Homework Set No. 11 NEEP 547
Due December 13, 2013 DLH

1. (5pts) Obtain two distinct Laurent expansions for $f(z) = (3z+1)/(z^2-1)$ around $z = 1$ and tell where each converges.

2. If C is the circle $|z-1| = \frac{3}{2}$, evaluate $\int_C f(z)dz$ using the Residue Theorem for each of the following:

   a). (3pts) $f(z) = \dfrac{z+1}{z^2\,(z+2)}$ b). (3pts) $f(z) = \dfrac{z^2}{(z^2+3z+2)^2}$ c). (3pts) $f(z) = \dfrac{1}{z(z^2+6z+4)}$

3. (4pts) Show that the following function has a simple pole at the origin and find its residue there:
$$f(z) = \frac{cosh(z) - 1}{sinh(z) - z}.$$

4. Evaluate the following integrals by the method of residues:

   $a)$. (5pts) $\displaystyle\int_0^{\pi} \frac{\cos(2\theta)\,d\theta}{4\cos(\theta)+5}$ $b)$. (5pts) $\displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)\,d\theta}{a+b\cos(\theta)}$ where $0 < b < a$ $c)$.(5pts) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^6}$

5. (6pts) Evaluate the following integral by integration around suitably indented contours in the complex plane:
$$\int_0^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx \qquad \text{where a} > 0 \text{ and b} > 0.$$

6. Evaluate the integrals:

   a). (6pts) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{px}-e^{qx}}{1-e^x}\,dx$ where $0 < p < 1$ and $0 < q < 1$ b). (5pts) $\displaystyle\int_0^{\infty} \frac{\ln(x^2+1)}{1+x^2}dx$

7. Determine the Laplace inversion of the following functions:

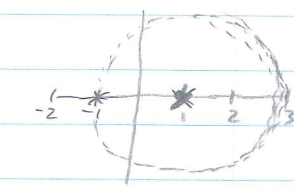
   a). (6pts) $F(s) = \dfrac{s+1}{s^2(s^2+s+1)}$ b). (6pts) $F(s) = \dfrac{1}{(s+b)\,\cosh(a\sqrt{s})}$

8. (8pts) In homework 9, problem 3, we solved for the deflection of the beam, y(x), in Fourier transform space using the following equation:
$$E\,I\frac{d^4\,y}{d\,x^4} + k\;y(x) = -p(x) \text{ where } p(x) = \begin{cases} 0 & \text{for } -\infty < x < -\ell \\ P_0(\ell+x)/\ell^2 & \text{for } -\ell < x < 0 \\ P_0(\ell-x)/\ell^2 & \text{for } 0 < x < \ell \\ 0 & \text{for } \ell < x < \infty. \end{cases}$$
and obtained the following integral:
$$y(x) = \frac{-2\,P_o}{\pi\,\ell^2}\int_0^{\infty}\left(\frac{1-\cos(\omega\ell)}{\omega^2}\right)\left(\frac{\cos(\omega x)}{EI\omega^4+k}\right)d\omega.$$
Evaluate the integral in the complex plane using the Residue theorem to obtain the complete solution for y(x).

1.) Obtain two distinct Laurent expansions for $f(z) = \frac{3z+1}{z^2-1}$ around $z=1$ and tell where each converges.

$$f(z) = \frac{3z+1}{z^2-1} = \frac{3z+1}{(z+1)(z-1)} = \frac{2}{z-1} + \frac{1}{z+1}$$ poles at 1 and -1

we will two expansions



(a) for $0 < |z-1| < 2$

$$f(z) = \frac{2}{z-1} + \frac{1}{z+1} = \frac{2}{z-1} + \frac{1}{z-1+2} = \frac{2}{z-1} + \left(\frac{1}{2}\right)\left(\frac{1}{1+\left(\frac{z-1}{2}\right)}\right)$$

less than 1 in the interval, use $\frac{1}{1-x} = 1+x+x^2+x^3\ldots$

$$= \frac{2}{z-1} + \left(\frac{1}{2}\right)\left(1 - \left(\frac{z-1}{2}\right) + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \left(\frac{z-1}{2}\right)^4 - \cdots\right)$$

$$= \frac{2}{z-1} + \left(\frac{1}{2}\right)\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{2}\right)^n$$

(b) for $|z-1| > 2$

$$f(z) = \frac{2}{z-1} + \frac{1}{z+1} = \frac{2}{z-1} + \frac{1}{z-1+2} = \frac{2}{z-1} + \frac{1}{(z-1)\left(1+\frac{2}{z-1}\right)}$$

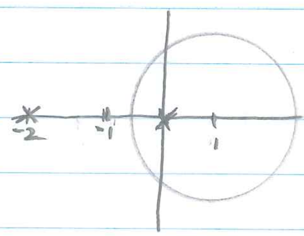
less than 1 in the interval, use $\frac{1}{1-x} = 1+x+x^2+\cdots$

$$= \frac{2}{z-1} + \left(\frac{1}{z-1}\right)\left(1 - \left(\frac{2}{z-1}\right) + \left(\frac{2}{z-1}\right)^2 - \left(\frac{2}{z-1}\right)^3 + \left(\frac{2}{z-1}\right)^4 + \cdots\right)$$

$$= \frac{2}{z-1} + \left(\frac{1}{z-1}\right)\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z-1}\right)^n$$

2). If C is the circle $|z-1| = 3/2$, evaluate $\int_C f(z)\,dz$ using the Residue Theorem for each of the following:

(a) $f(z) = \frac{z+1}{z^2(z+2)}$ we have poles at $z=-2$ and a second order pole at $z=0$



- the second order pole is in the circle $(z=0)$
- the simple pole $(z-2)$ is not

$\therefore \int_C f(z)\,dz = 2\pi i \, Res(z=0)$

$$Res(z=0) = \lim_{z\to 0} \left(\frac{1}{2-1}\right)\frac{d}{dz}\left(z^2 \frac{z+1}{z^2(z+2)}\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z+1}{z+2}\right) = \lim_{z\to 0}\left((z+2)^{-1} - (z+1)(z+2)^{-2}\right)$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \qquad \therefore \int_C f(z)\,dz = (2\pi i)\left(\frac{1}{4}\right) = \frac{\pi}{2} i$$

(b) $f(z) = \frac{z^2}{(z^2+3z+2)^2} = \frac{z^2}{(z+1)^2(z+2)^2}$ we have second order poles at $z=-1$ and $z=-2$. neither pole is within the circle

$\int_C f(z)\,dz = 0.$

(c) $f(z) = \frac{1}{z(z^2+6z+4)} = \frac{1}{z(z-(-3+\sqrt{5}))(z-(-3-\sqrt{5}))}$

- we have poles at $z=0$, $z=-3+\sqrt{5}$, $z=-3-\sqrt{5}$
- only the pole at $z=0$ lies within the circle

$\int_C f(z)\,dz = (2\pi i)\, Res(0)$

$$Res(0) = \lim_{z\to 0} z\left(\frac{1}{z(z-(-3+\sqrt{5}))(z-(-3-\sqrt{5}))}\right) = \frac{1}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{1}{9-5} = \frac{1}{4}$$

