Homework Set No. 2
Due September 20, 2013

NEEP 547
DLH

Nonlinear Eqs. reducible to first order:

1. (5pts) Find the general solution to the differential equation:
$y'' = \left[1 + (y')^2\right]^{3/2}$

2. (5pts) page 72, prob. 13c; Find the general solution to the differential equation:
$y\, y'' = y^2\, y' + (y')^2$

Linear Operators

3. (6pts) First factor the equation using operator notation and then find the general solution to the differential equation:
$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0$$

4. (6pts) First factor the equation using operator notation and then find the general solution to the differential equation:
$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 3\,x^2 - x$$

Linear dependent or independent solutions.

5. (4pts) page 69, prob. 8. : Show that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions of $x^2\, y'' - 2\,x\, y' + 2\, y = 0$ on [-1,1], but that $W(0) = 0$. Why does this not contradict Theorem 2.3.1 in this interval?
Theorem 2.3: Wronskian Test : Let y_1 and y_2 be solutions of $y'' + p(x)\, y' + q(x)\, y = 0$ on the open interval I. Then,
2.3.1. Either $W(x) = 0$ for all x in I, or $W(x) \neq 0$ for all x in I.
2.3.2. y_1 and y_2 are linearly independent on I if and only if $W(x) \neq 0$ on I.

6. (4pts) page 69, prob. 10: Show that $y_1(x) = 3\,e^{2x} - 1$ and $y_2(x) = e^{-x} + 2$ are solutions of $y\, y'' + 2\, y' - (y')^2 = 0$, but neither $2\, y_1$ nor $y_1 + y_2$ is a solution. Why does this not contradict Theorem 2.2?
Theorem 2.2: Let y_1 and y_2 be solutions of $y'' + p(x)\, y' + q(x)\, y = 0$ on an interval I. Then any linear combination of these solutions is also a solution.

Homogeneous Linear Differential Equations with Constant Coefficients:

7. (6pts) Solve the initial-value problem: $(D^3 - 6\,D^2 + 11\,D - 6)\, y = 0$ where $D^n = \frac{d^n}{dx^n}$; with conditions: $y = y' = 0$ and $y'' = 2$ when $x = 0$.

8. (6pts) Solve the initial-value problem: $8y''' - 4y'' + 6y' + 5y = 0$ with conditions: $y = 0, y'' = y' = 1$ when $x = 0$.

Nonhomogeneous Equations with Constant Coefficients

9. (6pts) O'Neil, page 93 prob. 16; find the general solution: $y'' - 2\,y' + y = 3\,x + 25\,\sin(3x)$

10. (7pts) find the general solution: $y'''' + 3\,y'' - 4\,y = \sinh(x) - \sin^2(x)$

1. Find the general solution to the differential equation:

$y'' = [1+(y')^2]^{3/2}$ $\qquad$ x is explicitly missing from the Eq.

let $v(y) = \frac{dy}{dx}$; $\frac{d^2y}{dx^2} = \frac{d}{dx}(v(y)) = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$

$y'' = [1+(y')^2]^{3/2} \Rightarrow v\frac{dv}{dy} = [1+v^2]^{3/2}$

$\Rightarrow \frac{v\,dv}{[1+v^2]^{3/2}} = dy \Rightarrow \int dy = \int \frac{v\,dv}{[1+v^2]^{3/2}} \Rightarrow y = -\frac{1}{(1+v^2)^{1/2}} + c_1$

$\Rightarrow (y-c_1) = -\frac{1}{(1+v^2)^{1/2}} \Rightarrow -(y-c_1) = \frac{-1}{(1+v^2)^{1/2}} \Rightarrow (1+v^2)^{1/2} = \frac{-1}{(y-c_1)}$

$(1+v^2) = \frac{1}{(y-c_1)^2} \Rightarrow v^2 = \frac{1}{(y-c_1)^2} - 1$ $\qquad$ recall $v = \frac{dy}{dx}$

$\frac{dy}{dx} = \sqrt{\frac{1}{(y-c_1)^2} - 1} = \sqrt{\frac{1-(y-c_1)^2}{(y-c_1)^2}} = \frac{\sqrt{1-(y-c_1)^2}}{y-c_1}$

