Homework Set No. 5
Due October 11, 2013

NEEP 547
DLH

1. (4pts) Find the inverse Laplace transform of the following function:

$$g(s) = \ln\left(\frac{s^2+1}{s(s+1)}\right)$$

2. (6pts) Use the Laplace transform to solve the following problem:
$(1-t)\,y'' + t\,y' - y = 0$ with initial conditions; $y(0) = 3$ and $y'(0) = -1$.

Periodic function

3. (8pts) Solve the initial value problem where is $f(t)$ is a periodic function:
$y' + 4\,y + 3\,\int_0^t y(t')\,dt' = f(t)$; $y(0) = 1$, with $f(t) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ -1 & \text{for } 2 < t < 4. \end{cases}$

Convolution Theorem

4. (10pts) Consider the equation of motion for a damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + 2\lambda\,\frac{dx}{dt} + \omega_o^2 x = \frac{f(t)}{m} \quad \text{where: } \lambda = \frac{\rho}{2m} \text{ and } \omega_o = \sqrt{\frac{k}{m}}.$$

with a general forcing function $f(t)$ and initial conditions $x(0) = 0$ and $x'(0) = v_0$. Express the general solution of the equation in terms of the convolution integral. Determine the solution for function a). $f(t) = P\,\delta(t)$ and b) $f(t) = F_0\,\sin(\omega\,t)$. Note: the oscillator is under damped, i.e. $\omega_o^2 > \lambda^2$.

5. (10pts) Heat Conduction Equation ($0 < x < \ell$ and $t > 0$):

$$\frac{\partial U}{\partial t} - k\frac{\partial^2 U}{\partial x^2} = a\,e^{-\alpha\,t} \quad \text{with I.C.: } U(x,0) = 0 \text{ and B.C.: } U(0,t) = U(\ell,t) = 0$$

where k, a and α are positive constants.

1) Find the Laplace Transform of the following function

$$g(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$$

$$= \left(\frac{1}{\ln(10)}\right)\left(\ln\left(\frac{s^2+1}{s(s+1)}\right)\right)$$

$$= \left(\frac{1}{\ln(10)}\right)\left(\ln(s^2+1) - \ln(s) - \ln(s+1)\right)$$

we will use the following property to aid in the inversion $\mathcal{L}^{-1}\{-F'(s)\} = t\, f(t) \therefore f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$

$$\mathcal{L}^{-1}\{g(s)\} = \left(\frac{1}{\ln(10)}\right)\left(\mathcal{L}^{-1}\left\{\frac{d}{ds}(\ln(s^2+1)) - \frac{d}{ds}(\ln(s)) - \frac{d}{ds}(\ln(s+1))\right\}\right)$$

$$g(t) = \left(\frac{1}{\ln(10)}\right)\left(\frac{-1}{t}\right)\left(\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\right)$$

$$= \left(\frac{1}{\ln(10)}\right)\left(\frac{1}{t}\right)\left(1 + e^{-t} - 2\cos(t)\right)$$

2) Solve the following: $(1-t)y'' + ty' - y = 0$; $y(0) = 3$; $y'(0) = -1$

$y'' - ty'' + ty' - y = 0$ Let's take the Laplace Transform

$$\left(s^2\bar{Y}(s) - sy(0) - y'(0)\right) + \frac{d}{ds}\left(s^2\bar{Y}(s) - sy(0) - y'(0)\right) - \frac{d}{ds}\left(s\bar{Y}(s) - y(0)\right) - \bar{Y}(s) = 0$$

$$\left(s^2\bar{Y}(s) - 3s + 1\right) + \left(2s\bar{Y}(s) + s^2\bar{Y}'(s) - y(0)\right) - \left(\bar{Y}(s) + s\bar{Y}'(s)\right) - \bar{Y}(s) = 0$$

$$\left(s^2\bar{Y}(s) - 3s + 1\right) + \left(2s\bar{Y}(s) + s^2\bar{Y}'(s) - 3\right) - \left(\bar{Y}(s) + s\bar{Y}'(s)\right) - \bar{Y}(s) = 0$$

$$\left(s^2 - s\right)\bar{Y}'(s) + \left(s^2 + 2s - 1 - 1\right)\bar{Y}(s) - 3s + 1 - 3 = 0$$

$$s(s-1)\bar{Y}'(s) + \left(s^2 + 2s - 2\right)\bar{Y}(s) = 3s + 2$$

$$\bar{Y}'(s) + \frac{(s^2 + 2s - 2)}{s(s-1)}\bar{Y}(s) = \frac{3s+2}{s(s-1)}$$

$$\bar{Y}'(s) + \left(\frac{s^2}{s(s-1)} + \frac{2(s-1)}{s(s-1)}\right)\bar{Y}(s) = \frac{3s+2}{s(s-1)}$$