$\int_C f(z)\,dz = (2\pi i)\left(\frac{1}{4}\right) = \frac{\pi}{2} i$

3) Show that the following function has a simple pole at the origin and find its residue there:

$$f(z) = \frac{\cosh(z) - 1}{\sinh(z) - z}.$$

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$\therefore \cosh(z) - 1 = \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \quad \text{and} \quad \sinh(z) - z = \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$\text{thus } f(z) = \frac{\cosh(z) - 1}{\sinh(z) - z} = \frac{\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots}{\frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots} = \frac{1}{z}\left[\frac{\frac{1}{2!} + \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots}{\frac{1}{3!} + \frac{z^2}{5!} + \frac{z^4}{7!} + \cdots}\right]$$

simple pole at $z = 0$

$$\operatorname{Res}(z=0) = \lim_{z \to 0}(z f(z)) = \left[\frac{\frac{1}{2!}}{\frac{1}{3!}}\right] = \frac{\frac{1}{2}}{\frac{1}{2\cdot 3}} = \frac{\frac{1}{2}}{\frac{1}{6}} = \frac{6}{2}$$

$$= 3$$

4a) Evaluate the following integral by the method of Residues:

$$\int_0^{\pi} \frac{\cos(2\theta)}{4\cos(\theta)+5}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\cos(2\theta)}{4\cos(\theta)+5}\, d\theta$$

$z = re^{i\theta}$; $r=1$, $z = e^{i\theta}$; $dz = ie^{i\theta}d\theta$; $d\theta = \frac{-i\,dz}{z}$

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}(z+z^{-1})$$

$$\cos(2\theta) = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{1}{2}(z^2+z^{-2})$$

$$\frac{1}{2}\int_c \frac{\frac{1}{2}(z^2+z^{-2})(-i\frac{dz}{z})}{4(\frac{1}{2})(z+z^{-1})+5} = \frac{-i}{4}\int_c \frac{(z^2+z^{-2})}{(2(z+z^{-1})+5)}\left(\frac{dz}{z}\right) = \frac{-i}{4}\int_c \left(\frac{1}{z^2}\right)\left(\frac{z^4+1}{2(z+z^{-1})+5}\right)\left(\frac{dz}{z}\right)$$

$$= \left(\frac{-i}{4}\right)\int_c \left(\frac{1}{z^2}\right)\left(\frac{(z^4+1)\,z}{2(z^2+1)+5z}\right)\left(\frac{dz}{z}\right) = \left(\frac{-i}{8}\right)\int_c \left(\frac{1}{z^2}\right)\left(\frac{z^4+1}{z^2+1+\frac{5}{2}z}\right)dz$$

$$= \left(\frac{-i}{8}\right)\int_c \frac{(z^4+1)}{(z^2)(z+2)(z+\frac{1}{2})}\,dz$$

the 2nd order pole $z=0$ and the simple pole $z=-\frac{1}{2}$ lie within the unit circle.

$$\therefore I = \left(-\frac{i}{8}\right)(2\pi i)\left(\text{Res}(0) + \text{Res}(z=-\tfrac{1}{2})\right)$$

$$\text{Res}(-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}} \left((z+\tfrac{1}{2})\left(\frac{1+z^4}{(z^2)(z+2)\cancel{(z+\frac{1}{2})}}\right)\right) = \frac{1+(-\frac{1}{2})^4}{(-\frac{1}{2})^2(2-\frac{1}{2})} = \frac{1+\frac{1}{16}}{(\frac{1}{4})(\frac{3}{2})} = \frac{\frac{17}{16}}{\frac{3}{8}} = \frac{17}{6}$$

$$\text{Res}(0) = \lim_{z\to 0} \frac{1}{(2-1)!}\frac{d}{dz}\left((z^2)\left(\frac{1+z^4}{(z)^2(z+2)(z+\frac{1}{2})}\right)\right) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{1+z^4}{(z+2)(z+\frac{1}{2})}\right)$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{1+z^4}{z^2+\frac{5}{2}z+1}\right) = \lim_{z\to 0}\frac{(4z^3)(z^2+\frac{5}{2}z+1)-(1+z^4)(2z+\frac{5}{2})}{(z^2+\frac{5}{2}z+1)^2}$$

$$= -\frac{(\frac{5}{2})}{(1)^2} = -\frac{5}{2}$$

$$I = \left(-\frac{i}{8}\right)(2\pi i)\left(\left(-\tfrac{5}{2}\right)+\left(\tfrac{17}{6}\right)\right) = \left(\frac{\pi}{4}\right)\left(\frac{17}{6}-\frac{15}{6}\right) = \left(\frac{\pi}{4}\right)\left(\frac{1}{3}\right) = \underline{\underline{\frac{\pi}{12}}}$$

4b). Evaluate the following integral by the method of Residues:

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta \quad \text{where } 0<b<a$$

we shall integrate around the unit circle: $z=re^{i\theta}$; $r=1$; $z=e^{i\theta}$; $dz=ie^{i\theta}d\theta$; $d\theta=-\frac{i\,dz}{z}$

$$\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{1}{2}\left(z+\frac{1}{z}\right);\quad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$I=\int_{C_1}\frac{\left(-\frac{i}{2}\left(z-\frac{1}{z}\right)\right)^2}{a+b\left(\frac{1}{2}\right)\left(z+\frac{1}{z}\right)}\left(-\frac{i\,dz}{z}\right)=\frac{i}{4}\int_{C_1}\frac{\left(z-\frac{1}{z}\right)^2}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)}\left(\frac{dz}{z}\right)$$

$$=\frac{i}{4}\int_{C_1}\frac{z^2-2+\frac{1}{z^2}}{z\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)}dz=\frac{i}{4}\int_{C_1}\frac{z^2-2+\frac{1}{z^2}}{\left(za+\frac{b}{2}(z^2+1)\right)}dz$$

$$=\frac{i}{2b}\int_C\frac{z^2-2+\frac{1}{z^2}}{z^2+2(a/b)z+1}dz=\frac{i}{2b}\left(\underset{①}{\int_{C_1}\frac{z^2}{z^2+2(a/b)+1}dz}-2\underset{②}{\int_{C_1}\frac{dz}{z^2+2(a/b)+1}}+\underset{③}{\int_{C_1}\frac{dz}{z^2(z^2+2(a/b)+1)}}\right)$$

① $\int_{C_1}\frac{z^2}{z^2+2(a/b)z+1}dz$ we have singularities at $z=-\frac{a}{b}\pm\sqrt{(a/b)^2-1}$

$z_1=-a/b+\sqrt{(a/b)^2-1}$, $z_2=-a/b-\sqrt{(a/b)^2-1}$

z_1 is in the unit circle and z_2 is out of the unit circle

$$\operatorname{Res}(z=z_1)=\lim_{z\to z_1}\left(\cancel{(z-z_1)}\left(\frac{z^2}{\cancel{(z-z_1)}(z-z_2)}\right)\right)=\frac{z_1^2}{z_1-z_2}=\frac{\left(-a/b+\sqrt{(a/b)^2-1}\right)^2}{-a/b+\sqrt{(a/b)^2-1}+(a/b)+\sqrt{(a/b)^2-1}}$$

$$=\frac{\left(-a/b+\sqrt{(a/b)^2-1}\right)^2}{2\sqrt{(a/b)^2-1}}$$

② $\int_{C_1}\frac{dz}{z^2+2(a/b)z+1}$ we have the same singularities as in ① above

$$\operatorname{Res}(z=z_1)=\lim_{z\to z_1}\left(\cancel{(z-z_1)}\left(\frac{1}{\cancel{(z-z_1)}(z-z_2)}\right)\right)=\frac{1}{z_1-z_2}=\frac{1}{-a/b+\sqrt{(a/b)^2-1}+a/b+\sqrt{(a/b)^2-1}}$$

$$=\frac{1}{2\sqrt{(a/b)^2-1}}$$

③ $\int_{C_1}\frac{dz}{z^2(z^2+2(a/b)z+1)}$ we have poles at $z=-a/b\pm\sqrt{(a/b)^2-1}$ and at $z=0$ (2nd order)