$\Rightarrow \frac{y-c_1}{\sqrt{1-(y-c_1)^2}}\,dy = dx$ $\qquad$ let $z = y-c_1 \Rightarrow dz = dy$

$\Rightarrow \frac{z\,dz}{\sqrt{1-z^2}} = dx \Rightarrow \int dx = \int \frac{z\,dz}{\sqrt{1-z^2}} \Rightarrow x = -(1-z^2)^{1/2} + c_2$

$x - c_2 = -(1-z^2)^{1/2} \Rightarrow (x-c_2)^2 = (1-z^2) \Rightarrow z^2 = 1-(x-c_2)^2$

$z = (1-(x-c_2)^2)$ $\quad$ recall $z = y - c_1$

$y - c_1 = (1-(x-c_2)^2)^{1/2}$

$y = c_1 + (1-(x-c_2)^2)^{1/2}$

2. Find the general solution to the differential equation:

$$y \frac{d^2y}{dx^2} = y^2 \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$ note that x is not present.

Let $\frac{dy}{dx} = v$; $\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$

$$y\frac{d^2y}{dx^2} = y^2\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 \Rightarrow yv\frac{dv}{dy} = y^2 v + v^2$$

$$\frac{dv}{dy} = y + \frac{v}{y} \Rightarrow \frac{dv}{dy} - \frac{1}{y}v = y$$

integrating factor $e^{-\int \frac{1}{y}dy} = e^{-\ln(y)} = \frac{1}{y}$

$$\left(\frac{1}{y}\right)\frac{dv}{dy} - \left(\frac{1}{y}\right)\left(\frac{1}{y}\right)v = \left(\frac{1}{y}\right)(y)$$

$$\frac{d(\frac{1}{y}v)}{dy} = 1 \Rightarrow d\left(\frac{1}{y}v\right) = dy \Rightarrow \frac{1}{y}v = y + c_1$$

$v = y^2 + c_1 y$ now $v = \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = y^2 + c_1 y \Rightarrow \frac{dy}{dx} - c_1 y = y^2$ Bernoulli's Eq.

let $u = \frac{1}{y}$; $du = -\frac{1}{y^2}dy$; $\frac{dy}{du} = -y^2$

$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = -y^2\frac{du}{dx}$ now to substitute into the D.E.

$\frac{dy}{dx} - c_1 y = y^2 \Rightarrow -y^2\frac{du}{dx} - c_1 y = y^2 \Rightarrow \frac{du}{dx} + c_1\left(\frac{1}{y}\right) = -1$ now $u = \frac{1}{y}$

$\therefore \frac{du}{dx} + c_1 u = -1$ find the integrating factor $e^{\int c_1 dx} = e^{c_1 x}$

$$\frac{d(e^{c_1 x}u(x))}{dx} = -e^{c_1 x} \Rightarrow \int d(e^{c_1 x}u(x)) = -\int e^{c_1 x}dx$$

$$e^{c_1 x}u(x) = -\frac{1}{c_1}e^{c_1 x} + c_2 \Rightarrow u(x) = c_2 e^{-c_1 x} - \frac{1}{c_1}$$

$$y(x) = \frac{1}{u(x)} \Rightarrow y(x) = \frac{1}{c_2 e^{-c_1 x} - \frac{1}{c_1}}$$

3. First factor the equation using operator notation and then find the general solution to the differential Eq.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \Rightarrow \left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx} - 1\right)y(x) = 0$$

let's try the general factorization $\left(x\frac{d}{dx}+a\right)\left(x\frac{d}{dx}-c\right)y(x)=0$

now to expand as we need to determine a and c.

