Homework Set No. 3
Due September 27, 2013

NEEP 547
DLH

Nonhomogeneous Equation (Variation of Parameters)

1. (6pts) page 93, prob. 22: Find the general solution to the differential equation: $x^2\, y'' + 3x\, y' + y = 4/x$

2. (6pts) Using the variation of parameters show that

$$y = c_1 \cosh(kx) + c_2 \sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-s))\, f(s)\, ds$$

is a complete solution of the equation $y'' - k^2\, y = f(x)$, where $k \neq 0$ and f is everywhere continuous. *Hint:* Introduce the dummy variable s in the integrals which define u_1 and u_2. Then move $y_1(x)$ and $y_2(x)$ into the integrands of the respective integrals and combine the two integrals.

Reduction of Order

3. (6pts) page 72, prob. 8.: Verify that the given function is a solution of the differential equation. Derive the equation satisfied by $u(x)$, give its solution and give the general solution of the second order equation: $y'' - (2x/(1+x^2))\, y' + (2/(1+x^2))\, y = 0$; $y_1(x) = x$.

4. (6pts) Use the one solution indicated to find the complete solution: $(2x - x^2)\, y'' + 2\,(x-1)\, y' - 2\, y = 0$; $y_1(x) = x - 1$

Euler Equation

5. (6pts) page 81, prob. 20: $x^2\, y'' - 9x\, y' + 24\, y = 0$; $y(1) = 1$, $y'(1) = 10$.

6. (6pts) To reduce the Euler equation to a linear equation, we use the substitution, $z = \ln(x)$ to convert the equation from $y(x)$ to an equation for $y(z)$. If we use the operator notation $D = \mathrm{d/dx}$ and $\mathcal{D} = \mathrm{d/dz}$, show that

$$\text{i).}\frac{dy}{dx} = Dy = \frac{1}{x}\mathcal{D}y \qquad \text{or} \qquad x\, Dy = \mathcal{D}y$$

$$\text{ii).}\frac{d^2y}{dx^2} = D^2y = \frac{1}{x^2}\left(\mathcal{D}^2y - \mathcal{D}y\right) \qquad \text{or} \qquad x^2D^2y = \mathcal{D}(\mathcal{D}-1)y$$

$$\text{iii). and hence, that} \qquad x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

7. (6pts) Find the complete solution of the equation: $x^3\, y''' + 4x^2y'' - 5xy' - 15y = x^4$

First Order Equation

8. (6pts) The differential equation below has the boundary condition $y(1) = b$. Find the only value of b for which $y(0)$ is finite.

$$\frac{dy}{dx} + (\frac{1}{x} - 1)y = \frac{e^{2x}}{x}.$$

1. Find the general solution to the differential Eq. $x^2y''+3xy'+y=4/x$

Homogeneous Eq.: $x^2y''+3xy'+y=0$ This is an Euler Eq.

let $z=\ln(x)$; $\frac{dz}{dx}=\frac{1}{x}$; now $\frac{dy}{dx}=\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)=\frac{1}{x}\frac{dy}{dz}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\left(\frac{dz}{dx}\right)\left(\frac{d}{dz}\left(\frac{dy}{dz}\right)\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)\frac{d^2y}{dz^2}=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$$

let's substitute into our D.E.

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}\right)+3x\left(\frac{1}{x}\frac{dy}{dz}\right)+y=0$$

$$\frac{d^2y}{dz^2}-\frac{dy}{dz}+3\frac{dy}{dz}+y=0 \Rightarrow \frac{d^2y}{dz^2}+2\frac{dy}{dz}+y=0$$

we have a linear second order Eq. with constant coefficients

let's assume $y=e^{mx}$. Our characteristic Eq is. $m^2+2m+1=0$; $(m+1)^2=0$

we have a repeated root $m=-1$

$y_h(z)=Ae^{-z}+Bze^{-z}$ substitute for z ($z=\ln(x)$)

$$=Ae^{-\ln(x)}+B\ln(x)e^{-\ln(x)}$$

$$=Ax^{-1}+Bx^{-1}\ln(x)$$

To find the particular solution we can write the Eq. as $y''+\frac{3}{x}y'+\frac{1}{x^2}y=\frac{4}{x^3}$

