

- 1. (5pts) Use the Laplace transform to solve the following problem:
 $y'' - 4y' + 4y = t^3 e^{2t}$ with initial conditions; $y(0) = 0$ and $y'(0) = 0$.
- 2. (6pts) Use the Laplace transform to solve the following problem:
 $y'' - 4y' + 3y = 1 - H(t-2) - H(t-4) + H(t-6)$, with initial conditions; $y(0) = 0$ and $y'(0) = 0$.
- 3. (6pts) Use the Laplace transform to solve the following problem:
 $y'' - 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$ with initial conditions; $y(0) = 1$ and $y'(0) = 0$.
- 4. (6pts) Solve the boundary valued problem using the Laplace Transform:
$$t \frac{d^2y}{dt^2} - (t+3) \frac{dy}{dt} + 4y = t-1 \quad \text{where: } y(0) = y(1) = 0.$$
- 5. (7pts) Solve the following system of equations for the unknown functions, $y(t)$ and $z(t)$:

$$3y' + 8y + 2z' + 5z = e^{-t}$$

$$y' + z' + z = 0$$

where the initial conditions are $y(0) = 2$ and $z(0) = -2$.

- 6. (8pts) page 148, prob. 12: Solve for the currents in the circuit of Figure 3.37, assuming that the currents and charges are initially zero and that $E(t) = 2H(t-4) - H(t-5)$.
- 7. (8pts) page 149, prob. 15: Solve for the displacement functions in the system of Figure 3.38 if $f_1(t) = 1 - H(t-2)$ and $f_2(t) = 0$. Assume zero initial displacements and velocities.

1.) Use the Laplace Transform to solve the following problem:

$$y'' - 4y' + 4y = t^3 e^{2t} \quad \text{with the initial conditions } y(0) = 0, y'(0) = 0.$$

$$y'' - 4y' + 4y = t^3 e^{2t} \Rightarrow \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3 e^{2t}\}$$

$$(s^2 \bar{y} - s y(0) - y'(0)) - 4(s \bar{y} - y(0)) + 4\bar{y} = \mathcal{L}\{t^3 e^{2t}\}$$

From Table 3.1 $\mathcal{L}\{t^n f(t)\} = (-1)^n f^{(n)}(s)$ $\therefore \mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$
 $\therefore \mathcal{L}\{t^3 e^{2t}\} = (-1)^3 \frac{d^3(s-2)^{-1}}{ds^3} = \frac{6}{(s-2)^4}$

$$(s^2 - 4s + 4) \bar{y}(s) = \frac{6}{(s-2)^4} \Rightarrow \bar{y}(s) = \frac{6}{(s-2)^4 (s-2)^2} = \bar{y}(s) = \frac{6}{(s-2)^6}$$

$$y(t) = \mathcal{L}\{\bar{y}(s)\} = 6 \mathcal{L}\left\{\frac{1}{(s-2)^6}\right\} = 6 e^{2t} \mathcal{L}\left\{\frac{1}{s^6}\right\}$$

now $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
 $\therefore \frac{1}{s^{n+1}} = \frac{1}{n!} t^n$

$$\begin{aligned} y(t) &= 6 e^{2t} \left(\frac{t^5}{5!}\right) = \frac{e^{2t} t^5}{20} \\ &= \left(\frac{t^5}{20}\right) e^{2t} \end{aligned}$$