$$\bar{Y}'(s) + \left(\frac{s}{s-1} + \frac{2}{s}\right)\bar{Y}(s) = \frac{3s+2}{s(s-1)}$$

$$\bar{Y}'(s) + \left(1 + \frac{1}{s-1} + \frac{2}{s}\right)\bar{Y}(s) = \frac{3s+2}{s(s-1)}$$ integrating factor $e^{\int\left(1 + \frac{1}{s-1} + \frac{2}{s}\right)ds}$

$$e^{s + \ln(s-1) + 2\ln(s)} = e^s(s-1)s^2$$

$$\int d\left(\left(\bar{Y}(s)\right)\left(e^s(s-1)s^2\right)\right) = \int\left(\frac{3s+2}{s(s-1)}\right)\left(e^s(s-1)s^2\right)ds$$

$$\bar{Y}(s)\left(e^s(s-1)s^2\right) = \int e^s(3s+2)s\,ds = \int e^s(3s^2+2s)\,ds = 3\int s^2e^s\,ds + 2\int se^s\,ds$$
$$= 3\left(s^2e^s - \int 2se^s\,ds\right) + 2\int se^s\,ds$$
$$= 3s^2e^s - 4\int se^s\,ds$$
$$= 3s^2e^s - 4\left(se^s - \int e^s\,ds\right)$$
$$= 3s^2e^s - 4se^s + 4e^s + c$$

$$\bar{Y}(s) = \frac{3s^2e^s}{e^s(s-1)s^2} - \frac{4se^s}{e^s(s-1)s^2} + \frac{4e^s}{e^s(s-1)s^2} + \frac{c}{e^s(s-1)s^2}$$

$$Y(s) = \frac{3}{(s-1)} - \frac{4}{s(s-1)} + \frac{4}{s^2(s-1)} + \frac{ce^{-s}}{s^2(s-1)}$$

$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s-1}\right)\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} + c\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}e^{-s}\right\}$$

$$= 3e^{t} - 4\int_0^t e^{t'}\,dt' + 4\mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{(s-1)}\right\} + c\mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{(s-1)}\right\}\Big|_{t' \to t-1} H(t-1)$$

$$= 3e^{t} - 4(e^{t} - 1) + 4(-1 - t + e^{t}) + c\left(-1 - (t-1) + e^{(t-1)}\right) H(t-1)$$

$$= 3e^{t} - \cancel{4e^{t}} + \cancel{4} - \cancel{4} - 4t + \cancel{4e^{t}} + c\left(-1 - t + 1 + e^{(t-1)}\right) H(t-1)$$

$$y(t) = 3e^{t} - 4t - c\left(t - e^{(t-1)}\right) H(t-1)$$

There is not enough information given to find c.

3) Solve the initial value problem where $f(t)$ is a periodic function

$$y' + 4y + 3\int_0^t y(t')\,dt' = f(t);\ y(0)=1 \text{ with } f(t) = \begin{cases} 1 & 0<t<2 \\ -1 & 2<t<4 \end{cases}$$

$$(s\bar{y}(s) - y(0)) + 4\bar{y}(s) + 3\,\mathcal{L}\left\{\int_0^t y(t')\,dt'\right\} = \mathcal{L}\{f(t)\}$$

now $\mathcal{L}\left\{\int_0^t y(t')\,dt'\right\} = \frac{1}{s}\bar{y}(s)$

and $\mathcal{L}\{f(t)\} = \left(\frac{1}{1-e^{-sk}}\right)\int_0^k f(t)e^{-st}\,dt \qquad k=4$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(\int_0^2 e^{-st'}dt' - \int_2^4 e^{-st'}dt'\right)$$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(-\frac{1}{s}e^{-st'}\Big|_0^2 - \left(-\frac{1}{s}\right)e^{-st'}\Big|_2^4\right)$$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(-\frac{1}{s}\right)\left(e^{-2s} - 1 - e^{-4s} + e^{-2s}\right)$$

$$= \left(\frac{1}{1-e^{-sk}}\right)\left(\frac{1}{s}\right)\left(1 - 2e^{-2s} + e^{-4s}\right)$$

$$= \left(\frac{1}{s}\right)\left(\frac{1}{1-e^{-4s}}\right)\left(1-e^{-2s}\right)^2 = \left(\frac{1}{s}\right)\left(\frac{(1-e^{-2s})^2}{(1-e^{-2s})(1+e^{-2s})}\right)$$