now $y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$

where $u_1(x)=\int^x \frac{-y_2(x')}{W[y_1,y_2]}\frac{f(x')}{a_0(x')}dx'$ and $u_2(x)=\int^x \frac{y_1(x')}{W[y_1,y_2]}\frac{f(x')}{a_0(x')}dx'$

and where $W[y_1,y_2]\equiv$ Wronskian; $y_1(x)=x^{-1}$, $y_2(x)=x^{-1}\ln(x)$, $f(x)=\frac{4}{x^3}$, $a_0(x)=1$

$$W=\begin{vmatrix} x^{-1} & x^{-1}\ln(x) \\ -x^{-2} & -x^{-2}\ln(x)+x^{-2}\end{vmatrix}=-x^{-3}\ln(x)+x^{-3}+x^{-3}\ln(x)=x^{-3}=\frac{1}{x^3}$$

$$u_1(x)=-\int\left(\frac{1}{x^{-3}}\right)\left(x^{-1}\ln(x)\right)\left(\frac{4}{x^3}\right)dx=-4\int\frac{\ln(x)}{x}dx=-2\left(\ln(x)\right)^2$$

$$u_2(x)=\int\left(\frac{1}{x^{-3}}\right)\left(x^{-1}\right)\left(\frac{4}{x^3}\right)dx=4\int\frac{1}{x}dx=4\ln(x)$$

$$y_p(x)=u_1y_1+u_2y_2=-2(\ln(x))^2\left(\frac{1}{x}\right)+4\ln(x)\left(\frac{\ln(x)}{x}\right)$$

$$=-\frac{2}{x}(\ln(x))^2+\frac{4}{x}(\ln(x))^2=\frac{2}{x}(\ln(x))^2$$

$$y(x)=\frac{A}{x}+\frac{B}{x}\ln(x)+\frac{2}{x}(\ln(x))^2$$

2. Using the variation of parameters method show that

$$y(x) = c_1 \cosh(kx) + c_2 \sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-s))\, f(s)\, ds$$

is a complete solution of the equation $y'' - ky = f(x)$, where $k \neq 0$ and f is everywhere continuous

The homogeneous Eq. is $y'' - k^2 y = 0$; we let $y(x) = e^{mx}$, Our characteristic Eq. is $m^2 - k^2 = 0 \Rightarrow m^2 = k^2 \therefore m_1 = k,\ m_2 = -k$

$$y_h(x) = Ae^{kx} + Be^{-kx} = a_1 \cosh(kx) + a_2 \sinh(kx) \quad ; \quad y_1(x) = \cosh(kx),\ y_2(x) = \sinh(kx)$$

to apply the variation of parameter method, we need to know the Wronskian

$$W = \begin{vmatrix} \cosh(kx) & \sinh(kx) \\ k\sinh(kx) & k\cosh(kx) \end{vmatrix} = k\cosh^2(kx) - k\sinh^2(kx) = k\left(\cosh^2(kx) - \sinh^2(kx)\right) = k$$

$$y_p(x) = u_1 y_1 + u_2 y_2$$

$$u_1(x) = -\int_0^x \frac{y_2(x')}{W[y_1, y_2]} \frac{f(x')}{a_0(x')}\, dx' \quad ; \quad u_2(x) = \int_0^x \frac{y_1(x')}{W[y_1, y_2]} \frac{f(x')}{a_0(x')}\, dx'$$