2) Use the Laplace Transform to solve the following problem:

$$y'' - 4y + 3y = 1 - H(t-2) - H(t-4) + H(t-6) \text{ where } y(0)=0, y'(0)=0$$

$$y'' - 4y + 3y = 1 - H(t-2) - H(t-4) + H(t-6)$$

$$(s^2 - 4s + 3)y = \frac{1}{s} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s} + \frac{1}{s}e^{-6s}$$

$$(s^2 - 4s + 3)y = \frac{1}{s} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s} + \frac{1}{s}e^{-6s}$$

$$\widehat{y}(s) = \frac{1}{s(s-1)(s-3)} - \frac{e^{-2s}}{s(s-1)(s-3)} - \frac{e^{-4s}}{s(s-1)(s-3)} + \frac{e^{-6s}}{s(s-1)(s-3)}$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{s}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{s-1}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{s-3}\right) - \left(\frac{1}{3}\right)\left(\frac{e^{-2s}}{s}\right) + \left(\frac{1}{2}\right)\left(\frac{e^{-2s}}{s-1}\right) - \left(\frac{1}{6}\right)\left(\frac{e^{-2s}}{s-3}\right) \\ + \left(\frac{1}{3}\right)\left(\frac{e^{-4s}}{s}\right) + \left(\frac{1}{2}\right)\left(\frac{e^{-4s}}{s-1}\right) - \left(\frac{1}{6}\right)\left(\frac{e^{-4s}}{s-3}\right) - \left(\frac{1}{3}\right)\left(\frac{e^{-6s}}{s}\right) + \left(\frac{1}{2}\right)\left(\frac{e^{-6s}}{s-1}\right) - \left(\frac{1}{6}\right)\left(\frac{e^{-6s}}{s-3}\right)$$

$$y(t) = \frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} - \left\{ \left[\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\right) e^{-2s} - \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\right) e^{-4s} \right. \right. \\ \left. \left. + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\right) e^{-6s} \right] \right\}$$

$$= \frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} - \left(\frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} \right) \Big|_{t \rightarrow t-2} H(t-2) - \left(\frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} \right) \Big|_{t \rightarrow t-4} H(t-4)$$

$$+ \left(\frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} \right) \Big|_{t \rightarrow t-6} H(t-6)$$

$$y(t) = \frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} - \left(\frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} \right) \Big|_{t \rightarrow t-2} H(t-2) - \left(\frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} \right) \Big|_{t \rightarrow t-4} H(t-4)$$

$$+ \left(\frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6} \right) \Big|_{t \rightarrow t-6} H(t-6)$$

- 3.) Use the Laplace Transform to solve the following problem:
 $y'' - 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$ where $y(0) = 1$, $y'(0) = 0$.

$$y'' - 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$$

$$(s^2 \tilde{y}(s) - s\tilde{y}(0) - \tilde{y}'(0)) - 4(s\tilde{y}(s) - \tilde{y}'(0)) + 13\tilde{y}(s) = e^{\pi s} + e^{3\pi s}$$

$$(s^2 - 4s + 13)\tilde{y}(s) - s + 4 = e^{\pi s} + e^{3\pi s}$$

$$\begin{aligned}\tilde{y}(s) &= \frac{s-4}{s^2 - 4s + 13} + \frac{e^{\pi s}}{s^2 - 4s + 13} + \frac{e^{3\pi s}}{s^2 - 4s + 13} \\ &= \frac{s-4}{(s-2)^2 + 3^2} + \frac{e^{\pi s}}{(s-2)^2 + 3^2} + \frac{e^{3\pi s}}{(s-2)^2 + 3^2} \\ &= \frac{s-2}{(s-2)^2 + 3^2} - \frac{2}{(s-2)^2 + 3^2} + \frac{e^{\pi s}}{(s-2)^2 + 3^2} + \frac{e^{3\pi s}}{(s-2)^2 + 3^2} \\ &= e^{2t} \left[L^{-1} \left\{ \frac{s}{s^2 + 3^2} \right\} - 2e^{2t} L^{-1} \left\{ \frac{1}{s^2 + 3^2} \right\} + (e^{2t} L^{-1} \left\{ \frac{1}{s^2 + 3^2} \right\}) \Big|_{t \rightarrow t-\pi} H(t-\pi) \right.\end{aligned}$$

