Homework Set No. 10
Due November 22, 2013

NEEP 547
DLH

1. Evaulate the following integrals:

   (a) (3pts) $\int_0^{1+i} e^{2z}\, dz$ and (b) (3pts) $\int_C \frac{z+4}{(z^2+1)(z-1)}\, dz$, where C is the circle $|z| = 2$.

2. What is the value of

$$\int_C \frac{\sin(2z)}{z^2 - 4\,z + 5}\, dz, \quad \text{for:}$$

   (a) (3pts) if C is the circle $|z| = 1$?
   (b) (3pts) if C is the circle $|z - 2i| = 3$?
   (c) (3pts) if C is the circle $|z - 1 + 2i| = 2$?

3. page 1005, problems 4, 6 and 10: Evaluate the $\int f(z)\, dz$ for the given function and closed (positively) oriented path (Γ).

   4). (3pts) $f(z) = \dfrac{2\,z^3}{(z-2)^2}$; where Γ is the rectangle having vertices $4 \pm i$ and $-4 \pm i$.
   6). (3pts) $f(z) = \dfrac{\cos(z-i)}{(z+2i)^3}$; where Γ is any path enclosing $-2i$.
   10). (3pts) $f(z) = (z-i)^2$ where Γ is the semicircle of radius 1 about 0 from i to $-i$.

4. Expand $f(z) = 1/(z^2 + 3\,z + 2)$ in a Taylor series (a) (3pts) about the point $z = 0$ and (b) (3pts) about the point $z = 2$. Determine the radius of convergence for each case.

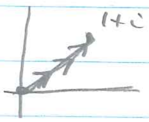
5. page 1018, problems 6 and 8: Find the Taylor series of the function about the indicated point and determine the radius of convergence: (6) (3pts) $f(z) = 1/(2 + z)$, point: $1 - 8i$ and (8) (3pts) $f(z) = 1 + 1/(2 + z^2)$, point: i.

6. ~~Obtain two distinct Laurent expansions for $f(z) = (3z + 1)/(z^2 - 1)$ around $z = 1$ and tell where each converges.~~

1. Evaluate the following integrals

(a) $\int_0^{1+i} e^{2z}\,dz$ and (b) $\int_C \frac{z+4}{(z^2+1)(z-1)}\,dz$ where C is the circle $|z|=2$

(a) $\int_0^{1+i} e^{2z}\,dz$

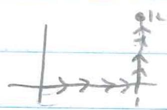


we will integrate along this line using paramaterization

let $z(t) = (1+i)t \Rightarrow dz = (1+i)\,dt$; $t=0,\ z=0$; $t=1,\ z=1+i$

$$\int_0^{1+i} e^{2z}\,dz = \int_0^1 e^{2(1+i)t}(1+i)\,dt = \frac{1}{2} e^{2(1+i)t}\Big|_0^1 = \frac{1}{2}\left(e^{2(1+i)}-1\right)$$

or we can integrate along the x-axis and y-axis as follows, as long as there are no singularities between this path and the previous path.



$$\int_0^{1+i} e^{2z}\,dz = \int_0^1 e^{2z}\,dz + \int_1^{1+i} e^{2z}\,dz$$

let $z=x$ $dz=dx$, $z=0, x=0$; $z=1, x=1$; let $z=1+iy$, $dz=i\,dy$, $z=1, y=0$; $z=1+i, y=1$

$$\int_0^{1+i} e^{2z}\,dz = \int_0^1 e^{2x}\,dx + \int_0^1 e^{2(1+iy)}\,i\,dy$$
$$= \frac{1}{2}e^{2x}\Big|_0^1 + \left(\frac{1}{2i}\right)(i)\,e^{2(1+iy)}\Big|_0^1 = \left(\frac{1}{2}\right)\left(e^{2}-1\right) + \left(\frac{1}{2}\right)\left(e^{2(1+i)} - e^{2}\right)$$
$$= \frac{1}{2}\left(e^{2(1+i)}-1\right)$$

Note: straightforward integration gives $\int_0^{1+i} e^{2z}\,dz = \frac{1}{2}e^{2z}\Big|_0^{1+i} = \frac{1}{2}\left(e^{2(1+i)}-1\right)$