$z_1=-a/b+\sqrt{(a/b)^2-1}$; $z_2=-a/b-\sqrt{(a/b)^2-1}$; $z_3=0$ (2nd order pole).

z_1 and z_3 are in the unit circle and z_2 is out of the unit circle

$$\operatorname{Res}(z=z_1) = \lim_{z\to z_1}\left((z-z_1)\left(\frac{1}{z^2(z-z_1)(z-z_2)}\right)\right) = \frac{1}{z_1^2(z_1-z_2)} = \left(\frac{1}{(-a/b+\sqrt{(a/b)^2-1})}\right)^2\left(\frac{1}{2\sqrt{(a/b)^2-1}}\right)$$

$$= \left(\frac{1}{(a/b)^2-2(a/b)\sqrt{(a/b)^2-1}+(\frac{a}{b})^2-1}\right)\left(\frac{1}{2\sqrt{(a/b)^2-1}}\right)$$

$$= \left(\frac{1}{2(a/b)^2-1-2(a/b)\sqrt{(a/b)^2-1}}\right)\left(\frac{1}{2\sqrt{(a/b)^2-1}}\right)$$

$$\operatorname{Res}(0) = \lim_{z\to 0}\left(\frac{d}{dz}\left(z^2\left(\frac{1}{z^2(z-z_1)(z-z_2)}\right)\right)\right) = \lim_{z\to 0}\left(\frac{d}{dz}\left(\frac{1}{z^2-(z_1+z_2)z+z_1z_2}\right)\right)$$

$$= \lim_{z\to 0}\left(-\frac{(2z-(z_1+z_2))}{(z^2-(z_1+z_2)z+z_1z_2)^2}\right) = \frac{(z_1+z_2)}{(z_1z_2)^2} = \frac{-a/b+\sqrt{(a/b)^2-1}-(a/b)-\sqrt{(a/b)^2-1}}{((-a/b)+\sqrt{(a/b)^2-1})((-a/b)-\sqrt{(a/b)^2-1})^2}$$

$$= \frac{-2(a/b)}{((a/b)^2-(a/b)^2+1)^2} = -2(a/b)$$

Lets add differently.

$$I = (2\pi i)\sum_n \operatorname{Res}(z_n) = (2\pi i)\left(\frac{i}{2b}\right)(1+2+3)$$

$$= \left(-\frac{\pi}{b}\right)\left(\frac{z_1^2}{z_1-z_2}-\frac{2}{z_1-z_2}+\frac{1}{z_1^2(z_1-z_2)}+\frac{(z_1+z_2)}{(z_1z_2)^2}\right)$$

where $z_1 = -a/b+\sqrt{(a/b)^2-1}$, $z_2 = -a/b-\sqrt{(a/b)^2-1}$, $z_1-z_2 = 2\sqrt{(a/b)^2-1}$, $z_1+z_2 = -2(a/b)$

$$I = \left(-\frac{\pi}{b}\right)\left(\frac{z_1^2}{z_1-z_2}-\frac{2}{z_1-z_2}+\frac{1}{z_1^2(z_1-z_2)}+\frac{(z_1+z_2)}{(z_1z_2)^2}\right)$$ note $(z_1z_2)^2=1$

$$= \left(-\frac{\pi}{b}\right)\left(\frac{z_1^2-2}{z_1-z_2}+\frac{z_2^2}{(z_1z_2)^2(z_1-z_2)}+\frac{(z_1+z_2)(z_1-z_2)}{(z_1z_2)^2(z_1-z_2)}\right)$$

$$= \left(-\frac{\pi}{b}\right)\left(\frac{z_1^2-2+\cancel{z_2^2}+z_1^2-\cancel{z_2^2}}{z_1-z_2}\right) = \left(-\frac{\pi}{b}\right)\left(\frac{2z_1^2-2}{z_1-z_2}\right) = \left(-\frac{2\pi}{b}\right)\left(\frac{z_1^2-1}{z_1-z_2}\right)$$

$$= \left(-\frac{2\pi}{b}\right)\left(\frac{(-a/b+\sqrt{(a/b)^2-1})^2-1}{2\sqrt{(a/b)-1}}\right) = \left(-\frac{2\pi}{b}\right)\left(\frac{(-a/b)^2-2(a/b)\sqrt{(a/b)^2-1}+(a/b)^2-1-1}{2\sqrt{(a/b)-1}}\right)$$

$$= \left(-\frac{2\pi}{b}\right)\left(\frac{2(a/b)^2-2-2(a/b)\sqrt{(a/b)^2-1}}{2\sqrt{(a/b)^2-1}}\right) = \left(-\frac{2\pi}{b}\right)\left(\frac{2}{2}\right)\left(\frac{(a/b)^2-1-(a/b)\sqrt{(a/b)^2-1}}{\sqrt{(a/b)^2-1}}\right)$$

$$= \left(-\frac{2\pi}{b}\right)\left(\frac{(a/b)^2-1}{\sqrt{(a/b)^2-1}}-\frac{(a/b)\sqrt{(a/b)^2-1}}{\sqrt{(a/b)^2-1}}\right) = \left(-\frac{2\pi}{b}\right)\left(\sqrt{(a/b)^2-1}-(a/b)\right)$$

$$= \left(-\frac{2\pi}{b}\right)\left(-\frac{1}{b}\right)\left(a-b\sqrt{(a/b)^2-1}\right) = \left(\frac{2\pi}{b^2}\right)\left(a-\sqrt{a^2-b^2}\right)$$

4c). Evaluate the integral $\int_{-\infty}^{\infty} \frac{x^2 dx}{1+x^6}$ note the denominator is more than two orders of magnitude large than the Numerator.

we take the integral to the complex plane

$\int_{-\infty}^{\infty} \frac{z^2}{1+z^6} dz$ and note $I =$ sum of the residues in the UHP.

we need to find the roots of $1+z^6=0$ $z^6=(-1) \Rightarrow z=(-1)^{1/6}$

$z_k = \sqrt[n]{r}\, e^{\frac{\theta+2\pi k i}{n}}$ $k=0,1,2,\dots(n-1)$ for our case $n=6, \theta=\pi$ $z=(1)e^{\frac{\pi+2\pi k}{6}i}$ $k=0,1,2,\dots 5$

$z_1 = e^{i\pi/6}$, $z_2 = e^{i\frac{3\pi}{6}}$, $z_3 = e^{i5\pi/6}$, $z_4 = e^{i\frac{7\pi}{6}}$, $z_5 = e^{i\frac{9\pi}{6}}$, $z_6 = e^{i\frac{11\pi}{6}}$

z_1, z_2 and z_3 are in the UHP

$$z_1 = e^{i\pi/6} = \cos(\pi/6) + i\sin(\pi/6) = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$z_2 = e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

$$z_3 = e^{i5\pi/6} = \cos(5\pi/6) + i\sin(5\pi/6) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$I = 2\pi i\,(\mathrm{Res}(z_1) + \mathrm{Res}(z_2) + \mathrm{Res}(z_3))$

For simple poles $\mathrm{Res}(a) = \left.\frac{N(z)}{D'(z)}\right|_{z=a}$

For our case $F(z) = \frac{z^2}{1+z^6}$, $\mathrm{Res}(z_n) = \frac{z_n^2}{6 z_n^5}$