$$= \left(\frac{1}{s}\right)\left(\frac{1-e^{-2s}}{1+e^{-2s}}\right)$$

thus $s\bar{y}(s) - 1 + 4\bar{y}(s) + \frac{3}{s}\bar{y}(s) = \left(\frac{1}{s}\right)\left(\frac{1-e^{-2s}}{1+e^{-2s}}\right)$

$$(s^2+4s+3)\bar{y}(s) = \left(\frac{1-e^{-2s}}{1+e^{-2s}}\right) + s$$

$$\bar{y}(s) = \frac{s}{(s+1)(s+3)} + \left(\frac{1-e^{-2s}}{1+e^{-2s}}\right)$$

$$= \frac{s}{(s+1)(s+3)} + \left(\frac{1}{(s+1)(s+3)}\right)\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} - 2e^{-10s} + \cdots\right)$$

$$y(t) = \mathcal{L}^{-1}\underbrace{\left\{\frac{s}{(s+1)(s+3)}\right\}}_{\text{use partial Fraction}} + \mathcal{L}^{-1}\left\{\left(\frac{1}{(s+1)(s+3)}\right)\underbrace{\left(1-2e^{-2s}+2e^{-4s}-2e^{-6s}+2e^{-8s}-\cdots\right)}_{\text{Will give Heaviside functions}}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ -\frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{3}{2}\left(\frac{1}{s+3}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+3}\right)\left(1 - 2e^{-2s} + 2e^{-4s} - 2e^{-6s} + 2e^{-8s} - \cdots\right)\right\}$$

$$= \frac{1}{2}\left(-e^{-t} + 3e^{-3t}\right) + \frac{1}{2}\left(e^{-t} - e^{-3t}\right)\Big| \left(1 - 2H(t-2) + 2H(t-4) - 2H(t-6)\right)$$

$t \to (t-1)$ or $(t-3)$ t is shifted

$$= \frac{1}{2}\left(-e^{-t} + 3e^{-3t}\right) + \frac{1}{2}\left(e^{-t} - e^{-3t}\right) - \frac{1}{2}(2)\left(e^{-(t-2)} - e^{-3(t-2)}\right)H(t-2) + \frac{1}{2}(2)\left(e^{-(t-4)} - e^{-2(t-4)}\right)\underbrace{}_{H(t-4)}$$

$$- \frac{1}{2}(2)\left(e^{-(t-6)} - e^{-3(t-6)}\right)H(t-6) + \frac{1}{2}(2)\left(e^{-(t-8)} - e^{-3(t-8)}\right)H(t-8) + \cdots$$

$$y(t) = \begin{cases} e^{-3t} & 0<t<2 \\ e^{-3t} - e^{-(t-2)} + e^{-3(t-2)} & 2<t<4 \\ e^{-3t} - e^{-(t-2)} + e^{-3(t-2)} + e^{-(t-4)} - e^{-3(t-4)} & 4<t<6 \\ \vdots & \vdots \end{cases}$$

$$y(t) = \begin{cases} e^{-3t} & 0<t<2 \\ e^{-3t} - e^{-(t-2)} + e^{-3(t-2)} & 2<t<4 \\ e^{-3t} - \left(e^{-(t-2)} - e^{-(t-4)}\right) + \left(e^{-3(t-2)} - e^{-3(t-4)}\right) & 4<t<6 \\ e^{-3t} - \left(e^{-(t-2)} - e^{-(t-4)} + e^{-(t-6)}\right) + \left(e^{-3(t-2)} - e^{-3(t-4)} + e^{-3(t-6)}\right) & 6<t<8 \end{cases}$$

$$y(t) = \begin{cases} e^{-3t} & 0<t<2 \\ e^{-3t} - e^{-(t-2)} + e^{-3(t-2)} & 2<t<4 \\ e^{-3t} - e^{-t}\left(e^{2} - e^{4}\right) + e^{-3t}\left(e^{3(2)} - e^{3(4)}\right) & 4<t<6 \\ e^{-3t} - e^{-t}\left(e^{2} - e^{4} + e^{6}\right) + e^{-3t}\left(e^{3(2)} - e^{3(4)} + e^{3(6)}\right) & 6<t<8 \end{cases}$$

$$y(t) = \begin{cases} e^{-3t} & 0<t<2 \\ 2e^{-3t} - e^{-t} + e^{-t}\left(1 - e^{2}\right) - e^{-3t}\left(1 - e^{6}\right) & 2<t<4 \\ 2e^{-3t} - e^{-t} + e^{-t}\left(1 - e^{2} + e^{4}\right) - e^{-3t}\left(1 - e^{3(2)} + e^{3(4)}\right) & 4<t<6 \\ 2e^{-3t} - e^{-t} + e^{-t}\left(1 - e^{2} + e^{4} - e^{6}\right) - e^{-3t}\left(1 - e^{3(2)} + e^{3(4)} - e^{3(6)}\right) & 6<t<8 \end{cases}$$