(b) $\int_C \frac{z+4}{(z^2+1)(z-1)}\,dz$, where C is the circle $|z|=2$, singularities at $z=-i, i, 1$

$$\int_C \frac{z+4}{(z^2+1)(z-1)}\,dz = \int_C \frac{z+4}{(z-i)(z+i)(z-1)}\,dz$$ the circle encloses all singularities

$$= \int_C \left(\left(-\tfrac{5}{2}\right)\left(\frac{z}{z^2+1}\right) - \left(\tfrac{3}{2}\right)\left(\frac{1}{z^2+1}\right) + \tfrac{5}{2}\left(\frac{1}{z-1}\right)\right)dz$$
$$= \int_C \left(-\tfrac{5}{4}\right)\left(\frac{1}{z+i}+\frac{1}{z-i}\right) - \left(\tfrac{3}{4}\right)\left(\frac{i}{z+i} - \frac{i}{z-i}\right) + \left(\tfrac{5}{2}\right)\left(\frac{1}{z-1}\right)dz$$
$$= \left(-\tfrac{5}{4}\right)\int_C \frac{dz}{z+i} - \left(\tfrac{5}{4}\right)\int_C \frac{dz}{z-i} - \left(\tfrac{3}{4}\right)\int_C \frac{i\,dz}{z+i} + \left(\tfrac{3}{4}\right)\int \frac{i\,dz}{z-i} + \left(\tfrac{5}{2}\right)\int_C \frac{dz}{z-1}$$

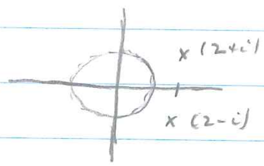
using $\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$ we get

$$\int_C \frac{z+4}{(z^2+1)(z-1)}\,dz = \left(\left(-\tfrac{5}{4}\right) - \left(\tfrac{5}{4}\right) - \left(\tfrac{3}{4}i\right) + \left(\tfrac{3}{4}i\right) + \tfrac{5}{2}\right)(2\pi i) = \left(-\tfrac{5}{2}+\tfrac{5}{2}\right)(2\pi i)$$
$$= \underline{\underline{0}}$$

2 What is the value of $\int_C \frac{\sin(2z)}{z^2-4z+5}\,dz = \int_C \frac{\sin(2z)}{(z-2)^2+1}\,dz = \int_C \frac{\sin(2z)}{(z-(2+i))(z-(2-i))}\,dz$

poles at $z=2+i$, $z=2-i$

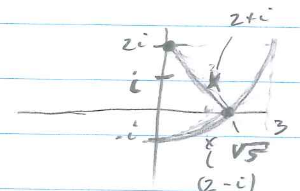
(a) if C is the circle $|z|=1$?



neither singularity is in the circle, hence

$$\int_C \frac{\sin(2z)}{(z-(2+i))(z-(2-i))}\,dz = \underline{0}$$

(b) if C is the circle $|z-2i|=3$?



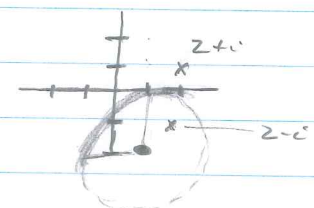
only the $z=2+i$ root is enclosed

$$\int_C \frac{f(z)}{(z-(2+i))}\,dz = 2\pi i f(z=2+i)$$

$$=(2\pi i)\frac{\sin(2(2+i))}{(2+i)-(2-i)} = (2\pi i)\frac{\sin(4+2i)}{2i} = \underline{\pi \sin(4+2i)}$$

$|z-(1-2i)|$

(c) if C is the circle $|z-1+2i|=2$?



only the $z=2-i$ root is enclosed

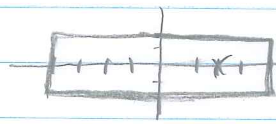
$$\int_C \frac{f(z)}{(z-(2-i))}\,dz = 2\pi i\, f(z=2-i)$$

$$=(2\pi i)\frac{\sin(2(2-i))}{(2-i)-(2+i)}$$

$$=(2\pi i)\frac{\sin(4-2i)}{-2i} = \underline{-\pi \sin(4-2i)}$$

3. Evaluate the $\int f(z)\,dz$ for the given function and closed (positively) oriented path (Γ).

(3.4) $f(z) = \dfrac{2z^3}{(z-2)^2} \Rightarrow \displaystyle\int_C \frac{2z^3}{(z-2)^2}\,dz$ where C = is the rectangle having vertices at $4+i$, $4-i$, $-4+i$, $-4-i$

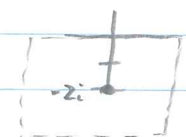


$z=2$ is a singular point and is enclosed in the rectangle.

$$\int_C \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i \left.\frac{df(z)}{dz}\right|_{z=z_0} \quad z_0 = 2,\ f(z) = 2z^3,\ \frac{df(z)}{dz} = 6z^2,\ \frac{df}{dz} = 6\cdot 4 = 24$$

$$= (2\pi i)(24) = \underline{48\pi i}$$

(3.6) $f(z) = \dfrac{\cos(z-i)}{(z+2i)^3} \Rightarrow \displaystyle\int_C \frac{\cos(z-i)}{(z+2i)^3}\,dz$ where Γ is any path enclosing $-2i$.