$\mathrm{Res}(z_n) = \frac{1}{6 z_n^3}$

$$I = 2\pi i\left(\tfrac{1}{6}\right)\left(\left(\frac{1}{\frac{\sqrt{3}}{2}+\frac{i}{2}}\right)^3 + \left(\frac{1}{i}\right)^3 - \left(\frac{1}{\frac{\sqrt{3}}{2}-\frac{i}{2}}\right)^3\right)$$

$$= \left(\frac{2\pi i}{6}\right)\left(\left(\frac{2}{\sqrt{3}+i}\right)^3 + (-i)^3 - \left(\frac{2}{\sqrt{3}-i}\right)^3\right) = \left(\frac{2\pi i}{6}\right)\left(\left(\frac{\sqrt{3}-i}{2}\right)^3 + (-i)^3 - \left(\frac{\sqrt{3}+i}{2}\right)^3\right)$$

$$= \left(\frac{\pi i}{3}\right)\left(\frac{1}{2^3}\right)\left((\sqrt{3}-i)^3 + (-2i)^3 - (\sqrt{3}+i)^3\right)$$

$$= \left(\frac{\pi i}{3}\right)\left(\frac{1}{8}\right)\left((-8i) + \cancel{(8i)} - \cancel{(8i)}\right) = \left(\frac{\pi i}{3}\right)\left(\frac{1}{8}\right)(-8i)$$

$= \pi/3$

Z

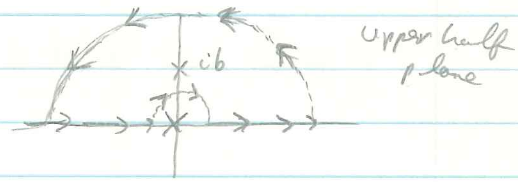
5. Evaluate the following integral by integration around a suitable indented contour in the complex plane:

$$\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,dx \qquad \text{where } a>0 \text{ and } b>0$$

$$\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \frac{1}{2} I_m \int_{-\infty}^{\infty} \frac{e^{iaz}}{z(z^2+b^2)}\,dz$$

we have singularites at $z=0$ and $z=\pm ib$

Our contour of integration is



we have $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iaz}}{z(z^2+b^2)}\,dz = \pi i \,\mathrm{Res}(z=0) + 2\pi i\, \mathrm{Res}(z=ib)$

let's use $\displaystyle \mathrm{Res}(z_n) = \frac{N(z)}{D'(z)} = \frac{e^{iaz}}{(z^2+b^2)+2z^2}$

$$\pi i\,\mathrm{Res}(0) = (\pi i)\left(\frac{1}{(0+b^2)+0}\right) = \frac{\pi i}{b^2}$$

$$2\pi i\,\mathrm{Res}(ib) = (2\pi i)\left(\frac{e^{-ab}}{(ib)^2+b^2+2(ib)^2}\right) = \frac{2\pi i e^{-ab}}{b^2-b^2-2b^2} = -\frac{\pi i e^{-ab}}{b^2}$$

$$I = \frac{1}{2} I_m\left(\frac{\pi i}{b^2} - \frac{\pi i e^{-ab}}{b^2}\right) = \underline{\underline{\frac{\pi}{2b^2}\left(1-e^{-ab}\right)}}$$

If we use our usual way of finding the residue, we have

$$\mathrm{Res}(0) = \lim_{z\to 0} \frac{\cancel{z}\, e^{iaz}}{\cancel{z}(z-ib)(z+ib)} = \frac{1}{(-ib)(ib)} = \frac{1}{b^2}$$

$$\mathrm{Res}(ib) = \lim_{z\to ib}\left(\frac{\cancel{(z-ib)}\, e^{iaz}}{z\cancel{(z-ib)}(z+ib)}\right) = \frac{e^{ia(ib)}}{(ib)(ib+ib)} = \frac{e^{-ab}}{(ib)(2ib)} = -\frac{e^{-ab}}{2b^2}$$

$$I = \frac{1}{2} I_m\left(\pi i\left(\frac{1}{b^2}\right) + 2\pi i\left(-\frac{e^{ab}}{2b^2}\right)\right) = \frac{1}{2} I_m\left(\frac{\pi i}{b^2} - \frac{\pi i e^{-ab}}{b^2}\right)$$

$$= \frac{\pi}{2b^2}\left(1-e^{-ab}\right)$$

6a) Evaluate $\int_{-\infty}^{\infty} \frac{e^{px}-e^{qx}}{1-e^{x}}\,dx$ where $0<p<1$ and $0<q<1$

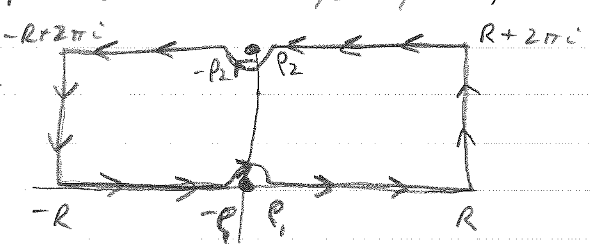
$$\int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^{x}}\,dx = \underbrace{\int_{-\infty}^{\infty}\frac{e^{px}}{1-e^{x}}\,dx}_{①} - \underbrace{\int_{-\infty}^{\infty}\frac{e^{qx}}{1-e^{x}}\,dx}_{②}$$

we will solve the first integral and deduce the result of the second

$$\int_{-\infty}^{\infty}\frac{e^{px}}{1-e^{x}}\,dx = \int_{-\infty}^{\infty}\frac{e^{pz}}{1-e^{z}}\,dz$$

the poles are at $z=0, 2\pi i, 4\pi i, \dots$

we will use the following contour



$$\int_C \frac{e^{pz}}{1-e^{z}}\,dz = \int_{-R}^{-\rho}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{C_{\rho_1}}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{\rho_1}^{R}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{R}^{R+2\pi i}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{R+2\pi i}^{\rho_2}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{C_{\rho_2}}\frac{e^{pz}}{1-e^{z}}\,dz$$
$$+ \int_{-\rho_2}^{-R+2\pi i}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{-R+2\pi i}^{-R}\frac{e^{pz}}{1-e^{z}}\,dz = 2\pi i \sum_n Res(z_n)$$

$$= \int_{-R}^{R}\frac{e^{px}}{1-e^{x}}\,dx - (\pi i)Res(0) + \int_{R}^{R+2\pi i}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{R+2\pi i}^{R-2\pi i}\frac{e^{pz}}{1-e^{z}}\,dz - (\pi i)Res(2\pi i)$$
$$+ \int_{-R+2\pi i}^{-R}\frac{e^{pz}}{1-e^{z}}\,dz = 2\pi i \sum_n Res(z_n)$$

let $t = z-2\pi i$

$dt = dz$ $\quad z=-R+2\pi i \quad t=-R$

$\qquad\qquad z = R+2\pi i \quad t = R$

$$\int_C \frac{e^{pz}}{1-e^{z}}\,dz = \int_{-R}^{R}\frac{e^{px}}{1-e^{x}}\,dx - (\pi i)Res(0) + \int_{R}^{R+2\pi i}\frac{e^{pz}}{1-e^{z}}\,dz + \int_{R}^{-R}\frac{e^{p(t+2\pi i)}}{1-e^{(t+2\pi i)}}\,dt - (\pi i)\,Res(2\pi i)$$
$$+ \int_{-R+2\pi i}^{-R}\frac{e^{pz}}{1-e^{z}}\,dz = 2\pi i \sum_n Res(z_n)$$

vanish
as $R\to\infty$
see class notes

$$= \int_{-\infty}^{\infty}\frac{e^{px}}{1-e^{x}}\,dx - e^{2p\pi i}\int_{-\infty}^{\infty}\frac{e^{pt}}{1-e^{t}\underbrace{e^{2\pi i}}_{1}}\,dt - (\pi i)Res(0) - (\pi i)Res(2\pi i) = 2\pi i\sum_n Res(z_n)$$

$$\Rightarrow \left(1-e^{p2\pi i}\right)\int_{-\infty}^{\infty}\frac{e^{px}}{1-e^{x}}\,dx = 2\pi i \sum_n Res(z_n) + (\pi i)Res(0) + (\pi i)Res(2\pi i)$$

$$\int_{-\infty}^{\infty}\frac{e^{px}}{1-e^{x}}\,dx = \left(\frac{1}{1-e^{p2\pi i}}\right)\left[2\pi i\sum_n Res(z_n) + (\pi i)Res(0) + (\pi i)\,Res(2\pi i)\right]$$