$$y(t) = 2e^{-3t} - e^{-t} + e^{-t}\left(\frac{1 + (-1)^{(n+1)}e^{2n}}{1 + e^{2}}\right) - e^{-3t}\left(\frac{1 + (-1)^{(n+1)}e^{6n}}{1 + e^{6}}\right) \qquad 2(n-1) < t < 2n$$

for $n = 1, 2, 3, \cdots$

4). Consider the equation of motion for a damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = \frac{f(t)}{m} \qquad \text{where } \lambda = \frac{\mathcal{f}}{2m} \text{ and } \omega_0 = \sqrt{\frac{k}{m}}$$

with a general forcing function $f(t)$ and initial conditions $x(0)=0$ and $x'(0) = V_0$. Express the general solution of the equation in terms of a convolution integral. Determine the solution for function a). $f(t) = P\,\delta(t)$ and b) $f(t) = F_0 \sin(\omega t)$. Note the oscillator is underdamped, i.e. $\omega_0^2 > \lambda^2$.

$$\frac{d^2x}{dt} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = \frac{f(t)}{m}$$

$$\left(s^2\bar{x}(s) - s x(0) - x'(0)\right) + 2\lambda\left(s\bar{x}(s) - x(0)\right) + \omega_0^2\bar{x}(s) = \bar{f}(s)\left(\frac{1}{m}\right)$$

$$\left(s^2\bar{x}(s) - V_0\right) + 2\lambda\left(s\bar{x}(s)\right) + \omega_0^2\bar{x}(s) = \bar{f}(s)\left(\frac{1}{m}\right)$$

$$\left(s^2 + 2\lambda s + \omega_0^2\right)\bar{x}(s) = \left(\frac{1}{m}\right)\bar{f}(s) + V_0$$

$$\bar{x}(s) = \left(\frac{\bar{f}(s)}{m} + V_0\right)\left(\frac{1}{s^2 + 2\lambda s + \omega_0^2}\right) = \left(\frac{\bar{f}(s)}{m} + V_0\right)\left(\frac{1}{(s+\lambda)^2 + (\omega_0^2 - \lambda^2)}\right)$$

$$= \frac{\bar{f}(s)}{m}\left(\frac{1}{(s+\lambda^2)^2 + k^2}\right) + \frac{V_0}{(s+\lambda)^2 + k^2} \qquad \text{where } k^2 = (\omega_0^2 - \lambda^2)$$

how to invert

$$x(t) = \left(\frac{1}{m}\right)\mathcal{L}^{-1}\left\{\left(\bar{f}(s)\right)\left(\frac{1}{(s+\lambda)^2 + k^2}\right)\right\} + V_0\,\mathcal{L}^{-1}\left\{\frac{1}{(s+\lambda)^2 + k^2}\right\}$$

$$= \left(\frac{1}{m}\right)\mathcal{L}^{-1}\left\{\bar{f}(s)\,\bar{g}(s)\right\} + V_0\,\mathcal{L}^{-1}\left\{\bar{g}(s)\right\} \qquad \text{where } \bar{g}(s) = \frac{1}{(s+\lambda)^2 + k^2}$$

$$\mathcal{L}^{-1}\left\{\bar{g}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+\lambda)^2 + k^2}\right\}$$

$$g(t) = e^{-\lambda t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\}$$

$$g(t) = e^{-\lambda t}\left(\frac{1}{k}\right)\sin(kt)$$

$$\therefore X(t) = \frac{1}{mk}\int_0^t f(\tau)\, e^{-\lambda(t-\tau)} \sin(k(t-\tau))\, d\tau$$

$$+ \frac{V_0}{k} e^{-\lambda t} \sin(kt) \qquad \text{where } k^2 = (\omega_0^2 - \lambda^2)$$

$$X(t) = \frac{e^{-\lambda t}}{mk}\int_0^t f(\tau)\, e^{\lambda \tau} \sin(k(t-\tau))\, d\tau + \frac{V_0}{k}\sin(kt)$$

$$X(t) = X_1(t) + X_2(t) \qquad \text{where } X_2(t) = \frac{V_0}{k}\sin(kt)$$

(a) $f(t) = P\,\delta(t)$ let's just work with the convolution part and add the second component later

$$X_1(t) = \frac{e^{-\lambda t}}{mk}\int_0^t P\delta(\tau)\, e^{\lambda\tau} \sin(k(t-\tau))\, d\tau$$

$$= \frac{P}{mk} e^{-\lambda t} \sin(kt)$$

$$\therefore X(t) = \frac{P}{mk} e^{-\lambda t} \sin(kt) + \frac{V_0}{k}\sin(kt)$$