we know

$$L^{-1} \left\{ \frac{s}{s^2 + 3^2} \right\} = \cos(3t)$$

$$L^{-1} \left\{ \frac{1}{s^2 + 3^2} \right\} = \frac{1}{3} \sin(3t)$$

$$\left. + (e^{2t} L^{-1} \left\{ \frac{1}{s^2 + 3^2} \right\}) \Big|_{t \rightarrow t-3\pi} H(t-3\pi) \right]$$

$$\text{Thus } y(t) = e^{2t} \cos(3t) - \frac{2}{3} e^{2t} \sin(3t) + \frac{e^{2(t-\pi)}}{3} \sin(3(t-\pi)) H(t-\pi)$$

$$+ \frac{e^{2(t-3\pi)}}{3} \sin(3(t-3\pi)) H(t-3\pi)$$

4). Solve the boundary valued problem

$$t \frac{d^2y}{dt^2} - (t+3) \frac{dy}{dt} + 4y = t-1 \quad \text{where: } y(0) = y(1) = 0$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}(s^2 f(s)) = -\frac{d}{ds}(f(s))$$

$$\mathcal{L}\{t \frac{d^2y}{dt^2}\} = -\frac{d}{ds}(s^2 \tilde{y}(s) - s y(0) - y'(0)) \\ = -2s \tilde{y}(s) - s^2 \frac{d\tilde{y}(s)}{ds}$$

$$\mathcal{L}\{t \frac{dy}{dt}\} = -\frac{d}{ds}(s \tilde{y}(s) - y(0)) = -\tilde{y}(s) - s \frac{d\tilde{y}(s)}{ds}$$

$$-2s \tilde{y}(s) - s^2 \frac{d\tilde{y}(s)}{ds} + \tilde{y}(s) + s \frac{d\tilde{y}(s)}{ds} = 3(s \tilde{y}(s) - y(0)) + 4\tilde{y}(s) = \frac{1}{s^2} - \frac{1}{s}$$

$$(-s^2 + s) \frac{d\tilde{y}(s)}{ds} - (2s + 3s) \tilde{y}(s) + 5\tilde{y}(s) = \frac{1}{s^2} - \frac{1}{s}$$

$$(-s^2 + s) \frac{d\tilde{y}(s)}{ds} - (5s - 5) \tilde{y}(s) = \frac{1}{s^2} - \frac{1}{s} \\ -s(s-1) \frac{d\tilde{y}(s)}{ds} - 5(s-1) \tilde{y}(s) = -\frac{(s-1)}{s^2} \Rightarrow \frac{d\tilde{y}(s)}{ds} - \frac{s(s-1)}{(-s)(s-1)} \tilde{y}(s) = \frac{-(s-1)}{(s^2)(-s)(s-1)}$$

$$\frac{d\tilde{y}(s)}{ds} + \frac{5}{s} \tilde{y}(s) = \frac{1}{s^3} \quad \text{First order Eq. Integrating factor, } e^{\int \frac{5}{s} ds} = e^{5 \ln(s)} = s^5$$

$$\int d(s^5 \tilde{y}(s)) = \int (s^5/s^3) ds \Rightarrow s^5 \tilde{y}(s) = \frac{1}{3} s^3 + C$$

$$\tilde{y}(s) = \frac{1}{3} s^2 + \frac{C}{s^5}$$

$$\mathcal{L}^{-1}\{\tilde{y}(s)\} = y(t) = \frac{t}{3} + \frac{t^4}{24} C \quad \text{what is } C?$$

$$y(1) = 0 = \frac{1}{3} + \frac{1}{24} C \Rightarrow C = -\frac{24}{3} = -8$$

$$y(t) = \frac{t}{3} - \frac{t^4}{3}$$

5) Solve the system of equations for the unknown functions $y(t)$ and $z(t)$.

