}{dx} + A x^2\phi(x) = \frac{dF(x)}{dx} \qquad I_f = e^{\int A x^2 dx} = e^{A\frac{x^3}{3}}$$

$$\int_a^x d\left(\phi(x)\,e^{\frac{A}{3}x^3}\right) = \int_a^x e^{\frac{A}{3}t^3}\frac{dF(t)}{dt}\,dt =$$

$$\phi(x)\,e^{\frac{A}{3}x^3} - \phi(a)\,e^{\frac{A}{3}a^3} = F(t)e^{\frac{A}{3}t^3}\Big|_a^x - \int_a^x F(t)\,d\left(e^{\frac{A}{3}t^3}\right)$$

$$\phi(x)e^{\frac{A}{3}x^3} - \phi(a)e^{\frac{A}{3}a^3} = F(x)e^{\frac{A}{3}x^3} - F(a)e^{\frac{A}{3}a^3} - \int_a^x F(t)e^{\frac{A}{3}t^3}(At^2)\,dt \qquad \phi(a) = F(a)$$

$$\phi(x) = F(x) - e^{-\frac{A}{3}x^3}\int_a^x (At^2)\,F(t)\,e^{\frac{A}{3}t^3}\,dt$$

$$\phi(x) = F(x) - A\int_a^x t^2 F(t)\,e^{\frac{A}{3}(t^3-x^3)}\,dt \qquad \text{now } F(t) = \frac{f(t)}{t e^{\lambda t}} \text{ and } \phi(x) = \frac{y(x)}{x e^{\lambda x}}$$

$$\frac{y(x)}{x e^{\lambda x}} = \frac{f(x)}{x e^{\lambda x}} - A\int_a^x t^2\,\frac{f(t)}{t e^{\lambda t}}\,e^{\frac{A}{3}(t^3-x^3)}\,dt$$

$$y(x) = f(x) - A\,x\,e^{\lambda x}\int_a^x t\,f(t)\,e^{-\lambda t}\,e^{\frac{A}{3}(t^3-x^3)}\,dt$$

$$y(x) = f(x)\;A\int_a^x x\,t\,e^{\frac{A}{3}(t^3-x^3)}\,e^{\lambda(x-t)}\,f(t)\,dt$$