(b) $f(t) = \sin(\omega t)$

$$X_1(t) = \frac{e^{-\lambda t}}{mk}\int_0^t e^{\lambda\tau} \sin(\omega\tau)\sin(k(t-\tau))\, d\tau \qquad \text{let's use a trig. identity}$$

$$= \frac{e^{-\lambda t}}{mk}\int_0^t e^{\lambda\tau}\sin(\omega\tau)\left(\sin(kt)\cos(k\tau) - \cos(kt)\sin(k\tau)\right) d\tau$$

$$= \frac{e^{-\lambda t}}{mk}\left[\sin(kt)\int_0^t e^{\lambda\tau}\sin(\omega\tau)\cos(k\tau)\, d\tau - \cos(kt)\int_0^t e^{\lambda\tau}\sin(\omega\tau)\sin(k\tau)\, d\tau\right]$$

now $\sin((\omega-k)\tau) = \sin(\omega\tau)\cos(k\tau) - \cos(\omega\tau)\sin(k\tau)$

and $\sin((\omega+k)\tau) = \sin(\omega\tau)\cos(k\tau) + \cos(\omega\tau)\sin(k\tau)$

let's add

$$\sin((\omega-k)\tau) + \sin((\omega+k)\tau) = 2\sin(\omega\tau)\cos(k\tau)$$

$$\therefore \frac{1}{2}\left(\sin((\omega-k)\tau) + \sin((\omega+k)\tau)\right) = \sin(\omega\tau)\cos(k\tau)$$

we can do the same for the second integral

$$\cos((\omega-k)\tau) = \cos(\omega\tau)\cos(k\tau) + \sin(\omega\tau)\sin(k\tau)$$
$$\underline{\cos((\omega+k)\tau) = \cos(\omega\tau)\cos(k\tau) - \sin(\omega\tau)\sin(k\tau)}$$

let's subtract

$$\cos((\omega-k)\tau) - \cos((\omega+k)\tau) = 2\sin(\omega\tau)\sin(k\tau)$$

$$\therefore \tfrac{1}{2}\left(\cos((\omega-k)\tau) - \cos((\omega+k)\tau)\right) = \sin(\omega\tau)\sin(k\tau)$$

Thus

$$X_1(t) = \frac{e^{-\lambda t}}{mK}\Big[\sin(kt)\int_0^t e^{\lambda\tau}\left(\tfrac{1}{2}\right)\left(\sin((\omega-k)\tau) + \sin((\omega+k)\tau)\right)d\tau$$
$$-\cos(kt)\int_0^t e^{\lambda\tau}\left(\tfrac{1}{2}\right)\left(\cos((\omega-k)\tau) - \cos((\omega+k)\tau)\right)d\tau\Big]$$

we have integrals of the form

$$\int_0^t e^{at'}\sin(bt')\,dt' \quad \text{and} \quad \int_0^t e^{at'}\cos(bt')\,dt'$$

these can be integrated by parts.

$$\int_0^t e^{at'}\sin(bt')\,dt' = \int_0^t \sin(bt')\,d\left(\tfrac{1}{a}e^{at'}\right) = \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \frac{b}{a}\int_0^t e^{at'}\cos(bt')\,dt'$$
$$= \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \frac{b}{a}\int_0^t \cos(bt')\,d\left(\tfrac{1}{a}e^{at'}\right)$$
$$= \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \frac{b}{a}\left(\tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \int_0^t \frac{b}{a}\sin(bt')e^{at'}\,dt'\right)$$
$$= \tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \frac{b}{a^2}e^{at'}\cos(bt')\Big|_0^t - \left(\frac{b}{a}\right)^2\int_0^t \sin(bt')e^{at'}\,dt'$$

$$\left(1+\left(\frac{b}{a}\right)^2\right)\int_0^t e^{at'}\sin(bt')\,dt' = \frac{1}{a}e^{at}\sin(bt) - \frac{b}{a^2}e^{at}\cos(bt) + \frac{b}{a^2}$$

$$\therefore \int_0^t e^{at'}\sin(bt')\,dt' = \frac{a^2}{a^2+b^2}e^{at}\left(\frac{1}{a}\sin(bt) + \frac{b}{a^2}\left(e^{-at} - \cos(bt)\right)\right)$$

$$= \frac{e^{at}}{a^2+b^2}\left(a\sin(bt) + b\left(e^{-at} - \cos(bt)\right)\right)$$