$y_1(x) = \cosh(kx)$, $a_0(x) = 1$
$y_2(x) = \sinh(kx)$

$$u_1(x) = -\int_0^x \frac{\sinh(kx')}{k} f(x')\, dx' = -\frac{1}{k}\int_0^x \sinh(kx') f(x')\, dx'$$

$$u_2(x) = \int_0^x \frac{\cosh(kx')}{k} f(x')\, dx' = \frac{1}{k}\int_0^x \cosh(kx') f(x')\, dx'$$

$$y_p(x) = u_1 y_1 + u_2 y_2 = -\frac{1}{k}\cosh(kx)\int_0^x \sinh(kx') f(x')\, dx' + \frac{1}{k}\sinh(kx)\int_0^x \cosh(kx') f(x')\, dx'$$

$$= \frac{1}{k}\left[\int_0^x \left(\sinh(kx)\cosh(kx') - \cosh(kx)\sinh(kx')\right) f(x')\, dx'\right.$$

let's apply the identity $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$

$$y_p(x) = \frac{1}{k}\int_0^x \sinh(k(x - x')) f(x')\, dx'$$

$$y(x) = a_1 \cosh(kx) + a_2 \sinh(kx) + \frac{1}{k}\int_0^x \sinh(k(x - x')) f(x')\, dx'$$

3. Verify that the given function is a solution of the differential equation. Derive the equation satisfied by $u(x)$, give its solution and give the general solution of the second order equation:

$$y'' - \frac{2x}{1+x^2} y' + \frac{2}{1+x^2} y = 0 \; ; \quad y_1(x) = x.$$

$y_1(x) = x$, $y_1'(x) = 1$, $y_1''(x) = 0$ let's insert into the D.E.

$$0 - \frac{2x}{1+x^2}(1) + \frac{2}{1+x^2}(x) = 0 \Rightarrow \frac{-2x}{1+x^2} + \frac{2x}{1+x^2} = 0$$ given solution satisfies the D.E.

$$y_2(x) = u(x)\, y_1(x) = u(x)\, x \; ; \; y_2'(x) = u'(x)\, x + u(x) \; ; \; y_2''(x) = u''(x)\, x + u'(x) + u'(x)$$
$$= u''(x)\, x + 2u'(x)$$

let's insert into the D.E.

$$u''(x)\, x + 2u'(x) - \left(\frac{2x}{1+x^2}\right)\left(u'(x)\, x + u(x)\right) + \left(\frac{2}{1+x^2}\right) u(x)\, x = 0$$

$$u''(x)\, x + 2u'(x) - \left(\frac{2x}{1+x^2}\right) u'(x)\, x - \cancel{\left(\frac{2x}{1+x^2}\right) u(x)} + \cancel{\left(\frac{2}{1+x^2}\right) u(x)\, x} = 0$$

$$x\, u''(x) + \left(2 - \left(\frac{2x^2}{1+x^2}\right)\right) u'(x) = 0$$ let $v(x) = u'(x)$; $v'(x) = u''(x)$

$$x\, v'(x) + \left(2 - \left(\frac{2x^2}{1+x^2}\right)\right) v(x) = 0 \Rightarrow v'(x) + \left(\frac{2}{x} - \left(\frac{2x}{1+x^2}\right)\right) v(x) = 0$$

$$\frac{dv}{dx} = -\left(\frac{2}{x} - \left(\frac{2x}{1+x^2}\right)\right) v(x) = 0 \Rightarrow \int \frac{dv}{v} = \int \left(-\frac{2}{x} + \frac{2x}{1+x^2}\right) dx$$

$$\ln(v(x)) = -2\ln(x) + \ln(1+x^2) \Rightarrow v(x) = \frac{1+x^2}{x^2} = \frac{1}{x^2} + 1$$

$$v(x) = \frac{du(x)}{dx} = \frac{1}{x^2} + 1 \Rightarrow \int du(x) = \int \left(\frac{1}{x^2} + 1\right) dx = \int \frac{1}{x^2}\, dx + \int dx$$

$$u(x) = -\frac{1}{x} + x$$

$$y_2(x) = u(x)\, y_1(x) = \left(-\frac{1}{x} + x\right)(x) = x^2 - 1$$

$$\therefore y(x) = a_1 y_1(x) + a_2 y_2(x) = a_1 x + a_2(x^2 - 1)$$

4). Use the one solution indicated and find the complete solution.