$$\left(x\frac{d}{dx}\left(x\frac{dy}{dx}-cy\right)+a\left(x\frac{dy}{dx}-cy\right)\right)=0$$

$$\left(x\frac{d}{dx}(x)\frac{dy}{dx}+x^2\frac{d}{dx}\left(\frac{dy}{dx}\right)-cx\frac{dy}{dx}+ax\frac{dy}{dx}-acy\right)=0$$

$$\left(x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}-cx\frac{dy}{dx}+ax\frac{dy}{dx}-acy\right)=0$$

$$\left(x^2\frac{d^2}{dx^2}+(1-c+a)x\frac{d}{dx}-ac\right)y(x)=0$$

comparing this to the orginal Eq. we have

$(1-c+a)=1$ and $ac=1 \Rightarrow a-c=0$ and $ac=1$

which gives $a=c$ and $a^2=1$ or $a=\pm1$ and $c=\pm1$

let's choose $a=1=c$. Our factorization becomes

$$\left(x\frac{d}{dx}+1\right)\underbrace{\left(x\frac{d}{dx}-1\right)y(x)}_{v(x)}=0 \Rightarrow \left(x\frac{d}{dx}+1\right)v(x)=0 \Rightarrow x\frac{dv}{dx}+v(x)=0$$

$$x\,dv=-v(x)\,dx \Rightarrow \frac{dv}{v}=\frac{-dx}{x} \Rightarrow \ln|v|=-\ln|x|+c=\ln\left(\frac{1}{x}\right)+c$$

$$\Rightarrow v(x)=\frac{C_1}{x}$$

now $\left(x\frac{d}{dx}-1\right)y(x)=v(x) \Rightarrow x\frac{dy}{dx}-y(x)=\frac{C_1}{x} \Rightarrow \frac{dy}{dx}-\frac{1}{x}y(x)=\frac{C_1}{x^2}$

our integrating factor is $e^{-\int\frac{1}{x}dx}=e^{\ln(x^{-1})}=\frac{1}{x}$

$$\frac{1}{x}\left(\frac{dy}{dx}-\frac{1}{x}y(x)\right)=\frac{C_1}{x^3} \Rightarrow \frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y(x)=\frac{C_1}{x^3}=\frac{d\left(\frac{y}{x}\right)}{dx}=\frac{C_1}{x^3}$$

$$\int d\left(\frac{y}{x}\right)=\int\frac{C_1}{x^3}dx \Rightarrow \frac{y}{x}=\frac{-2C_1}{x^2}+C_2 \Rightarrow y(x)=\frac{C_3}{x}+C_2x$$

if $a=-1$ would have been chosen, the factored Eq is

$\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}+1\right)y(x)=0$ which gives the same result.

4. First factor the equation using operator notation and then find the general solution to the differential equation.

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=3x^2-x \Rightarrow \left[x\frac{d^2}{dx^2}+\frac{d}{dx}\right]y=3x^2-x$$

① $\left(x\frac{d}{dx}+1\right)\left(\frac{d}{dx}\right)y=3x^2-x$ or $\left(\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)y=3x^2-x$ ②

Both factorizations are correct.

① $\left(x\frac{d}{dx}+1\right)\underbrace{\left(\frac{d}{dx}\right)y}_{v(x)}=3x^2-x \Rightarrow \left(x\frac{d}{dx}+1\right)v(x)=3x^2-x \Rightarrow x\frac{dv}{dx}+v(x)=3x^2-x$

$\frac{dv}{dx}+\frac{v(x)}{x}=3x-1$ our integrating factor is $e^{\int\frac{1}{x}dx}=e^{\ln(x)}=x$

$\therefore x\frac{dv}{dx}+v(x)=3x^2-x \Rightarrow \frac{d(xv)}{dx}=3x^2-x \Rightarrow \int d(xv)=\int(3x^2-x)dx$

$xv=x^3-\frac{1}{2}x^2+c_1 \Rightarrow v(x)=x^2-\frac{x}{2}+\frac{c_1}{x}$

now $v(x)=\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=x^2-\frac{x}{2}+\frac{c_1}{x} \Rightarrow \int dy=\int\left(x^2-\frac{x}{2}+\frac{c_1}{x}\right)dx$

$y(x)=\frac{x^3}{3}-\frac{x^2}{4}+c_1\ln(x)+c_2$

② $\left(\frac{d}{dx}\right)\underbrace{\left(x\frac{d}{dx}\right)y}_{v(x)}=3x^2-x \Rightarrow \left(\frac{d}{dx}\right)v(x)=3x^2-x \Rightarrow \frac{dv}{dx}=3x^2-x$