There are no poles inside the contour $\therefore$ $\text{Res}(z_n) = 0$

$$\text{Res}(0) = \left.\frac{N(z)}{D'(z)}\right|_{z=0} = \left.\frac{e^{pz}}{-e^{z}}\right|_{z=0} = \frac{e^{0}}{-e^{0}} = -1$$

$$\text{Res}(2\pi i) = \left.\frac{N(z)}{D'(z)}\right|_{z=2\pi i} = \left.\frac{e^{pz}}{-e^{z}}\right|_{z=2\pi i} = \frac{e^{p2\pi i}}{-e^{2\pi i}} = -e^{p2\pi i}$$

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx = \left(\frac{1}{1-e^{p2\pi i}}\right)\left[\pi i(-1) + \pi i(-e^{p2\pi i})\right]$$

$$= \left(\frac{-\pi i}{1-e^{p2\pi i}}\right)\left(1+e^{p2\pi i}\right) = -\pi i\left(\frac{1+e^{p2\pi i}}{1-e^{p2\pi i}}\right) = \left(\frac{e^{p\pi i}+e^{-p\pi i}}{e^{p\pi i}-e^{p\pi i}}\right)$$

$$= (-\pi i)\left(\frac{e^{p\pi i}+e^{-p\pi i}}{2}\right)\left(\frac{2}{e^{-p\pi i}-e^{p\pi i}}\right)$$

$$= \pi\underbrace{\left(\frac{e^{p\pi i}+e^{-p\pi i}}{2}\right)}_{\cos p\pi}\underbrace{\left(\frac{2i}{e^{p\pi i}-e^{-p\pi i}}\right)}_{1/\sin p\pi} = \pi\,\frac{\cos\pi p}{\sin\pi p}$$

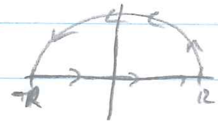
$$\int_{-\infty}^{\infty} \frac{e^{px}}{1-e^{x}}\,dx = \pi\cot(\pi p)$$

we infer from the above analysis $\displaystyle\int_{-\infty}^{\infty} \frac{e^{qx}}{1-e^{x}}\,dx = \pi\cot(\pi q)$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{px}-e^{qx}}{1-e^{x}}\,dx = \pi\cot(\pi p) - \pi\cot(\pi q)$$

$$= \pi\left(\cot(\pi p) - \cot(\pi q)\right)$$

6b). Evaluate $\int_0^\infty \frac{\ln(x^2+1)}{1+x^2}\,dx$



of the function $\frac{\ln(z+i)}{(z^2+1)}$

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \int_0^\infty \frac{\ln(z^2+1)}{z^2+1}\,dz = \int_0^\infty \frac{\ln((z+i)(z-i))}{z^2+1}\,dz$$

$$= \int_0^\infty \frac{\ln(z+i)}{z^2+1}\,dz + \int_0^\infty \frac{\ln(z-i)}{z^2+1}\,dz$$

$$= \underbrace{\frac{1}{2}\int_{-\infty}^\infty \frac{\ln(z+i)}{z^2+1}\,dz}_{I_1} + \underbrace{\frac{1}{2}\int_{-\infty}^\infty \frac{\ln(z-i)}{z^2+1}\,dz}_{I_2}$$

we have poles at $z = i$ and $-i$

$$I_1 = \frac{1}{2}\int_C \frac{\ln(z+i)}{z^2+1}\,dz = \frac{1}{2}\int_C \frac{\ln(z+i)}{(z+i)(z-i)}\,dz$$

use the UHP for pole at $z = i$

$$= \left(\tfrac{1}{2}\right)(2\pi i)\,\mathrm{Res}(i) = \pi i \lim_{z\to i}\left(\frac{(z-i)\ln(z+i)}{(z-i)(z+i)}\right) = (\pi i)\,\frac{\ln(2i)}{2i} = \frac{\pi}{2}\ln(2i)$$

$$I_2 = \tfrac{1}{2}\int_C \frac{\ln(z-i)}{z^2+1}\,dz = \tfrac{1}{2}\int_C \frac{\ln(z-i)}{(z+i)(z-i)}\,dz$$

use the LHP for pole at $z = -i$

$$= \left(\tfrac{1}{2}\right)(-2\pi i)\,\mathrm{Res}(-i) = -\pi i \lim_{z\to -i}\left(\frac{(z+i)\ln(z-i)}{(z+i)(z-i)}\right) = (-\pi i)\left(\frac{\ln(-2i)}{-2i}\right) = \frac{\pi}{2}\ln(-2i)$$

$$I = I_1 + I_2 = \tfrac{\pi}{2}\left(\ln(2i) + \ln(-2i)\right) = \tfrac{\pi}{2}\left(\ln(2) + \ln(i) + \ln(2) + \ln(-i)\right)$$

$$= \left(\tfrac{\pi}{2}\right)(2\ln(2)) + \left(\tfrac{\pi}{2}\right)\left(\ln(i) + \ln\left(\tfrac{1}{i}\right)\right) = \pi\ln(2) + \frac{\pi}{2}\ln\left(\frac{i}{i}\right)$$

$$= \pi\ln(2) + \tfrac{\pi}{2}\ln(1) = \underline{\underline{\pi\ln(2)}}$$

7a) $F(s) = \frac{s+1}{s^2(s^2+s+1)}$ we have a 2^{nd} order pole at $s=0$ and simple poles at $s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\mathcal{L}^{-1}(F(s)) = \text{Res}(F(s)e^{st})$ we will use partial Fractions to help

$$\frac{s+1}{s^2(s^2+s+1)} = \frac{(A+Bs)}{s^2} + \frac{C+Ds}{s^2+s+1} \qquad A=1,\ B=0,\ C=-1,\ D=0$$

$$= \frac{1}{s^2} - \frac{1}{s^2+s+1}$$

$$\mathcal{L}^{-1}(F(s)) = \text{Res}(s=0) - \text{Res}\left(s=-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) - \text{Res}\left(s=-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)$$

$$\text{Res}(s=0) = \lim_{s\to 0} \frac{1}{2-1} \frac{d}{ds}\left((s^2)\left(\frac{e^{st}}{s^2}\right)\right) = \lim_{s\to 0} \frac{d}{ds}(e^{st})$$

$$= \lim_{s\to 0} te^{st} = t$$

$$\text{Res}\left(s=-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) = \lim_{s\to -\frac{1}{2}-\frac{\sqrt{3}}{2}i} \left(\cancel{\left(s-\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right)} \frac{e^{st}}{\cancel{\left(s-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)}\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\right)$$