$$3y' + 8y + 2z' + 5z = e^{-t}$$

$$y' + z' + z = 0$$

$$y(0) = 2 \text{ and } z(0) = -2$$

$$\begin{aligned} 3(s\bar{y}(s) - y(0)) + 8\bar{y}(s) + 2(s\bar{z}(s) - z(0)) + 5\bar{z}(s) &= \frac{1}{s+1} \\ s\bar{y}(s) - y(0) + s\bar{z}(s) - z(0) + \bar{z}(s) &= 0 \end{aligned}$$

$$(3s+8)\bar{y}(s) + (2s+5)\bar{z}(s) = \frac{1}{s+1} + 2$$

$$s\bar{y}(s) + (s+1)\bar{z}(s) = 0 \Rightarrow \bar{z}(s) = -\frac{s}{s+1}\bar{y}(s)$$

$$\therefore (3s+8)\bar{y}(s) = \frac{(2s+5)s}{s+1}\bar{y}(s) = \frac{1}{s+1} + 2$$

$$(3s+8)(s+1)\bar{y}(s) - (2s^2+5s)\bar{y}(s) = 1 + 2(s+1)$$

$$(3s^2+8s+3s+8 - 2s^2-5s)\bar{y}(s) = (s+2)(s+1)$$

$$(s^2+6s+8)\bar{y}(s) = 2s+3$$

↙ partial fractions

$$\bar{y}(s) = \frac{2s+3}{(s^2+6s+8)} = \frac{2s+3}{(s+2)(s+4)} = \frac{1}{s+4} + \frac{1}{s+2} - \frac{3}{(s+2)(s+4)} = \left(\frac{5}{2}\right)\left(\frac{1}{s+4}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{s+2}\right)$$

$$y(t) = \frac{5}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$

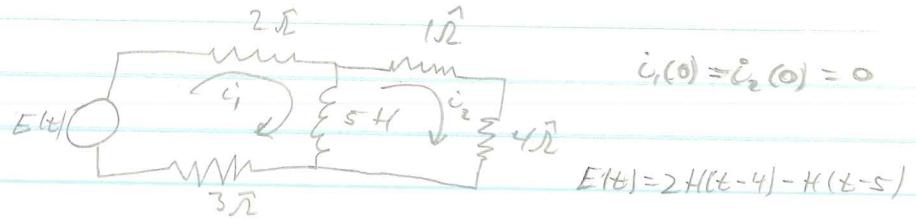
$$\begin{aligned} \bar{z}(s) &= \left(-\frac{s}{s+1}\right)\bar{y}(s) = \left(-\frac{s}{s+1}\right)\left(\frac{2s+3}{(s+2)(s+4)}\right) = -\left(\frac{s(s+1)+s(s+2)}{(s+1)(s+2)(s+4)}\right) \\ &= -\frac{s}{(s+2)(s+4)} - \frac{s}{(s+1)(s+4)} = \left(\frac{1}{3}\right)\left(\frac{1}{s+4}\right) + \left(\frac{1}{s+2}\right) - \left(\frac{10}{3}\right)\left(\frac{1}{s+4}\right) \end{aligned}$$

$$z(t) = \frac{1}{3}e^{-4t} + e^{-2t} - \left(\frac{10}{3}\right)e^{-4t}$$

In summary: $y(t) = \frac{1}{2}e^{-2t} + \frac{5}{2}e^{-4t}$

$$z(t) = \frac{1}{3}e^{-4t} + e^{-2t} - \left(\frac{10}{3}\right)e^{-4t}$$

6) page 148, prob. 12



$$1^{\text{st}} \text{ loop} \quad 2\bar{i}_1(s) + 5 \frac{d}{dt}(i_1(t) - i_2(t)) + 3i_1(t) = E(t) = 2H(t-4) - H(t-5)$$

$$2^{\text{nd}} \text{ loop} \quad i_2(t) + 4i_2(t) + 5 \frac{d}{dt}(i_2(t) - i_1(t)) = 0$$