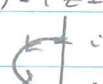


the singularity at $z=-2i$ is enclosed in the square

$$\oint \frac{f(z)}{(z-z_0)^3}\,dz = (2\pi i)\left(\frac{1}{2!}\right)\left.\frac{d^2 f(z)}{dz^2}\right|_{z=z_0} = (2\pi i)\left(\frac{1}{2}\right)\frac{d^2}{dz^2}\left(\cos(z-i)\right)\Big|_{z=-2i}$$

$$= (\pi i)(-\cos(z-i))\Big|_{z=-2i} = -(\pi i)\cos(-2i-i) = -\pi i\,(\underline{\cos(-3i)}) = \underline{(-\pi i)(\cos(3i))}$$

(3.10) $f(z) = (z-i)^2$ where Γ is the semi-circle of radius 1 about 0 from i to $-i$

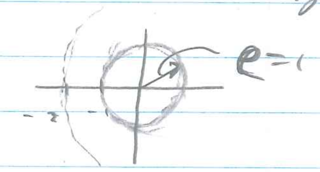


there are no singularities enclosed

hence $\int_C f(z)\,dz = \underline{0}$

4. Expand $f(z) = \dfrac{1}{z^2+3z+2}$ in a Taylor series and determine the radius of convergence

(a) about the point $z=0$.



$$f(z) = \frac{1}{(z+2)(z+1)} \quad \text{singularities at } z=-1,-2$$

Radius of convergence is $\underline{\rho = 1}$

$$f(z) = \frac{1}{(z+2)(z+1)} = \frac{1}{z+1} - \frac{1}{z+2} \qquad \text{will use } \frac{1}{1-u} = 1+u+u^2+\cdots \text{ for } |u|<1$$

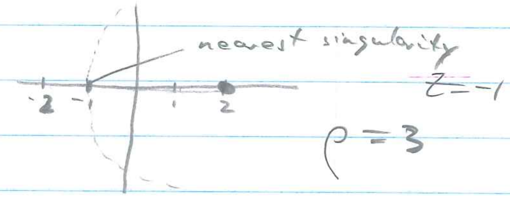
$$= \frac{1}{1-(-z)} - \frac{1}{2}\left(\frac{1}{1-(-\frac{z}{2})}\right)$$

$$= \left(1 - z + z^2 - z^3 + \cdots\right) - \left(\tfrac{1}{2}\right)\left(1 - \tfrac{z}{2} + \left(\tfrac{z}{2}\right)^2 - \left(\tfrac{z}{2}\right)^3 \cdots\right)$$

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n - \left(\tfrac{1}{2}\right)\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{z}{2}\right)^n = \sum_{n=0}^{\infty}(-1)(z^n)\left(\tfrac{1}{2}\right)\left(1 - \tfrac{1}{2^{n+1}}\right)$$

(b) about the point $z=2$.

Radius of convergence is $\rho = \underline{3}$



$$f(z) = \frac{1}{z+1} - \frac{1}{z+2} = \frac{1}{(z-2)+3} - \frac{1}{(z-2)+4}$$

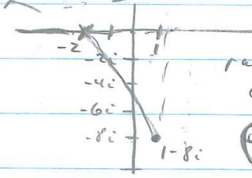
$$= \frac{1}{3+(z-2)} - \frac{1}{4+(z-2)} = \left(\tfrac{1}{3}\right)\left(\frac{1}{1+\left(\frac{z-2}{3}\right)}\right) - \left(\tfrac{1}{4}\right)\left(\frac{1}{1+\left(\frac{z-2}{4}\right)}\right)$$

$$= \left(\tfrac{1}{3}\right)\left(1 - \left(\tfrac{z-2}{3}\right) + \left(\tfrac{z-2}{3}\right)^2 - \left(\tfrac{z-2}{3}\right)^3 + \cdots\right) - \left(\tfrac{1}{4}\right)\left(1 - \left(\tfrac{z-2}{4}\right) + \left(\tfrac{z-2}{4}\right)^2 - \left(\tfrac{z-2}{4}\right)^3 + \cdots\right)$$

$$= \left(\tfrac{1}{3}\right)\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{z-2}{3}\right)^n - \left(\tfrac{1}{4}\right)\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{z-2}{4}\right)^2 = \sum_{n=0}^{\infty}(-1)^n(z-2)^n\left(\tfrac{1}{3^{n+1}} - \tfrac{1}{4^{n+1}}\right)$$