$$= \lim_{s\to -\frac{1}{2}-\frac{\sqrt{3}}{2}i} \left(\frac{e^{st}}{\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\right) = \frac{e^{-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}}{\cancel{-\frac{1}{2}}-\frac{\sqrt{3}}{2}i+\cancel{\frac{1}{2}}-\frac{\sqrt{3}}{2}i}$$

$$= e^{-\frac{1}{2}t} e^{-\frac{\sqrt{3}}{2}it} / (-\sqrt{3}i)$$

$$\text{Res}\left(s=-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = \lim_{s\to -\frac{1}{2}+\frac{\sqrt{3}}{2}i} \left(\cancel{\left(s-\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)} \frac{e^{st}}{\cancel{\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\left(s-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)}\right)$$

$$= \lim_{s\to -\frac{1}{2}+\frac{\sqrt{3}}{2}i} \left(\frac{e^{st}}{\left(s-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)}\right) = \frac{e^{-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)t}}{\cancel{-\frac{1}{2}}+\frac{\sqrt{3}}{2}i+\cancel{\frac{1}{2}}+\frac{\sqrt{3}}{2}i}$$

$$= e^{-\frac{1}{2}t} e^{\frac{\sqrt{3}}{2}it} / \sqrt{3}i$$

$$\mathcal{L}^{-1}(F(s)) = t - \left(e^{-t/2} e^{-\frac{\sqrt{3}}{2}it} / -\sqrt{3}i + e^{-t/2} e^{\frac{\sqrt{3}}{2}it} / \sqrt{3}i\right)$$

$$= t - \frac{2}{\sqrt{3}} e^{-t/2} \left(\frac{e^{\frac{\sqrt{3}}{2}it} - e^{-\frac{\sqrt{3}}{2}it}}{2i}\right)$$

$$= t - \frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t$$

7b) $F(s) = \dfrac{1}{(s+b)\cosh(a\sqrt{s})}$ we need to find the poles. one pole is at $s=-b$

now to find the poles for $\cosh(a\sqrt{s})$

$\cosh(a\sqrt{s}) = \cos(ia\sqrt{s})$, so we are looking for the poles of

$\cos(ia\sqrt{s})$ $\therefore$ $ia\sqrt{s} = \pm\frac{n}{2}\pi$ $n=1,3,5,\ldots$

or $\sqrt{s} = \pm\dfrac{(2n+1)\pi}{2ai}$ $n=0,1,2,3,\ldots$

$s = -\left(\dfrac{(2n+1)\pi}{2a}\right)^2$ $n=0,1,2,3,\ldots$

$\mathcal{L}^{-1}\{F(s)\} = \sum_n \text{Res}\,(F(s)e^{st})$

$$\text{Res}(-b) = \lim_{s\to -b}\left((s+b)\left(\frac{e^{st}}{(s+b)\cosh(a\sqrt{s})}\right)\right) = \frac{e^{-bt}}{\cosh(a\sqrt{-b})} = \frac{e^{-bt}}{\cosh(ia\sqrt{b})} = \frac{e^{-bt}}{\cos(i\,i\,a\sqrt{b})}$$
$$= \frac{e^{-bt}}{\cos(a\sqrt{b})}$$

we will now use the following procedure to find the residue: $\text{Res}(z_n) = \left.\dfrac{N(z)}{D'(z)}\right|_{z_n}$

$$\frac{N(s)}{D(s)} = \frac{e^{st}}{(s+b)\cosh(a\sqrt{s})}, \quad \frac{N(s)}{D'(s)} = \frac{e^{st}}{\cosh(a\sqrt{s}) + (s+b)(\frac{a}{2}s^{-1/2})\sinh(a\sqrt{s})}$$

$$\text{Res}\left(-\left(\frac{(2n+1)\pi}{2a}\right)^2\right)_{n=0,1,2,\ldots} = \lim_{s\to z_n}\left(\frac{e^{st}}{\cosh(a\sqrt{s}) + (s+b)(a/2\, s^{-1/2})\sinh(a\sqrt{s})}\right)$$

$$= \frac{e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}}{\cosh\left(a\frac{(2n+1)\pi}{2a}\right) + \left(-\left(\frac{(2n+1)\pi}{2a}\right)^2 + b\right)(a/2)\left(\frac{2a}{(2n+1)\pi i}\right)\sinh\left(a\frac{(2n+1)\pi}{2a}\right)}$$

$\cosh\left(\frac{(2n+1)\pi}{2}\right) = 0$

$\sinh\left(\frac{(2n+1)}{2}\pi\right)$
$= i\sin\left((2n+1)\frac{\pi}{2}\right)$
$= i(-1)^n$

$$= \frac{e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}}{\left(-\left(\frac{(2n+1)\pi}{2a}\right)^2 + b\right)\left(\frac{a^2}{(2n+1)\pi i}\right)\sinh\left(\frac{(2n+1)}{2}\pi\right)}$$

$$= \frac{e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}(4)(2n+1)\pi}{(4a^2b-(2n+1)^2\pi^2)(i)(i\sin((2n+1)\pi/2))} = \frac{(4\pi)(2n+1)e^{-\left(\frac{(2n+1)\pi}{2a}\right)^2 t}}{(4a^2b-(2n+1)^2\pi^2)(-1)(-1)^n}$$

$$= -\frac{(-1)^n(4\pi)(2n+1)e^{-\left(\frac{(2n+1)}{2a}\pi\right)^2 t}}{(4a^2b-(2n+1)^2\pi^2)}$$

thus $$\mathcal{L}^{-1}\left\{\frac{1}{(s+b)\cosh(a\sqrt{s})}\right\} = \frac{e^{-bt}}{\cos(a\sqrt{b})} - 4\pi\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)e^{-\left(\frac{(2n+1)}{2a}\pi\right)^2 t}}{(4a^2b-(2n+1)^2\pi^2)}$$

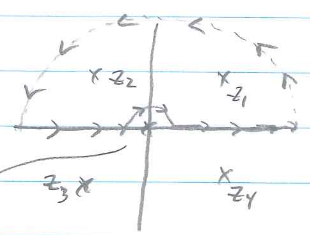
8). $y(x) = \frac{-2P_o}{\pi \ell^2}\int_0^\infty \left(\frac{1-\cos(w\ell)}{w^2}\right)\left(\frac{\cos(wx)}{EIw^4+k}\right)dw$

Evaluate the integral in the complex plane using the Residue theorem to obtain the complete solution for y(x).

$$I = \int_0^\infty \left(\frac{1-\cos(w\ell)}{w^2}\right)\left(\frac{\cos(wx)}{EI\,w^4+k}\right)dw = \frac{1}{2}\int_{-\infty}^\infty \left(\frac{\cos(wx)-\cos(w\ell)\cos(wx)}{(w^2)(EI\,w^4+k)}\right)dw$$

$$= \underbrace{\frac{1}{2}\int_{-\infty}^\infty \frac{\cos(wx)}{(w^2)(EI\,w^4+k)}dw}_{I_1} \underbrace{-\frac{1}{2}\int_{-\infty}^\infty \frac{\cos(wx)\cos(w\ell)}{(w^2)(EI\,w^4+k)}dw}_{I_2}$$ let $z=w$

$$I_1 = \left(\frac{1}{2EI}\right)Re\int_{-\infty}^\infty \frac{e^{izx}\,dw}{(z^2)(z^4+\frac{k}{EI})} = \left(\frac{1}{2EI}\right)Re\int_{-\infty}^\infty \frac{e^{izx}\,dz}{(z^2)(z^4+4b^4)}$$ where $4b^2 = \left(\frac{k}{EI}\right)$



need to find the poles: 2^{nd} order pole at $z=0$.