Take the Laplace Transform of both loop equations

$$1^{\text{st}} \quad 2\bar{i}_1(s) + 5(s\bar{i}_1(s) - i_1(0)) - 5(s\bar{i}_2(s) - i_2(0)) + 3\bar{i}_1(s) = \left(\frac{2}{s}\right)\bar{e}^{-4s} - \left(\frac{1}{s}\right)\bar{e}^{-5s}$$

$$2^{\text{nd}} \quad 5\bar{i}_2(s) + 5(s\bar{i}_2(s) - i_2(0)) - 5(s\bar{i}_1(s) - i_1(0)) = 0$$

$$1^{\text{st}} \text{ loop: } 5(s+1)\bar{i}_1(s) - 5s\bar{i}_2(s) = 2\frac{e^{-4s}}{s} - \left(\frac{1}{s}\right)\bar{e}^{-5s}$$

$$2^{\text{nd}} \text{ loop: } 5(s+1)\bar{i}_2(s) - 5s\bar{i}_1(s) = 0 \Rightarrow \bar{i}_1(s) = \frac{(s+1)}{s} \bar{i}_2(s)$$

$$5(s+1)\bar{i}_1(s) - 5s\bar{i}_2(s) = 2\frac{e^{-4s}}{s} - \left(\frac{1}{s}\right)\bar{e}^{-5s}$$

$$5(s+1)\left(\frac{s+1}{s}\bar{i}_2(s) - 5s\bar{i}_2(s)\right) =$$

$$5\left(\frac{(s+1)^2}{s} - \frac{5s^2}{s}\right)\bar{i}_2(s) =$$

$$5\left(\frac{2s+1}{s}\right)\bar{i}_2(s) =$$

$$\bar{i}_2(s) = \left(\frac{1}{s}\right)\left(\frac{s}{2s+1}\right)\left(2\frac{e^{-4s}}{s} - \left(\frac{1}{s}\right)\bar{e}^{-5s}\right)$$

$$= \left(\frac{1}{s}\right)\left(\frac{1}{2s+1}\right)\left(2e^{-4s} - e^{-5s}\right)$$

and

$$\bar{i}_1(s) = \left(\frac{1}{s}\right)\bar{i}_2(s) = \left(\frac{1}{s}\right)\left(\frac{s+1}{s(2s+1)}\right)\left(2e^{-4s} - e^{-5s}\right)$$

now to invert

$$\Rightarrow \mathcal{L}^{-1}\{\bar{i}_2(s) = \left(\frac{1}{s}\right)\left[\frac{1}{(s+1)}\bar{e}^{-4s} - \frac{1}{2}\left(\frac{1}{s+1}\right)\bar{e}^{-5s}\right]\}$$

$$i_2(t) = \left(\frac{1}{s}\right)\left[e^{-\frac{1}{2}t}\left[H(t-4) - \frac{1}{2}e^{-\frac{1}{2}t}\right]H(t-5)\right]$$

$$i_1(t) = \left(\frac{1}{s}\right)\left[e^{-\frac{1}{2}(t-4)}H(t-4) - \frac{1}{2}e^{-\frac{1}{2}(t-5)}H(t-5)\right]$$

$$\tilde{c}_1(s) = \left(\frac{1}{5}\right) \left[\left(\frac{1}{2s+1} + \frac{1}{s(2s+1)} \right) (2e^{-4s} - e^{-5s}) \right]$$

$$= \left(\frac{1}{5}\right) \left[\left(\frac{1}{2s+1} + \frac{1}{s} - \frac{2}{2s+1} \right) (2e^{-4s} - e^{-5s}) \right]$$

$$2As + A + Bs = 1$$

$$2A + B = 0 \quad B = -2A$$

$$A = 1$$

$$B = -2$$

$$\mathcal{L}^{-1}\{\tilde{c}_1(s)\} = \left(\frac{1}{5}\right) \left[\left(\frac{1}{s} - \left(\frac{1}{2}\right) \left(\frac{1}{s+1/2}\right) \right) k e^{-4s} - \left(\frac{1}{s} - \left(\frac{1}{2}\right) \left(\frac{1}{s+1/2}\right) \right) e^{-5s} \right] \}$$