5. Find the Taylor series of the function about the indicated point and determine the radius of convergence

(6) $f(z) = \frac{1}{2+z}$ and the point $z = 1-8i$



radius of convergence $\rho = \sqrt{(3)^2+8^2} = \underline{\sqrt{73}}$

$$f(z) = \frac{1}{2+z} = \frac{1}{(2+1-8i)+(z-(1-8i))}$$

$$= \frac{1}{3-8i+(z-(1-8i))}$$

$$= \left(\frac{1}{3-8i}\right)\left(\frac{1}{1+\frac{(z-(1-8i))}{3-8i}}\right) = \left(\frac{3+8i}{73}\right)\left(1-\left(\frac{z-(1-8i)}{3-8i}\right)+\left(\frac{z-(1-8i)}{3-8i}\right)^2-(\ \ )^3+\cdots\right.$$

$$= \left(\frac{3+8i}{73}\right)\left(1-\left(\frac{z-(1-8i)}{3-8i}\right)+\left(\frac{z-(1-8i)}{3-8i}\right)^2-\left(\frac{z-(1-8i)}{3-8i}\right)^3+\cdots\right)$$

$$= \left(\frac{3+8i}{73}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-(1-8i)}{3-8i}\right)^n = \left(\frac{3+8i}{73}\right)\sum_{n=0}^{\infty}(-1)\left(\frac{3+8i}{73}\right)^n(z-(1-8i))^n$$

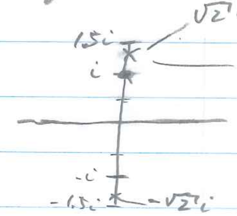
Radius of convergence $\left|\frac{z-(1-8i)}{3-8i}\right| < 1 \Rightarrow |z-(1-8i)| < |3-8i|$

recall $|z|^2 = zz^* \therefore |z| = \sqrt{zz^*} \Rightarrow |3-8i| = \sqrt{(3-8i)(3+8i)} = \sqrt{73}$

$\therefore |z-(1-8i)| < \sqrt{73}$ our radius of convergence.

(8) $f(z) = 1 + \frac{1}{2+z^2}$ and the point $z = i$

$$= \frac{z^2+3}{z^2+2} = \frac{z^2+3}{(z+\sqrt{2}i)(z-\sqrt{2}i)}$$



$$f(z) = 1+\frac{1}{(z+\sqrt{2}i)(z-\sqrt{2}i)}$$

radius of convergence $\rho = \underline{(\sqrt{2}-1)i}$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{z+\sqrt{2}i}\right)-\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{z-\sqrt{2}i}\right)$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{z+\sqrt{2}i}-\frac{1}{z-\sqrt{2}i}\right)$$

recall to expand about $z=i$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{(z-i)+(1+\sqrt{2})i}-\frac{1}{(z-i)+(1-\sqrt{2})i}\right)$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left(\frac{1}{(1+\sqrt{2})i\left(1+\frac{(z-i)}{(1+\sqrt{2})i}\right)}-\frac{1}{(1-\sqrt{2})i\left(1+\frac{(z-i)}{(1-\sqrt{2})i}\right)}\right)$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left[\left(\frac{1}{(1+\sqrt{2})i}\right)\left(1-\frac{(z-i)}{(1+\sqrt{2})i}+\frac{(z-i)^2}{((1+\sqrt{2})i)^2}-\cdots\right)-\left(\frac{1}{(1-\sqrt{2})i}\right)\left(1-\frac{(z-i)}{(1-\sqrt{2})i}+\frac{(z-i)^2}{((1-\sqrt{2})i)^2}-\cdots\right)\right]$$

$$= 1+\left(\frac{i}{2\sqrt{2}}\right)\left[\frac{-i}{(1+\sqrt{2})}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{(1+\sqrt{2})i}\right)^n+\frac{i}{(1-\sqrt{2})}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{(1-\sqrt{2})i}\right)^n\right]$$

$$= 1+\left(\frac{1}{2\sqrt{2}+4}\right)\sum_{n=0}^{\infty}(-1)^n(-i)^n\left(\frac{z-i}{1+\sqrt{2}}\right)^n-\left(\frac{1}{2\sqrt{2}-4}\right)\sum_{n=0}^{\infty}(-1)(-i)^n\left(\frac{z-i}{1-\sqrt{2}}\right)^n$$

Radius of convergence is equal to the smaller of $\frac{|z-i|}{|1+\sqrt{2}|} < 1$ or $\frac{|z-i|}{|1-\sqrt{2}|} < 1$

$|z-i| < |1+\sqrt{2}|$ or $|z-i| < |1-\sqrt{2}|$ smaller value $\rho = |1-\sqrt{2}| = \underline{\sqrt{2}-1}$