and $z^4+4b^4 \Rightarrow z^4 = -4b^4 \Rightarrow z = (-4b^4)^{1/4}$

$z_1 = \sqrt{2}\,b\,e^{\frac{\pi}{4}i}$, $z_2 = \sqrt{2}\,b\,e^{\frac{3}{4}\pi i}$, $z_3 = \sqrt{2}\,b\,e^{\frac{5\pi}{4}i}$, $z_4 = \sqrt{2}\,b\,e^{\frac{7\pi}{4}i}$

z_1 & z_2 are in the UHP, z_3 & z_4 are in the LHP

$z_1 = \sqrt{2}\,b\,e^{\pi/4\, i} = \sqrt{2}\,b\left(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right) = \sqrt{2}\,b\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = b(1+i)$

indented contour

$z_2 = \sqrt{2}\,b\,e^{3\pi/4\, i} = \sqrt{2}\,b\left(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right) = \sqrt{2}\,b\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -b(1-i)$

$$I_1 = \left(\frac{1}{2EI}\right)Re\int_{-\infty}^\infty \frac{e^{izx}\,dz}{(z^2)(z^4+4b^4)} = \left(\frac{1}{2EI}\right)Re\left(\underbrace{\frac{1}{4b^4}\int_{-\infty}^\infty \frac{e^{izx}}{z^2}dz}_{①} - \underbrace{\frac{1}{4b^4}\int_{-\infty}^\infty \frac{z^2e^{izx}}{z^4+4b^4}dz}_{②}\right)$$

① $\frac{1}{4b^4}\int_{-\infty}^\infty \frac{e^{izx}}{z^2}dz = \left(\frac{1}{4b^4}\right)(i\pi)\,Res(0)$ $\qquad Res(0) = \frac{1}{(2-1)!}\frac{d}{dz}\left(z^2\frac{e^{izx}}{z^2}\right)\Big|_{z\to 0} = \frac{d}{dz}e^{izx}\Big|_{z\to 0} = ix$

$= \left(\frac{1}{4b^4}\right)(i\pi)(ix)$

$= -\frac{\pi x}{4b^4}$

we have simple poles

$\therefore Res(z_n) = \frac{N(z)}{D'(z)} = \frac{e^{izx}z^2}{4z^3} = \frac{e^{izx}}{4z}$

② $-\frac{1}{4b^4}\int_{-\infty}^\infty \frac{e^{izx}z^2}{z^4+4b^4}dz = \left(-\frac{1}{4b^4}\right)(2\pi i)\left(Res(z_1) + Res(z_2)\right)$

$Res(z_1) + Res(z_2) = \left(\frac{e^{i(b+ib)x}}{4b(1+i)} - \frac{e^{i(-b+ib)x}}{4b(1-i)}\right) = \frac{1}{4(2b)}\left((1-i)\,e^{-b(1-i)} - (1+i)\,e^{-b(1+i)}\right)$

② $= \left(-\frac{1}{4b^4}\right)(2\pi i)\left(\frac{1}{8b}\right)\left((1-i)e^{-b(1-i)} - (1+i)e^{-b(1+i)}\right)$

$$② = \left(\frac{-\pi i}{16b^5}\right)\left((1-i)\,e^{-b(1-i)x} - (1+i)\,e^{-b(1+i)x}\right)$$

$$= \left(\frac{-\pi i}{16b^5}\right)(-i)\left(e^{-b(1-i)x} + e^{-b(1+i)x}\right) - \left(\frac{\pi i}{16b^5}\right)\left(e^{-b(1-i)x} - e^{-b(1+i)x}\right)$$

$$= \left(-\frac{\pi e^{-bx}}{8b^5}\right)\left(\frac{e^{ibx}+e^{-ibx}}{2}\right) + \left(\frac{\pi e^{-bx}}{8b^5}\right)\left(\frac{e^{ibx}-e^{-ibx}}{2i}\right)$$

$$= \left(-\frac{\pi e^{-bx}}{8b^5}\right)\cos(bx) + \left(\frac{\pi e^{-bx}}{8b^5}\right)(\sin(bx))$$

$$= \left(\frac{\pi e^{-bx}}{8b^5}\right)(-\cos(bx) + \sin(bx))$$

$$I_1 = \left(\frac{1}{2EI}\right)\mathrm{Re}\,(① + ②) = \left(\frac{1}{2EI}\right)\left[\frac{\pi}{8b^5}\left(e^{-bx}(-\cos(bx)+\sin(bx)) - 2bx\right)\right]$$

$$I_2 = \left(-\frac{1}{2EI}\right)\int_{-\infty}^{\infty}\frac{\cos(zx)\cos(zl)}{(z^2)(z^4+4b^4)}\,dz$$

$$\begin{aligned} &\cos((x+l)z) = \cos(zx)\cos(zl) - \sin(zx)\sin(zl)\\ + &\cos((x-l)z) = \cos(zx)\cos(zl) + \sin(zx)\sin(zl)\\ \hline &\cos((x+l)z) + \cos((x-l)z) = 2\cos(zx)\cos(zl)\end{aligned}$$

$$= \left(-\frac{1}{2EI}\right)\left(\frac{1}{2}\right)\int_{-\infty}^{\infty}\frac{\cos((x+l)z) + \cos((x-l)z)}{(z^2)(z^4+4b^4)}\,dz$$

$$= \left(-\frac{1}{4EI}\right)\int_{-\infty}^{\infty}\frac{\cos((x+l)z)}{(z^2)(z^4+4b^4)}\,dz + \left(-\frac{1}{4EI}\right)\int_{-\infty}^{\infty}\frac{\cos((x-l)z)}{(z^2)(z^4+4b^4)}\,dz$$

$$= \left(-\frac{1}{4EI}\right)\mathrm{Re}\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(x+l)z}}{(z^2)(z^4+4b^4)}\,dz}_{①} + \left(-\frac{1}{4EI}\right)\mathrm{Re}\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(x-l)z}}{(z^2)(z^4+4b^4)}\,dz}_{②}$$

① since $(x+l>0)$, the evaluation of ① is similar to the evaluation of I_1, where the x in I_1 becomes $(x+l)$ in ①.

$$① = \left(\frac{\pi}{8b^5}\right)\left[e^{-b(x+l)}\left(-\cos(b(x+l)) + \sin(b(x+l))\right) - 2b(x+l)\right]$$

② There are three cases to consider (1) $(x-l)>0$, (2) $(x-l)=0$, and (3) $(x-l)<0$.
Case (1) $(x-l)>0$, the evaluation of ② is similar to the evaluation of I_1, where the x in I_1 becomes $(x-l)$ in ② for case 1.

$$\text{Case (1)} = \left(\frac{\pi}{8b^5}\right)\left[e^{-b(x-l)}\left(\cos(b(x-l)) + \sin(b(x-l))\right) - 2b(x-l)\right] \quad \text{for } (x-l)>0.\ x>l$$

Case (2) $|x-\ell|=0$ $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(z^2)(z^4+4b^4)}\,dz = \underbrace{\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{1}{z^2}\,dz}_{①} - \underbrace{\frac{1}{4b^2}\int_{-\infty}^{\infty}\frac{z^2}{z^4+4b^4}\,dz}_{②}$