$$\begin{aligned} i_1(t) &= \left(\frac{1}{5}\right) \left[\left(1 - \left(\frac{1}{2}\right) e^{-\frac{1}{2}(t-4)} \right) H(t-4) - \left(1 - \left(\frac{1}{2}\right) e^{-\frac{1}{2}(t-5)} \right) H(t-5) \right] \\ &= \frac{2}{5} \left(1 - \left(\frac{1}{2}\right) e^{-\frac{1}{2}(t-4)} \right) H(t-4) - \left(\frac{1}{5}\right) \left(1 - \left(\frac{1}{2}\right) e^{-\frac{1}{2}(t-5)} \right) H(t-5) \\ &= \frac{1}{5} \left(2 - e^{-\frac{1}{2}(t-4)} \right) H(t-4) - \left(\frac{1}{10}\right) \left(2 - e^{-\frac{1}{2}(t-5)} \right) H(t-5) \end{aligned}$$

Thus $i_1(t) = \frac{1}{5} \left(2 - e^{-\frac{1}{2}(t-4)} \right) H(t-4) - \frac{1}{10} \left(2 - e^{-\frac{1}{2}(t-5)} \right) H(t-5)$

$$i_1(t) = \frac{1}{5} e^{-\frac{1}{2}(t-4)} H(t-4) - \frac{1}{10} e^{-\frac{1}{2}(t-5)} H(t-5)$$

7) page 149, prob 15

Equations of motion

$$m_1 \frac{d^2x_1}{dt^2} + (k_1 + k_2)x_1 - k_2 x_2 = f_1(t)$$

$$m_2 \frac{d^2x_2}{dt^2} + (k_2 + k_3)x_2 - k_2 x_1 = f_2(t)$$



$$k_1 = 6$$

$$m_1 = 1 \quad f_1(t) = 1 - H(t-2)$$

$$k_2 = 2$$

$$m_2 = 1 \quad f_2(t) = 0$$

$$k_3 = 3$$

NEED

$$1^{\text{st}} \text{ Eq} \Rightarrow \frac{d^2x_1}{dt^2} + 8x_1 - 2x_2 = 1 - H(t-2)$$

$$2^{\text{nd}} \text{ Eq} \Rightarrow \frac{d^2x_2}{dt^2} + 5x_2 - 2x_1 = 0$$

Take the Laplace Transform of both equations

$$1^{\text{st}} \text{ Eq} \Rightarrow (s^2\bar{x}_1 - s\bar{x}_1(0) - \bar{x}_1'(0)) + 8\bar{x}_1(s) - 2\bar{x}_2(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$2^{\text{nd}} \text{ Eq} \Rightarrow (s^2\bar{x}_2 - s\bar{x}_2(0) - \bar{x}_2'(0)) + 5\bar{x}_2(s) - 2\bar{x}_1(s) = 0$$

$$(s^2 + 8)\bar{x}_1(s) - 2\bar{x}_2(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$(s^2 + 5)\bar{x}_2(s) - 2\bar{x}_1(s) = 0 \Rightarrow \bar{x}_1(s) = \left(\frac{s^2 + 5}{2}\right)\bar{x}_2(s)$$

$$(s^2 + 8)\left(\frac{s^2 + 5}{2}\right)\bar{x}_2(s) - 2\bar{x}_2(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$(s^2 + 8)(s^2 + 5) - 4 \bar{x}_2(s) = 2\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right)$$

$$\bar{x}_2(s) = \left(\frac{2}{s}\right)\left(\frac{1}{(s^2 + 8)(s^2 + 5) - 4}\right)(1 - e^{-2s})$$