$$\int_0^t e^{at'}\cos(bt')\,dt' = \int_0^t \cos(bt')\,d\left(\tfrac{1}{a}e^{at'}\right) = \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a}\int_0^t e^{at'}\sin(bt')\,dt'$$

$$= \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a}\int_0^t \sin(bt')\,d\left(\tfrac{1}{a}e^{at'}\right)$$

$$= \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a}\left(\tfrac{1}{a}e^{at'}\sin(bt')\Big|_0^t - \int_0^t \tfrac{b}{a}e^{at'}\cos(bt')\,dt'\right)$$

$$= \tfrac{1}{a}e^{at'}\cos(bt')\Big|_0^t + \tfrac{b}{a^2}e^{at'}\sin(bt')\Big|_0^t - \left(\tfrac{b}{a}\right)^2\int_0^t \cos(bt')\,dt'$$

$$\left(1+\left(\tfrac{b}{a}\right)^2\right)\int_0^t e^{at'}\cos(bt')\,dt' = \tfrac{1}{a}e^{at}\cos(bt) - \tfrac{1}{a} + \tfrac{b}{a^2}e^{at}\sin(bt)$$

$$\therefore \int_0^t e^{at'}\cos(bt')\,dt' = \frac{a^2}{a^2+b^2}e^{at}\left(\tfrac{1}{a}\cos(bt) - \frac{e^{-at}}{a} + \tfrac{b}{a^2}\sin(bt)\right)$$

$$= \frac{e^{at}}{a^2+b^2}\left(b\sin(bt) - a\left(e^{-at} - \cos(bt)\right)\right)$$

we have four integrals $a=\lambda$ and b is either $(\omega-k)$ or $(\omega+k)$.

$$X_1(t) = \frac{e^{-\lambda t}}{2mk}\Big[\sin(kt)\int_0^t e^{\lambda\tau}\left(\sin((\omega-k)\tau)+\sin((\omega+k)\tau)\right)d\tau$$

$X(t) = X_1(t) + X_2(t)$

$$-\cos(kt)\int_0^t e^{\lambda\tau}\left(\cos((\omega-k)\tau)-\cos((\omega+k)\tau)\right)d\tau\Big]$$ how to substitute

$$X_1(t) = \left(\frac{\sin(kt)}{2mk}\right)\left[\frac{a}{a^2+(\omega-k)^2}\sin((\omega-k)t) + \frac{(\omega-k)}{a^2+(\omega-k)^2}\left(e^{-\lambda t} - \cos((\omega-k)t)\right)\right]$$

$$+\left(\frac{\sin(kt)}{2mk}\right)\left[\frac{a}{a^2+(\omega+k)^2}\sin((\omega+k)t) + \frac{(\omega+k)}{a^2+(\omega+k)^2}\left(e^{-\lambda t} - \cos((\omega+k)t)\right)\right]$$

$$-\left(\frac{\cos(kt)}{2mk}\right)\left[\frac{(\omega-k)}{a^2+(\omega-k)^2}\sin((\omega-k)t) - \frac{a}{a^2+(\omega-k)^2}\left(e^{-\lambda t} - \cos((\omega-k)t)\right)\right]$$

$$+\left(\frac{\cos(kt)}{2mk}\right)\left[\frac{(\omega+k)}{a^2+(\omega+k)^2}\sin((\omega+k)t) - \frac{a}{a^2+(\omega+k)^2}\left(e^{-\lambda t} - \cos((\omega+k)t)\right)\right]$$

$$+\frac{v_0}{k}e^{-\lambda t}\sin(kt)$$

where $a=\lambda$, $k^2=(\omega_0^2-\lambda^2)$

Problem 5.) Heat Conduction Equation ($0 < x < l$ and $t > 0$):

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = a e^{-\alpha t} \quad \text{with I.C.: } u(x,0) = 0 \text{ and B.C.: } u(0,t) = u(l,t) = 0$$

where k, a, and α are constants.

Take Laplace Transform of the equation and B.C.

$$s\bar{u}(x,s) - u(x,0) - k\frac{d^2\bar{u}}{dx^2} = a\left(\frac{1}{s+\alpha}\right) \qquad \text{now } u(x,0) = 0$$

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u}(x,s) = -\frac{a}{k}\left(\frac{1}{s+\alpha}\right)$$

$$\bar{u}_H(x,s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x}$$

now to find the particular solution let $\bar{u}_p = C$

$$-\frac{s}{k}C = -\frac{a}{k}\left(\frac{1}{s+\alpha}\right) \Rightarrow C = \frac{a}{s}\left(\frac{1}{s+\alpha}\right)$$

$$\therefore \bar{u}(x,s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right)$$