$\int dv(x)=\int(3x^2-x)dx \Rightarrow v(x)=x^3-\frac{1}{2}x^2+c_1$

now $v(x)=\left(x\frac{d}{dx}\right)y \Rightarrow x\frac{dy}{dx}=x^3-\frac{1}{2}x^2+c_1 \Rightarrow \frac{dy}{dx}=x^2-\frac{1}{2}x+\frac{c_1}{x}$

$\int dy=\int\left(x^2-\frac{1}{2}x+\frac{c_1}{x}\right)dx \Rightarrow y(x)=\frac{x^3}{3}-\frac{x^2}{4}+c_1\ln(x)+c_2$

5. page 69, prob 8: Show that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent solutions of $x^2y'' - 2xy' + 2y = 0$ on $[-1,1]$, but that $W(0) = 0$. Why does this not contradict Theorem 2.3.1 in this interval?

We will use the Wronskian to determine linear independence

$$W[y_1, y_2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$W \neq 0 \therefore$ solutions are linearly independent

However, we note that $W(0) = 0$.

The fact that $W(0) = 0$ at $x = 0$ leads us to the following

conclusion that the solutions are linearly independent on $[-1,1]$

except at $0 \therefore$ the interval should be written as $[-1,0), (0,1]$

6. page 69, prob 10: Show that $y_1(x) = 3e^{2x}-1$ and $y_2(x) = e^{-x}+2$ are solutions of $yy'' + 2y' - (y')^2 = 0$, but neither $2y_1$ nor $y_1 + y_2$ is a solution. Why does this not contradict Theorem 2.2?

$y_1(x) = 3e^{2x}-1$, $y_1' = 6e^{2x}$, $y_1'' = 12e^{2x}$

$(3e^{2x}-1)(12e^{2x}) + 2(6e^{2x}) - (6e^{2x})^2 \stackrel{?}{=} 0$

$\cancel{36e^{4x}} - \cancel{12e^{2x}} + \cancel{12e^{2x}} - \cancel{36e^{4x}} = 0$ ✓ solution satisfies the equation

$y_2(x) = e^{-x}+2$, $y_2' = -e^{-x}$, $y_2'' = e^{-x}$

$(e^{-x}+2)e^{-x} + 2(-e^{-x}) - (-e^{-x})^2 \stackrel{?}{=} 0$

$e^{-2x} + \cancel{2e^{-x}} - \cancel{2e^{-x}} - e^{-2x} = 0$ ✓ solution satisfies the equation

let's check the Wronskian

$$W[y_1, y_2] \quad \begin{bmatrix} 3e^{2x}-1 & e^{-x}+2 \\ 6e^{2x} & -e^{-x} \end{bmatrix} = (3e^{2x}-1)(-e^{-x}) - (6e^{2x})(e^{-x}+2)$$
$$= -3e^{x} + e^{-x} - 6e^{x} - 12e^{2x} -$$
$$= -9e^{x} + e^{-x} - 12e^{2x}$$

$W[y_1, y_2] \neq 0$ linearly independent solution

$y_3 = 2y_1 = 6e^{2x} - 2$ = new solution. Will it work?

$y_3(x) = 6e^{2x} - 2$, $y_3' = 12e^{2x}$, $y_3'' = 24e^{2x}$

$(6e^{2x}-2)(24e^{2x}) + 2(12e^{2x}) - (12e^{2x})^2 \stackrel{?}{=} 0$

$\cancel{144e^{4x}} - 48e^{2x} + 24e^{2x} - \cancel{144e^{4x}} = -24e^{2x}$ does not satisfy the equation

Thus $2y_1$ is not a solution

$y_3 = y_1 + y_2 = 3e^{2x} - 1 + e^{-x} + 2$, $y_3' = 6e^{2x} - e^{-x}$, $y_3'' = 12e^{2x} + e^{-x}$