$$(2x-x^2)y'' + 2(x-1)y' - 2y = 0 \;;\; y_1(x) = x-1$$

$$y_2(x) = u(x)y_1(x) = u(x)(x-1), \quad y_2'(x) = u'(x)(x-1) + u(x)$$
$$y_2''(x) = u''(x)(x-1) + u'(x) + u'(x)$$
$$= u''(x)(x-1) + 2u'(x)$$

$$(2x-x^2)\left(u''(x)(x-1) + 2u'(x)\right) + 2(x-1)\left(u'(x-1) + u(x)\right) - 2\left(u(x)(x-1)\right) = 0$$

$$(2x-x^2)(x-1)u''(x) + 2(2x-x^2)u'(x) + 2(x-1)^2u'(x) + 2(x-1)\cancel{u(x)} - 2\cancel{u(x)}(x-1) = 0$$

$$(2x-x^2)(x-1)u''(x) + \left(2(2x-x^2) + 2(x-1)^2\right)u'(x) = 0$$
$$(2x-x^2)(x-1)u''(x) + 2\left[(2x-x^2)+(x-1)^2\right]u'(x) = 0 \qquad \text{let } v(x) = u'(x)$$
$$v'(x) = u''(x)$$

$$(2x-x^2)(x-1)v'(x) + 2\left[(2x-x^2)+(x-1)^2\right]v(x) = 0$$

$$v'(x) = -\frac{2\left[(2x-x^2)+(x-1)^2\right]}{(2x-x^2)(x-1)}v(x) \Rightarrow \frac{dv(x)}{dx} = -\frac{2\left[(2x-x^2)+(x-1)^2\right]}{(2x^2-x^2)(x-1)}v(x)$$

$$\frac{dv(x)}{v(x)} = -\frac{2\left[(2x-x^2)+(x-1)^2\right]}{(2x-x^2)(x-1)}dx = \left[-2\left(\frac{1}{x-1}\right) - 2\left(\frac{x-1}{2x-x^2}\right)\right]dx$$

$$\frac{dv(x)}{v(x)} = -2\int\frac{1}{x-1}dx + \int\frac{2x-2}{x^2-2x}dx$$

$$\ln(v(x)) = -2\ln(x-1) + \ln(x^2-2x) \Rightarrow v(x) = (x-1)^{-2}(x^2-2x)$$

$$v(x) = \frac{du(x)}{dx} = \frac{x^2-2x}{(x-1)^2} \Rightarrow \int du(x) = \int\frac{x^2-2x}{(x-1)^2}dx$$

$$u(x) = \int\frac{x^2-2x+1}{(x-1)^2}dx - \int\frac{1}{(x-1)^2}dx = \int\frac{(x-1)^2}{(x-1)^2}dx - \int\frac{1}{(x-1)^2}dx$$

$$= \int dx - \int\frac{1}{(x-1)^2}dx = x + \frac{1}{(x-1)}$$

$$y_2(x) = u(x)y_1(x) = \left(x + \frac{1}{(x-1)}\right)(x-1) = x^2 - x + 1$$

$$\therefore y(x) = a_1y_1(x) + a_2y_2(x) = \underline{a_1(x-1) + a_2(x^2-x+1)}$$

5. Solve the Eq. $x^2y'' - 9xy' + 24y = 0$; $y(1) = 1$, $y'(1) = 10$.

Euler Eq. $\Rightarrow$ let $z = \ln(x)$; $\frac{dz}{dx} = \frac{1}{x}$; now $\frac{dy}{dx} = \left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right) = \frac{1}{x}\frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{dz}{dx}\right)\left(\frac{d}{dz}\left(\frac{dy}{dz}\right)\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \left(\frac{1}{x}\right)\left(\frac{1}{x}\right)\frac{d^2y}{dz^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

let's substitute in the D.E.