$$\bar{x}_1(s) = \left(\frac{s^2 + 5}{2}\right)\bar{x}_2(s) = \left(\frac{s^2 + 5}{2}\right)\left(\frac{2}{s}\right)\left(\frac{1}{(s^2 + 8)(s^2 + 5) - 4}\right)(1 - e^{-2s})$$

$$= \left(\frac{1}{s}\right)\left(\frac{s^2 + 5}{(s^2 + 8)(s^2 + 5) - 4}\right)(1 - e^{-2s})$$

$$(s^2 + 8)(s^2 + 5) - 4 \Rightarrow s^4 + 13s^2 + 40 - 4 \Rightarrow s^4 + 13s^2 + 36$$

now to invert $\tilde{X}_2(s)$

$$\begin{aligned}\tilde{X}_2(s) &= \left(\frac{1}{5}\right)\left(\frac{2}{s^4+13s^2+36}\right)(1-e^{-2s}) \\ &= \left(\frac{2}{5}\right)\left(\frac{1}{(s^2+4)(s^2+9)}\right)(1-e^{-2s})\end{aligned}$$

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$As^3 + Bs^2 + 9A + 9B + Cs^3 + Ds^2 + 4Cs + 4D = 1$$

$$(A+C)=0, (B+D)=0; (9A+4C)=0; 9B+4D=1$$

$$A=-C; B=-D; 9A=-4C; 9B+4D=1$$

$$A=0, C=0$$

$$9B-4B=1 \Rightarrow B=\frac{1}{5}$$

$$D=-\frac{1}{5}$$

$$\tilde{X}_2(s) = \left(\frac{2}{5}\right)\left(\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4}\right) - \frac{1}{5}\left(\frac{1}{s^2+9}\right)\right)(1-e^{-2s})$$

$$= \left(\frac{2}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)(1-e^{-2s})$$

$$\frac{1}{5}\tilde{f}(s) = \int_0^t f(\tau) d\tau$$

$$\tilde{X}_2(s) = \left(\frac{2}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right) - \left(\frac{2}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)e^{-2s} \quad \left\{ \right.$$

$$X_2(t) = \frac{2}{5} \int_0^t \left(\frac{1}{2} \sin(2t) - \frac{1}{3} \sin(3t) - (\frac{1}{2} \sin(2(t-2)) - \frac{1}{3} \sin(3(t-2))) H(t-2) dt'$$

$$= \frac{2}{15} \int_0^t \left(\frac{3}{2} \sin(2t) - \sin(3t) \right) dt' - \frac{2}{15} \int_0^t \left(\frac{3}{2} \sin(2(t-2)) - \sin(3(t-2)) \right) dt' H(t-2)$$

$$= \frac{2}{15} \left(-\frac{3}{4} \cos(2t) + \frac{1}{3} \cos(3t) \right) \Big|_0^t - \frac{2}{15} \left(-\frac{3}{4} \cos(2(t-2)) + \frac{1}{3} \cos(3(t-2)) \right) \Big|_0^t H(t-2)$$

$$= \frac{2}{15} \left(-\frac{3}{4} \cos(2t) + \frac{1}{3} \cos(3t) + \frac{3}{4} - \frac{1}{3} \right) - \frac{2}{15} \left(-\frac{3}{4} \cos(2(t-2)) + \frac{1}{3} \cos(3(t-2)) + \frac{3}{4} - \frac{1}{3} \right) H(t-2)$$

$$= \frac{2}{15} \left(\frac{5}{12} - \frac{3}{4} \cos(2t) + \frac{1}{3} \cos(3t) \right) - \frac{2}{15} \left(\frac{5}{12} - \frac{3}{4} \cos(2(t-2)) + \frac{1}{3} \cos(3(t-2)) \right) H(t-2)$$