$(3e^{2x} - 1 + e^{-x} + 2)(12e^{2x} + e^{-x}) + 2(6e^{2x} - e^{-x}) - (6e^{2x} - e^{-x})^2 \stackrel{?}{=} 0$

$\cancel{36e^{4x}} - \cancel{12e^{2x}} + 12e^{x} + 24e^{2x} + 3e^{x} - e^{-x} + \cancel{e^{-2x}} + \cancel{2e^{-x}} + 12e^{2x} - \cancel{2e^{-x}} - \cancel{36e^{4x}} - \cancel{e^{-2x}} + 12e^{x} \stackrel{?}{=} 0$

$12e^{x} + 24e^{2x} + 3e^{x} - e^{-x} + 12e^{x} = 24e^{2x} + 27e^{x} - e^{-x} \neq 0$

does not satisfy the equation

The D.E. is a nonlinear Eq. The theorem is for linear Eqs.

7. Solved the initial-value problem: $(D^3-6D^2+11D-6)y=0$ where $D^n = \frac{d^n}{dx^n}$; with conditions: $y=y'=0$ and $y''=2$ when $x=0$.

we have a linear O.D.E. with constant coefficients

let $y(x) = e^{mx}$

Our characteristic Eq. is: $(m^3-6m^2+11D-6)=0$

factoring we have: $(m-2)(m^2-4m+3) = (m-2)(m-1)(m-3) = 0$

our roots are $m_1=2$, $m_2=1$ and $m_3=3$

$y(x) = C_1e^x + C_2e^{2x} + C_3e^{3x}$ we need $y' = C_1e^x + 2C_2e^{2x} + 3C_3e^{3x}$

and $y'' = C_1e^x + 4C_2e^{2x} + 9C_3e^{3x}$

Now to determine the unknown coefficients

$y(0) = C_1 + C_2 + C_3 = 0$

$y'(0) = C_1 + 2C_2 + 3C_3 = 0$

$y''(0) = C_1 + 4C_2 + 9C_3 = 2$

After some algebra, we get $C_3=1$, $C_2=-2$, and $C_1=1$

$\therefore \underline{\underline{y(x) = e^x - 2e^{2x} + e^{3x}}}$

we can check to see if it satisfy the D.E.:

$y(x) = e^x - 2e^{2x} + e^{3x}$	$y''' \Rightarrow e^x - 16e^{2x} + 27e^{3x}$
$y'(x) = e^x - 4e^{2x} + 3e^{3x}$	$-6y'' \Rightarrow -6e^x + 48e^{2x} - 54e^{3x}$
$y''(x) = e^x - 8e^{2x} + 9e^{3x}$	$+11y' \Rightarrow 11e^x - 44e^{2x} + 33e^{3x}$
$y'''(x) = e^x - 16e^{2x} + 27e^{3x}$	$-6y \Rightarrow -6e^x + 12e^{2x} - 6e^{3x}$
	$= 0$ $\quad 0 + 0 + 0 = 0$

checks

8. Solve the initial-value problem: $8y''' - 4y'' + 6y' + 5y = 0$
with conditions: $y = 0,\ y'' = y' = 1$ when $x = 0$.

$8y''' - 4y'' + 6y' + 5y = 0$ let $y = e^{mx}$,
using this, we obtain the following characteristic Eq.

$8m^3 - 4m^2 + 6m + 5 = 0 \Rightarrow (2m+1)(4m^2 - 4m + 5) = 0$ $\quad (2m+1) = 0 \quad m = -\frac{1}{2}$

$4m^2 - 4m + 5 = 0 \Rightarrow m^2 - m + \frac{5}{4} = 0$ now to complete the square

$m^2 - m + (\frac{1}{2})^2 = -\frac{5}{4} + (\frac{1}{2})^2 \Rightarrow (m - \frac{1}{2})^2 = -1$

$m - \frac{1}{2} = \pm\sqrt{-1} = \pm i \Rightarrow m = \frac{1}{2} \pm i$

our roots are $m_1 = -\frac{1}{2}$, $m_2 = \frac{1}{2} + i$, $m_3 = \frac{1}{2} - i$