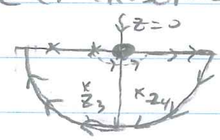
① $\displaystyle\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{1}{z^2}\,dz = \left(\frac{1}{4b^4}\right)(i\pi)\,Res(0)$ $\displaystyle Res(0) = \frac{1}{(2-1)!}\frac{d}{dz}\left(\frac{z^2}{z^2}\right)\Bigg|_{z\to 0} = \frac{d}{dz}(1) = 0$

② $\displaystyle -\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{z^2}{z^4+4b^4}\,dz = \left(-\frac{1}{4b^4}\right)(2\pi i)\left(Res(z_1)+Res(z_2)\right)$ $z_1 = b(1+i)$, $z_2 = -b(1-i)$ $\displaystyle Res(z_n) = \frac{N(z)}{D'(z)} = \frac{z^2}{4z^3} = \frac{1}{4z}$

$$\left(Res(z_1)+Res(z_2)\right) = \left(\frac{1}{4b(1+i)} - \frac{1}{4b(1-i)}\right) = \left(\frac{1}{8b}\right)\left((1-i)-(1+i)\right) = \left(\frac{-2i}{8b}\right) = -\frac{i}{4b}$$

$$\therefore \left(-\frac{1}{4b^4}\right)(2\pi i)\left(-\frac{i}{4b}\right) = \left(-\frac{1}{4b^4}\right)\left(\frac{-\pi}{2b}\right) = -\frac{\pi}{8b^5}$$

Case (3) $|x-\ell|<0$ $\displaystyle\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{(z^2)(z^4+4b^4)}\,dz$ need to integrate in the LHP. need z_3 & z_4



$z_3 = \sqrt{2}\,b\,e^{\frac{5\pi}{4}i} = \sqrt{2}\,b\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right) = \sqrt{2}\,b\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -b(1+i)$

$z_4 = \sqrt{2}\,b\,e^{\frac{7\pi}{4}i} = \sqrt{2}\,b\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right) = \sqrt{2}\,b\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = b(1-i)$

$$\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{(z^2)(z^4+4b^4)}\,dz = \underbrace{\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{z^2}\,dz}_{①} - \underbrace{\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{z^2 e^{i(x-\ell)z}}{z^4+4b^4}\,dz}_{②}$$

① $\displaystyle\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{z^2}\,dz = \left(\frac{1}{4b^4}\right)(-i\pi)\,Res(0)$ $\displaystyle Res(0) = \frac{1}{(2-1)!}\frac{d}{dz}\left(\frac{z^2 e^{i(x-\ell)z}}{z^2}\right)\Bigg|_{z\to 0} = \frac{d}{dz}e^{i(x-\ell)z}\Bigg|_{z\to 0} = i(x-\ell)$

$\displaystyle = \left(\frac{1}{4b^4}\right)(-i\pi)(i(x-\ell))$

$\displaystyle = \frac{\pi}{4b^4}(x-\ell) =$

② $\displaystyle -\frac{1}{4b^4}\int_{-\infty}^{\infty}\frac{e^{i(x-\ell)z}}{z^4+4b^4}\,dz = \left(-\frac{1}{4b^4}\right)(-2\pi i)\left(Res(z_3)+Res(z_4)\right)$ $\displaystyle\therefore Res(z_n) = \frac{N(z)}{D'(z)} = \frac{z^2 e^{i(x-\ell)z}}{4z^3} = \frac{e^{i(x-\ell)z}}{4z}$

$$Res(z_3)+Res(z_4) = \left(-\frac{e^{i(x-\ell)(-b)(1+i)}}{4b(1+i)} + \frac{e^{i(x-\ell)(b)(1-i)}}{4b(1-i)}\right)$$

$$= \left(-\frac{e^{b(x-\ell)(1-i)}}{4b(1+i)} + \frac{e^{b(x-\ell)(1+i)}}{4b(1-i)}\right) = \left(\frac{1}{8b}\right)\left((1+i)e^{b(x-\ell)(1+i)} - (1-i)e^{b(x-\ell)(1-i)}\right)$$

② $\displaystyle = \left(-\frac{1}{4b^4}\right)(-2\pi i)\left(\frac{1}{8b}\right)\left((1+i)e^{b(x-\ell)(1+i)} - (1-i)e^{b(x-\ell)(1-i)}\right)$

$$= \left(\frac{\pi i}{16b^5}\right)\left((1+i)e^{b(x-\ell)(1+i)} - (1-i)e^{b(x-\ell)(1-i)}\right)$$

$$= \left(\frac{\pi i}{16b^5}\right)\left(e^{b(x-\ell)(1+i)} - e^{b(x-\ell)(1-i)}\right) + \left(\frac{\pi i}{16b^5}\right)(i)\left(e^{b(x-\ell)(1+i)} + e^{b(x-\ell)(1-i)}\right)$$

$$= \left(\frac{\pi i}{16b^5}\right) e^{-b(\ell-x)} \left(e^{-ib(\ell-x)} - e^{ib(\ell-x)}\right) - \left(\frac{\pi}{16b^5}\right) e^{-b(\ell-x)} \left(e^{-ib(\ell-x)} + e^{ib(\ell-x)}\right)$$ for $(x-\ell)<0$ or $(\ell-x)>0$

$$= \left(\frac{\pi}{8b^5}\right) e^{-b(\ell-x)} \left(\frac{e^{ib(\ell-x)} - e^{-ib(\ell-x)}}{2i}\right) - \left(\frac{\pi}{8b^5}\right) e^{-b(\ell-x)} \left(\frac{e^{ib(\ell-x)} + e^{-ib(\ell-x)}}{2}\right)$$

$$= \left(\frac{\pi}{8b^5}\right) e^{-b(\ell-x)} \left(\sin(b(\ell-x)) - \cos(b(\ell-x))\right)$$

Case(3): $\int_{-\infty}^{\infty} \frac{e^{i(x-\ell)z}}{(z^2)(z^4+4b^4)}\,dz = ① + ②$

$$= \frac{\pi}{4b^4}(x-\ell) + \left(\frac{\pi}{8b^5}\right) e^{-b(\ell-x)} \left(\sin(b(\ell-x)) - \cos(b(\ell-x))\right)$$

$$= \left(\frac{\pi}{8b^5}\right)\left[e^{-b(\ell-x)}\left(\sin(b(\ell-x)) - \cos(b(\ell-x))\right) - 2b(\ell-x)\right]$$ for $(\ell-x)>0$

$$I_2 = \left(\frac{1}{-4EI}\right)\left(\frac{\pi}{8b^5}\right)\Bigg[\left[e^{-b(x+\ell)}\left(\sin(b(x+\ell)) - \cos(b(x+\ell))\right) - 2b(x+\ell)\right]$$

$$+ \begin{cases} \left[e^{-b(x-\ell)}\left(\sin(b(x-\ell)) + \cos(b(x-\ell))\right) - 2b(x-\ell)\right] & \text{for } x>\ell \\ (-1) & \text{for } x=\ell \\ \left[e^{-b(\ell-x)}\left(\sin(b(\ell-x)) - \cos(b(\ell-x))\right) - 2b(\ell-x)\right] & \text{for } \ell>x \end{cases}\Bigg]$$

recall

$$I_1 = \left(\frac{1}{2EI}\right)\left(\frac{\pi}{8b^5}\right)\left[e^{-bx}\left(\sin(bx) - \cos(bx)\right) - 2bx\right]$$

$$y(x) = \frac{-2P_0}{\pi \ell^2}\left[I_1 + I_2\right]$$ where $4b^4 = \left(\frac{k}{EI}\right)$