Let's transform the B.C. $\mathcal{L}\{u(0,t)\} = \bar{u}(0,s) = 0$

$\mathcal{L}\{u(l,t)\} = \bar{u}(l,s) = 0$

Thus $\bar{u}(0,s) = A(1) + B(1) + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0 \Rightarrow A = -\frac{a}{s}\left(\frac{1}{s+\alpha}\right) - B$

$$\bar{u}(l,s) = Ae^{\sqrt{\frac{s}{k}}l} + Be^{-\sqrt{\frac{s}{k}}l} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0$$

$$\Rightarrow \left(-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - B\right)e^{\sqrt{\frac{s}{k}}l} + Be^{-\sqrt{\frac{s}{k}}l} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0$$

$$-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{\sqrt{\frac{s}{k}}l} - Be^{\sqrt{\frac{s}{k}}l} + Be^{-\sqrt{\frac{s}{k}}l} + \frac{a}{s}\left(\frac{1}{s+\alpha}\right) = 0$$

$$-B\left(e^{\sqrt{\frac{s}{k}}l} - e^{-\sqrt{\frac{s}{k}}l}\right) = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{\sqrt{\frac{s}{k}}l}\right)$$

$$B = \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{\sqrt{\frac{s}{k}}l}\right)\left(\frac{1}{e^{\sqrt{\frac{s}{k}}l} - e^{-\sqrt{\frac{s}{k}}l}}\right) = \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{\sqrt{\frac{s}{k}}l}\right)\left(\frac{e^{-\sqrt{\frac{s}{k}}l}}{1 - e^{-2\sqrt{\frac{s}{k}}l}}\right)$$

$$B = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{-\sqrt{\frac{s}{k}}l}\right)\left(\frac{1}{1 - e^{-2\sqrt{\frac{s}{k}}l}}\right) = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1 + e^{-\sqrt{\frac{s}{k}}l}}\right)$$

$$A = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - B = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)$$

$$U(x,s) = \left(-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)\right)e^{\sqrt{\frac{s}{k}}\,x} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)e^{-\sqrt{\frac{s}{k}}\,x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - \frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)e^{\sqrt{\frac{s}{k}}\,x} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)e^{-\sqrt{\frac{s}{k}}\,x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1+e^{-\sqrt{\frac{s}{k}}\,l} - 1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)e^{\sqrt{\frac{s}{k}}\,x} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)e^{-\sqrt{\frac{s}{k}}\,x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)e^{-\sqrt{\frac{s}{k}}(l-x)} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{\frac{s}{k}}\,l}}\right)e^{-\sqrt{\frac{s}{k}}\,x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}(l-x)}\left(1 - e^{-\sqrt{\frac{s}{k}}\,l} + \left(e^{-\sqrt{\frac{s}{k}}\,l}\right)^2 - \left(e^{-\sqrt{\frac{s}{k}}\,l}\right)^3 + \cdots\right)$$

$$- \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}\,x}\left(1 - e^{-\sqrt{\frac{s}{k}}\,l} + \left(e^{-\sqrt{\frac{s}{k}}\,l}\right)^2 - \left(e^{-\sqrt{\frac{s}{k}}\,l}\right)^3 + \cdots\right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}(l-x)}\sum_{n=0}^{\infty}(-1)^n e^{-n\sqrt{\frac{s}{k}}\,l} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)e^{-\sqrt{\frac{s}{k}}\,x}\sum_{n=0}^{\infty}(-1)^n e^{-n\sqrt{\frac{s}{k}}\,l}$$

$$= \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{\frac{s}{k}}((n+1)l-x)} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{\frac{s}{k}}(nl+x)}$$

let $\beta = \frac{((n+1)l-x)}{\sqrt{k}}$ and $\epsilon = \frac{(nl+x)}{\sqrt{k}}$

$$= \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\beta\sqrt{s}} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\sum_{n=0}^{\infty}(-1)^n e^{-\epsilon\sqrt{s}}$$

now to find the inverse Laplace Transform

$$\mathcal{L}\left\{\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\right\} = \frac{a}{\alpha}\left(1-e^{-\alpha t}\right) \qquad \mathcal{L}\left\{\left(\frac{1}{s+\alpha}\right)\right\} = e^{-\alpha t}$$

$$\mathcal{L}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right) \qquad \mathcal{L}\left\{f_1(s)\,f_2(s)\right\} = \int_0^t F_1(t-\tau)\,F_2(\tau)\,d\tau$$