$y(x) = a_1 e^{-\frac{x}{2}} + a_2 e^{(\frac{1}{2}+i)x} + a_3 e^{(\frac{1}{2}-i)x} = a_1 e^{-\frac{x}{2}} + e^{\frac{x}{2}}(a_2 e^{ix} + a_3 e^{-ix})$

$y(x) = a_1 e^{-\frac{x}{2}} + e^{\frac{x}{2}}(a_4\cos(x) + a_5\sin(x))$

now to find the coefficients a_1, a_4 and a_5 using the conditions.

now $y'(x) = -\frac{a_1}{2} e^{-\frac{x}{2}} + \frac{1}{2} e^{x/2}(a_4\cos(x) + a_5\sin(x)) - e^{\frac{x}{2}}(a_4\sin(x) - a_5\cos(x))$

$y''(x) = \frac{a_1}{4} e^{-\frac{x}{2}} + \frac{1}{4} e^{\frac{x}{2}}(a_4\cos(x) + a_5\sin(x)) - \frac{e^{x/2}}{2}(a_4\sin(x) - a_5\cos(x))$
$\qquad - \frac{1}{2} e^{\frac{x}{2}}(a_4\sin(x) - a_5\cos(x)) - e^{x/2}(a_4\cos(x) + a_5\sin(x))$

$y(0) = a_1 + a_4 = 0 \Rightarrow a_1 = -a_4$

$y'(0) = -\frac{1}{2}a_1 + \frac{1}{2}a_4 + a_5 = 1$

$y''(0) = \frac{1}{4}a_1 + \frac{1}{4}a_4 + \frac{a_5}{2} + \frac{a_5}{2} - a_4 = 1 \Rightarrow -\frac{a_4}{4} + \frac{a_4}{4} + a_5 - a_4 = 1 \Rightarrow a_5 - a_4 = 1$

$\frac{a_4}{2} + \frac{a_4}{2} + a_5 = 1 \Rightarrow a_4 + a_5 = 1$ $\quad \therefore a_5 - a_4 = 1$

$a_5 + a_4 = 1$

$2a_5 = 2 \therefore a_5 = 1$

$y(x) = e^{x/2}\sin(x)$

$a_4 + a_5 = 1 \Rightarrow a_4 + 1 = 1$

$\therefore a_4 = 0$ and $a_1 = 0$

9. page 93, prob. 16, find the general solution: $y''-2y'+y=3x+25\sin(3x)$

Let's find the homogeneous solution

$$y''(x)-2y'+y=0 \Rightarrow (D^2-2D+1)y=0 \Rightarrow (D-1)^2y=0$$

characteristic eq. $(m-1)^2=0 \therefore m=1,1$: we have a repeated root.

$$y_h(x)=c_1e^x+c_2xe^x$$

the forcing function $f(x)=3x+25\sin(3x)$. We will assume the following particular solution $y_p(x)=A+Bx+Cx^2+D\sin(3x)+E\cos(3x)$

$$y_p'(x)=B+2Cx+3D\cos(3x)-3E\sin(3x)$$
$$y_p''(x)=2C-9D\sin(3x)-9E\cos(3x)$$

Let's insert into the D.E.

$$2C-9D\sin(3x)-9E\cos(3x)-2(B+2Cx+3D\cos(3x)-3E\sin(3x))$$
$$+A+Bx+Cx^2+D\sin(3x)+E\cos(3x)=3x+25\sin(3x)$$

$$(2C-2B+A)+(-4C+B)x+Cx^2+(-9D+6E+D)\sin(3x)+(-9E-6D+E)\cos(3x)$$
$$=3x+25\sin(3x)$$

comparing coefficients

$C=0$, $2C-2B+A=0 \Rightarrow 2B=A$

$-4C+B=3 \Rightarrow B=3 \therefore A=6$

$$\left.\begin{array}{l}-8D+6E=25\\-8E-6D=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}-8D+6E=25\\-6D-8E=0\end{array}\right\} \Rightarrow \begin{array}{l}-8D+6E=25\\-3D-4E=0\end{array}$$