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right) - 9x\left(\frac{1}{x}\right)\frac{dy}{dz} + 24y = 0$$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 9\frac{dy}{dz} + 24y = 0 \Rightarrow \frac{d^2y}{dz^2} - 10\frac{dy}{dz} + 24y = 0$$

we have a linear second order Eq. with constant coefficients.

let's assume $y = e^{mx}$. Our characteristic Eq. is $m^2 - 10m + 24 = 0$

$(m-6)(m-4) = 0$; our roots are $m = 6$ and $m = 4$

$y(z) = Ae^{6z} + Be^{4z}$ recall $z = \ln(x)$

$= Ae^{6\ln(x)} + Be^{4\ln(x)}$

$y(x) = Ax^6 + Bx^4$ $y'(x) = 6Ax^5 + 4Bx^3$

now to find the coefs. A and B

$y(1) = A + B = 1$ $A = 1 - B$ $A = 3$

$y'(1) = 6A + 4B = 10$ $6(1-B) + 4B = 10 \Rightarrow 6 - 6B + 4B = 10 \Rightarrow -2B = 4$ $B = -2$

$y(x) = 3x^6 - 2x^4$

2

6. To reduce the Euler equation to a linear equation, we use the substitution, $z = \ln(x)$ to convert the equation from $y(x)$ to an equation for $y(z)$. If we use the operator notation $D = \frac{d}{dx}$ and $\mathcal{D} = d/dz$ show that,

i) $\frac{dy}{dx} = Dy = \frac{1}{x}\mathcal{D}y$ or $xDy = \mathcal{D}y$

$z = \ln(x)$ $\quad dz = \frac{1}{x}\,dx$ $\quad \therefore \frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz} = \frac{1}{x}\frac{dy}{dx}$$

$$Dy = \frac{1}{x}\mathcal{D}y \Rightarrow xDy = \mathcal{D}y$$

ii) $\frac{d^2y}{dx^2} = D^2y = \frac{1}{x^2}(\mathcal{D}^2y - \mathcal{D}y)$ or $x^2D^2y = \mathcal{D}(\mathcal{D}-1)y$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{dz}{dx}\frac{d}{dz}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} = \frac{1}{x^2}(\mathcal{D}^2y - \mathcal{D})y$$

$$= \frac{1}{x^2}\mathcal{D}(\mathcal{D}-1)y$$

$$\Rightarrow \frac{d^2y}{dx^2} = D^2y = \frac{1}{x^2}\mathcal{D}(\mathcal{D}-1)y$$

$$\therefore x^2D^2 = \mathcal{D}(\mathcal{D}-1)$$

iii) $\frac{d^3y}{dx^3} = D^3y = \frac{1}{x^3}(\mathcal{D})(\mathcal{D}-1)(\mathcal{D}-2)y$ or $x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dz} + \left(\frac{1}{x^2}\right)\frac{d^2y}{dz^2}\right)$$

$$= \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dz}\right) + \frac{d}{dx}\left(\frac{1}{x^2}\frac{d^2y}{dz^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \frac{1}{x^2}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{2}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dz^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \left(\frac{1}{x^2}\right)\frac{dz}{dx}\frac{d}{dz}\left(\frac{dy}{dz}\right) - \frac{2}{x^3}\frac{d^2y}{dz^2} + \left(\frac{1}{x^2}\right)\frac{dz}{dx}\frac{d}{dz}\left(\frac{d^2y}{dz^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \left(\frac{1}{x^3}\right)\frac{d^2y}{dz^2} - \frac{2}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^3}\frac{d^3y}{dz^3}$$

$$= \frac{1}{x^3}\frac{d^3y}{dz^3} - \frac{3}{x^3}\frac{d^2y}{dz^2} + \frac{2}{x^3}\cdot\frac{dy}{dz}$$

$$x^3\frac{d^3y}{dx^3} = x^3Dy = (\mathcal{D}^3 - 3\mathcal{D}^2 + 2\mathcal{D})y$$

$$\therefore x^3Dy = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

7. Find the complete solution of the equation

$$x^3y''' + 4x^2y'' - 5xy' - 15y = x^4$$

using the results from problem 6

let $z = \ln(x)$

$e^z = x$

$$[D(D-1)(D-2) + 4D(D-1) - 5D - 15]y = e^{4z}$$
$$[D(D^2-3D+2) + 4D^2 - 4D - 5D - 15]y = e^{4z}$$
$$[D^3 - 3D^2 + 2D + 4D^2 - 9D - 15]y = e^{4z}$$
$$[D^3 + D^2 - 7D - 15]y = e^{4z}$$