$$U(x,t) = \mathcal{L}\left\{\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\right\} - a\sum_{n=0}^{\infty}(-1)^n\,\mathcal{L}\left\{\left(\frac{1}{s+\alpha}\right)\left(\frac{e^{-\beta\sqrt{s}}}{s}\right)\right\} - a\sum_{n=0}^{\infty}(-1)^n\,\mathcal{L}\left\{\left(\frac{1}{s+\alpha}\right)\left(\frac{e^{-\epsilon\sqrt{s}}}{s}\right)\right\}$$

$$= \frac{a}{\alpha}\left(1-e^{-\alpha t}\right) - a\sum_{n=0}^{\infty}(-1)^n\int_0^t e^{-\alpha(t-\tau)}\operatorname{erfc}\left(\frac{\beta}{2\sqrt{\tau}}\right)d\tau$$

$$- a\sum_{n=0}^{\infty}(-1)^n\int_0^t e^{-\alpha(t-\tau)}\operatorname{erfc}\left(\frac{\epsilon}{2\sqrt{\tau}}\right)d\tau$$

$$= \frac{a}{\alpha}\left(1-e^{-\alpha t}\right) - a e^{-\alpha t}\sum_{n=0}^{\infty}(-1)^n\int_0^t e^{\alpha\tau}\left(\operatorname{erfc}\left(\frac{\beta}{2\sqrt{\tau}}\right) + \operatorname{erfc}\left(\frac{\epsilon}{2\sqrt{\tau}}\right)\right)d\tau$$

Alternative solution

$$B = \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(1 - e^{\sqrt{\frac{s}{k}}\,l}\right)\left(\frac{1}{e^{\sqrt{\frac{s}{k}}\,l} - e^{-\sqrt{s/k}\,l}}\right) = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)$$

$$A = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - B = -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)$$

$$U(x,s) = \left(-\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)\right)e^{\sqrt{\frac{s}{k}}\,x} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)e^{-\sqrt{\frac{s}{k}}\,x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= -\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)e^{-\sqrt{s/k}\,(l-x)} - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right)e^{-\sqrt{s/k}\,x} + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

now $\left(\frac{1}{1+e^{-\sqrt{s/k}\,l}}\right) = \frac{e^{\sqrt{\frac{s}{k}}\frac{l}{2}}}{e^{\sqrt{s/k}\,l/2} + e^{-\sqrt{s/k}\,l/2}} = \frac{e^{\sqrt{s/k}\,l/2}}{\cosh(\sqrt{s/k}\,l/2)}$

$$= \left(-a/s\right)\left(\frac{1}{s+\alpha}\right)\left[\frac{e^{-\sqrt{s/k}\,(l-x-l/2)}}{\cosh(\sqrt{s/k}\,l/2)} + \frac{e^{-\sqrt{s/k}\,(x-l/2)}}{\cosh(\sqrt{s/k}\,l/2)}\right] + \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left[\frac{e^{\sqrt{s/k}\,(x-l/2)} + e^{-\sqrt{s/k}\,(x-l/2)}}{\cosh(\sqrt{s/k}\,l/2)}\right]$$

$$= \left(a/s\right)\left(\frac{1}{s+\alpha}\right) - \left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left[\frac{\cosh(\sqrt{s/k}\,(l/2-x))}{\cosh(\sqrt{s/k}\,l/2)}\right]$$

now to find the inverse Laplace transforms

$$U(x,t) = \mathcal{L}\left\{\left(a/s\right)\left(\frac{1}{s+\alpha}\right)\right\} - \mathcal{L}\left\{\left(\frac{a}{s}\right)\left(\frac{1}{s+\alpha}\right)\left[\frac{\cosh(\sqrt{\frac{s}{k}}\,(l/2-x))}{\cosh(\sqrt{s/k}\,l/2)}\right]\right\}$$

Use Bromwich inversion theorem $\to$ Residues

$$= a\,\mathrm{Res}\left(\frac{e^{st}}{s(s+\alpha)}\right) - a\,\mathrm{Res}\left(\frac{e^{st}}{s(s+\alpha)}\left[\frac{\cosh(\sqrt{s/k}\,(l/2-x))}{\cosh(\sqrt{s/k}\,l/2)}\right]\right)$$

roots are 0 and $-\alpha$ roots are $0, -\alpha,$

$\cosh(a\sqrt{s}) = \cos(ia\sqrt{s})$ so we are looking for the poles of $\cos(ia\sqrt{s})$

$ia\sqrt{s} = \pm n/2\,\pi \Rightarrow \sqrt{s} \pm \frac{n\pi}{2ai}$ $n = 1,3,5,\ldots$

or $\sqrt{s} = \frac{\pm(2n+1)\pi}{2ai}$ $n = 0,1,2,\ldots$

$s = -\left(\frac{(2n+1)\pi}{2a}\right)^2$ $n = 0,1,2,\ldots$