$$\left.\begin{array}{l}-24D+18E=75\\-24D-32E=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}-24D+18E=75\\24D+32E=0\end{array}\right\} \quad 50E=75 \Rightarrow E=\frac{75}{50}=\frac{3}{2}$$

$$-3D-4E=0 \Rightarrow 3D=-4E \Rightarrow D=\frac{-4}{3}E=\left(\frac{-4}{3}\right)\left(\frac{3}{2}\right)=-2$$

$$y(x)=y_h(x)+y_p(x)=c_1e^x+c_2xe^x+6+3x-2\sin(3x)+\frac{3}{2}\cos(3x)$$

10. Find the general solution for the differential equation:
$y'''' + 3y'' - 4y = \sinh(x) - \sin^2(x)$. ✓

Let's find the homogeneous solution. Assume our solution goes as
$y(x) = e^{mx}$
our characteristic Eq. is $(m^4 + 3m^2 - 4) = 0 \Rightarrow (m^2+4)(m^2-1) = 0$
$\Rightarrow m^2 = -2$ and $m^2 = 1$ our roots are: $m_1 = 1, m_2 = -1, m_3 = 2i, m_4 = -2i$

$$y_h(x) = a_1 e^x + a_2 e^{-x} + a_3 e^{+2ix} + a_4 e^{-2ix}$$
$$= a_1 e^x + a_2 e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)$$

Now to find the particular solution: Note: $\sinh(x) = \frac{e^x - e^{-x}}{2}$; $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$
$\therefore \quad f(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x} - \frac{1}{2} + \frac{1}{2}\cos(2x)$

It should be noted that three of our homogeneous solutions appear in $f(x)$. They are $y_1 = e^x$, $y_2 = e^{-x}$ and $y_3 = \cos(2x)$

Given that $f(x)$ is quite long and complex, we will use the superposition principle to find the total particular by combining the particular solutions of y_{p_1} and y_{p_2} which correspond to
$f_1(x) = \frac{1}{2} + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ and $f_2(x) = \cos(2x)$

assume $y_{p_1}(x) = A + Bxe^x + Cxe^{-x}$

$$y_{p_1}'(x) = Bxe^x + Be^x - Cxe^{-x} + Ce^{-x}$$
$$y_{p_1}''(x) = Bxe^x + Be^x + Be^x + Cxe^{-x} - Ce^{-x} - Ce^{-x}$$
$$= Bxe^x + 2Be^x + Cxe^{-x} - 2Ce^{-x}$$
$$y_{p_1}'''(x) = Bxe^x + Be^x + 2Be^x - Cxe^{-x} + Ce^{-x} + 2Ce^{-x}$$
$$= Bxe^x + 3Be^x - Cxe^{-x} + 3Ce^{-x}$$
$$y_{p_1}''''(x) = Bxe^x + Be^x + 3Be^x + Cxe^{-x} - Ce^{-x} - 3Ce^{-x}$$
$$= Bxe^x + 4Be^x + Cxe^{-x} - 4Ce^{-x}$$

Let's insert into our D.E. $y'''' + 3y'' - 4y = f_1(x) = -\frac{1}{2} + \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

$$Bxe^x + 4Be^x + Cxe^{-x} - 4Ce^{-x} + 3(Bxe^x + 2Be^x + Cxe^{-x} - 2Ce^{-x})$$
$$- 4(A + Bxe^x + Cxe^{-x}) = -\frac{1}{2} + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$$\cancel{(B+3B-4B)xe^x} + (4B+6B)e^x + \cancel{(C+3C-4C)xe^{-x}} + (-4C-2C)e^{-x} - 4A =$$
$$= -\frac{1}{2} + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$\Rightarrow 10Be^x - 10Ce^{-x} - 4A = \tfrac{1}{2} + \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x}$

$$\left.\begin{array}{lll} -4A = -\tfrac{1}{2} & \Rightarrow & A = \tfrac{1}{8} \\ 10B = \tfrac{1}{2} & \Rightarrow & B = \tfrac{1}{20} \\ -10C = -\tfrac{1}{2} & \Rightarrow & C = \tfrac{1}{20} \end{array}\right\} \quad \begin{aligned} y_{p_1}(x) &= A + Bxe^x + Cxe^{-x} \\ &= \tfrac{1}{8} + \tfrac{1}{20}xe^x + \tfrac{1}{20}xe^{-x} \end{aligned}$$