$$X_2(t) = \left(\frac{1}{18} - \frac{1}{10} \cos(2t) + \frac{2}{45} \cos(3t) \right) - \left(\frac{1}{18} - \frac{1}{10} \cos(2(t-2)) + \frac{2}{45} \cos(3(t-2)) \right) H(t-2)$$

now to invert $\tilde{X}_1(s)$

$$\tilde{X}_1(s) = \left(\frac{1}{5}\right)\left(\frac{s^2+5}{(s^2+4)(s^2+9)}\right)(1-e^{-2s})$$

$$\frac{s^2+5}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$As^3 + Bs^2 + 9A + 9B + Cs^3 + Ds^2 + 4Cs + 4D = s^2 + 5$$

$$(A+C)=0; (B+D)=1; (9A+4C)=0; 9B+4D=5$$

$$A=C=0; B+D=1 \Rightarrow B=1-D \quad 9B+4D=5$$

$$\begin{aligned}B &= 1-D \\ &= 1 - \frac{4}{5} = \frac{1}{5} \\ B &= \frac{1}{5}\end{aligned}$$

$$\begin{aligned}9-9D+4D &= 5 \\ -5D &= -4 \\ D &= \frac{4}{5}\end{aligned}$$

$$X_1(s) = \left(\frac{1}{5}\right)\left(\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{s^2+9}\right)\right)(1 - e^{-2s})$$

$$= \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} + \frac{4}{s^2+9}\right)(1 - e^{-2s})$$

$$\mathcal{L}^{-1}\{X_1(s)\} = \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} + \frac{4}{s^2+9}\right) - \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} + \frac{4}{s^2+9}\right)e^{-2s}$$

$$\begin{aligned} Y_1(t) &= \frac{1}{5} \int_0^t \left(\frac{1}{8} \sin(2t) + \left[\sin(3t) - \left(\frac{3}{8} \sin(2(t-2)) + \sin(3(t-2)) \right) H(t-2) \right] dt \right) dt \\ &= \frac{4}{15} \int_0^t \left(\frac{3}{8} \sin(2t) + \sin(3t) \right) dt - \frac{4}{15} \int_{t-2}^t \left(\frac{3}{8} \sin(2(t-2)) + \sin(3(t-2)) \right) dt H(t-2) \\ &= \frac{4}{15} \left(-\frac{3}{16} \cos(2t) - \frac{1}{3} \cos(3t) \right) \Big|_0^t - \frac{4}{15} \left(-\frac{3}{16} \cos(2(t-2)) - \frac{1}{3} \cos(3(t-2)) \right) \Big|_2^t H(t-2) \\ &= \frac{4}{15} \left(-\frac{3}{16} \cos(2t) - \frac{1}{3} \cos(3t) + \frac{3}{76} + \frac{1}{3} \right) - \frac{4}{15} \left(-\frac{3}{16} \cos(2(t-2)) - \frac{1}{3} \cos(3(t-2)) + \frac{3}{76} + \frac{1}{3} \right) H(t-2) \\ &= \frac{4}{15} \left(\frac{25}{48} - \frac{3}{16} \cos(2t) + \frac{1}{3} \cos(3t) \right) - \frac{4}{15} \left(\frac{25}{48} - \frac{3}{16} \cos(2(t-2)) - \frac{1}{3} \cos(3(t-2)) \right) H(t-2) \end{aligned}$$

$$Y_1(t) = \left(\frac{5}{36} - \frac{1}{10} \cos(2t) - \frac{4}{45} \cos(3t) \right) + \left(\frac{25}{72} + \frac{1}{20} \cos(2(t-2)) + \frac{4}{45} \cos(3(t-2)) \right) H(t-2)$$

and

$$Y_2(t) = \left(\frac{1}{18} - \frac{1}{10} \cos(2t) + \frac{2}{45} \cos(3t) \right) - \left(\frac{1}{18} - \frac{1}{10} \cos(2(t-2)) + \frac{2}{45} \cos(3(t-2)) \right) H(t-2)$$