homogeneous Eq.: $[D^3 + D^2 - 7D - 15]y = 0$ let $y(z) = e^{mz}$

characteristic Eq: $m^3 + m^2 - 7m - 15 = 0$

$(m-3)(m^2+4m+5) = 0 \Rightarrow (m-3)(m+2-i)(m+2+i) = 0$

$m = 3, \ m = -2-i, \ m = -2+i$

$$y_h(z) = c_1e^{3z} + c_2e^{(-2-i)z} + c_3e^{(-2+i)z}$$
$$= c_1e^{3z} + e^{-2z}(c_2e^{-iz} + c_3e^{iz})$$
$$= c_1e^{3z} + e^{-2z}(c_4\cos(z) + c_5\sin(z))$$

$y_p(z) = Ae^{4z}$, $y_p'(z) = 4Ae^{4z}$, $y_p''(z) = 16Ae^{4z}$, $y_p'''(x) = 64Ae^{4z}$

$$64Ae^{4z} + 16Ae^{4z} - 7(4Ae^{4z}) - 15Ae^{4z} = e^{4z}$$

$64A + 16A - 28A - 15A = 1 \Rightarrow 80A - 43A = 1 \Rightarrow A = \frac{1}{37}$

$$y(z) = y_h(z) + y_p(z) = c_1e^{3z} + e^{-2z}(c_4\cos(z) + c_5\sin(z)) + \frac{1}{37}e^{4z}$$

recall $z = \ln(x)$

$$y(x) = c_1e^{3\ln(x)} + e^{-2\ln(x)}(c_4\cos(\ln(x)) + c_5\sin(\ln(x))) + \frac{1}{37}e^{4\ln(x)}$$

$$\Rightarrow y(x) = c_1x^3 + x^{-2}(c_4\cos(\ln(x)) + c_5\sin(\ln(x))) + \frac{1}{37}x^4$$

8). The differential equation below has the boundary condition $y(1) = b$. Find the only value of b for which $y(0)$ is finite.

$$\frac{dy}{dx} + \left(\frac{1}{x} - 1\right) y = \frac{e^{2x}}{x}$$

1^{st} order equation. will solve using an integrating factor.

$$e^{\int (\frac{1}{x} - 1) dx} = e^{(\ln(x) - x)} = xe^{-x}$$

$$xe^{-x} \frac{dy}{dx} + xe^{-x}\left(\frac{1}{x} - 1\right) y = \frac{e^{2x}}{x}(xe^{-x})$$

$$\frac{d(xe^{-x}y)}{dx} = e^x \Rightarrow \int d(xe^{-x}y) = \int e^x dx$$

$$x e^{-x} y(x) = e^x + C \Rightarrow y(x) = \frac{e^{2x}}{x} + \frac{C}{x} e^x$$

$$y(1) = e^2 + C e^1 = b \Rightarrow Ce = b - e^2 \Rightarrow C = be^{-1} - e$$

$$y(x) = \frac{e^{2x}}{x} + \frac{be^{-1} - e}{x} e^x = \frac{e^x}{x}(e^x + be^{-1} - e)$$

now $y(0) = \frac{e^0}{0}(e^0 + be^{-1} - e) = \frac{1}{0}(1 + be^{-1} - e)$

is infinite unless $(1 + be^{-1} - e) = 0$

$$\left(1 + \frac{b}{e} - e\right) = 0 \Rightarrow \frac{b}{e} = e - 1 \Rightarrow b = e(e-1) = e^2 - e$$

if we insert this into our eq. we obtain

$$y(x) = \frac{e^x}{x}\left(e^x + e^{-1}(e(e-1)) - e\right) = \frac{e^x}{x}(e^x - 1)$$

$$\lim_{x \to 0} y(x) = \lim_{x \to 0} \frac{e^x}{x}(e^x - 1) = \frac{0}{0} \text{ indeterminant}$$

l'Hopitals rule $\lim_{x \to 0} \frac{\frac{d}{dx}(e^x(e^x - 1))}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{e^x(e^x - 1) + e^x e^x}{1} = \underline{\underline{1}}$ finite limit