now to find the particular solution for $f_2(x) = \tfrac{1}{2}\cos(2x)$

$$\begin{aligned}
y_{p_2}(x) &= Dx\cos(2x) + Ex\sin(2x) \\
y'_{p_2}(x) &= -2Dx\sin(2x) + D\cos(2x) + 2Ex\cos(2x) + E\sin(2x) \\
y''_{p_2}(x) &= -4Dx\cos(2x) - 2D\sin(2x) - 2D\sin(2x) - 4Ex\sin(2x) + 2E\cos(2x) + 2E\cos(2x) \\
&= -4Dx\cos(2x) - 4D\sin(2x) - 4Ex\sin(2x) + 4E\cos(2x) \\
y'''_{p_2}(x) &= 8Dx\sin(2x) - 4D\cos(2x) - 8D\cos(2x) - 8Ex\cos(2x) - 4E\sin(2x) - 8E\sin(2x) \\
&= 8Dx\sin(2x) - 12D\cos(2x) - 8Ex\cos(2x) - 12E\sin(2x) \\
y''''_{p_2}(x) &= 16Dx\cos(2x) + 8D\sin(2x) + 24D\sin(2x) + 16Ex\sin(2x) - 8E\cos(2x) - 24E\cos(2x) \\
&= 16Dx\cos(2x) + 32D\sin(2x) + 16Ex\sin(2x) - 32E\cos(2x)
\end{aligned}$$

Let's insert into our D.E. $y'''' + 3y'' - 4y = f_2(x) = \tfrac{1}{2}\cos(2x)$

$$\begin{aligned}
&16Dx\cos(2x) + 32D\sin(2x) + 16Ex\sin(2x) - 32E\cos(2x) \\
&\quad + 3\left(-4Dx\cos(2x) - 4D\sin(2x) - 4Ex\sin(2x) + 4E\cos(2x)\right) \\
&\quad - 4\left(Dx\cos(2x) + Ex\sin(2x)\right) = \tfrac{1}{2}\cos(2x)
\end{aligned}$$

$$\cancel{(16D - 12D - 4D)x\cos(2x)} + (32D - 12D)\sin(2x) + \cancel{(16E - 12E - 4E)x\sin(2x)}$$

$$(-32E + 12E)\cos(2x) = \tfrac{1}{2}\cos(2x)$$

$$20D\sin(2x) - 20E\cos(2x) = \tfrac{1}{2}\cos(2x)$$

$$\left.\begin{array}{lll} 20D = 0 & \Rightarrow & D = 0 \\ -20E = \tfrac{1}{2} & \Rightarrow & E = -\tfrac{1}{40} \end{array}\right\} \quad \begin{aligned} y_{p_2}(x) &= Dx\cos(2x) + Ex\sin(2x) \\ &= -\tfrac{1}{40}x\sin(2x) \end{aligned}$$

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \tfrac{1}{8} + \tfrac{1}{20}xe^x + \tfrac{1}{20}xe^{-x} - \tfrac{1}{40}x\sin(2x)$$

$\therefore y(x) = a_1e^x + a_2e^{-x} + C_3\cos(2x) + C_4\sin(2x) + \tfrac{1}{8} + \tfrac{1}{20}xe^x + \tfrac{1}{20}xe^{-x} - \tfrac{1}{40}x\sin(2x)$

or $y(x) = a_1e^x + a_2e^{-x} + C_3\cos(2x) + C_4\sin(2x) + \tfrac{1}{8} + \tfrac{1}{10}x\cosh(x) - \tfrac{1}{40}x\sin(2x)$

or $Y(x) = C_1\cosh(x) + C_2\sinh(x) + C_3\cos(2x) + C_4\sin(2x) + \tfrac{1}{8} + \tfrac{1}{10}x\cosh(x) - \tfrac{1}{40}x\sin(2x)$