Homework Set No. 1
Due September 13, 2013

NEEP 547
DLH

Separable Eqs.; Solve the initial valued problems:

1. (4pts) O'Neal, page 20, prob. 14: $2yy' = e^{x-y^2}$; $y(4) = -2$
2. (4pts) O'Neal, page 20, prob. 15: $y\,y' = 2\,x\sec(3y)$; $y(2/3) = \pi/3$

Exact Differential Eqs.; Solve the initial valued problems:

3. (5pts) $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$; $y(1) = \ln(2)$
4. (5pts) O'Neal, page 32, prob. 14: $e^y + (xe^y - 1)y' = 0$; $y(5) = 0$

General Integrating Factor:

5. (6pts) $(3x - y)\,dx + (3y + x)\,dy = 0$
6. (6pts) O'Neal, page 38, prob. 17; Solve the initial valued problem: $2xy + 3y' = 0$; y(0) = 4 (Hint; try $\mu(x,y) = y^a\,e^{bx^2}$, where a and b are constants)
7. (6pts) O'Neal, page 38, prob. 20; Solve the initial valued problem: $3x^2y + y^3 + 2xy^2y' = 0$; $y(2) = 1$

Homogenous, Bernoulli and Ricatti Eqs.:

8. (5pts) O'Neal, page 45 prob. 12; find the general solution: $x^3y' = x^2y - y^3$
9. (5pts) O'Neal, page 45, prob. 17; find the general solution: $y' = \frac{3x-y-9}{x+y+1}$
10. (5pts) Find the general solution: $(2x^2 - y^2)\,dx + 3xy\,dy = 0$
11. (4pts) Show that if one solution, say $y = u(x)$, of the Riccati equation $y' = P(x)y^2 + Q(x)y + R(x)$ is known, then the substitution $y = u + \frac{1}{z}$ will transform this equation into a linear first-order equation in the new dependent variable z.

1. page 20, prob. 14: $2yy' = e^{x-y^2}$; $y(4) = -2$

$$2y\frac{dy}{dx} = e^{x-y^2} \Rightarrow 2y\,dy = e^x e^{-y^2}dx \Rightarrow 2y e^{y^2} dy = e^x dx$$

$$\int 2y e^{y^2} dy = \int e^x dx \Rightarrow \int_{-2}^{y} 2y' e^{y'^2} dy' = \int_4^x e^{x'} dx'$$

$$e^{y'^2}\Big|_{-2}^{y} = e^{x'}\Big|_4^x \Rightarrow e^{y^2} - e^{(-2)^2} = e^x - e^4$$

$$e^{y^2} - e^4 = e^x - e^4 \Rightarrow e^{y^2} = e^x \Rightarrow y^2 = x \Rightarrow y = -\sqrt{x}$$

2. page 20, prob. 15: $yy' = 2x\sec(3y)$; $y(\frac{2}{3}) = \pi/3$

$$y\frac{dy}{dx} = 2x\sec(3y) \Rightarrow y\cos(3y)\,dy = 2x\,dx$$

$$\int y\cos(3y)\,dy = \int 2x\,dx \Rightarrow \int_{\pi/3}^{y} y'\cos(3y')\,dy' = \int_{2/3}^{x} 2x'\,dx'$$

$$y'\left(\tfrac{1}{3}\sin(3y')\right)\Big|_{\pi/3}^{y} - \int_{\pi/3}^{y} \tfrac{1}{3}\sin(3y)\,dy = (x'^2)\Big|_{2/3}^{x}$$

$$\frac{y}{3}\sin(3y) - \frac{\pi}{9}\sin(\pi) + \frac{1}{9}\cos(3y')\Big|_{\pi/3}^{y} = x^2 - \frac{4}{9}$$

$$\frac{y}{3}\sin(3y) - 0 + \frac{1}{9}\cos(3y) - \frac{1}{9}\cos(\pi) = x^2 - \frac{4}{9}$$

$$\frac{y}{3}\sin(3y) + \frac{1}{9}\cos(3y) + \frac{1}{9} = x^2 - \frac{4}{9}$$

$$\frac{y}{3}\sin(3y) + \frac{1}{9}\cos(3y) = x^2 - \frac{5}{9}$$

3. $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0; \quad y(1) = \ln(2)$

$$\left.\begin{aligned} M(x,y) &= (2xy + e^y), \quad \frac{\partial M}{\partial y} = 2x + e^y \\ N(x,y) &= (x^2 + xe^y), \quad \frac{\partial N}{\partial x} = 2x + e^y \end{aligned}\right\} \text{ Eq. is exact}$$

$$\phi(x,y) = \int_{x_0}^{x} M(x', y_0)\,dx' + \int_{y_0}^{y} N(x, y')\,dy' = 0$$ From class notes

$$= \int_{x_0}^{x} (2x'y_0 + e^{y_0})\,dx' + \int_{y_0}^{y} (x^2 + xe^{y'})\,dy' = 0$$

$$= \int_{1}^{x} (2x'\ln(2) + e^{\ln(2)})\,dx' + \int_{\ln(2)}^{y} (x^2 + xe^{y'})\,dy' = 0$$

$$= \int_{1}^{x} (2x'\ln(2) + 2)\,dx' + \int_{\ln(2)}^{y} (x^2 + xe^{y'})\,dy' = 0$$

$$= \left(x'^2\ln(2) + 2x'\right)\Big|_{1}^{x} + \left(x^2y' + xe^{y'}\right)\Big|_{\ln(2)}^{y}$$

$$= x^2\ln(2) + 2x - \ln(2) - 2 + x^2y + xe^y - x^2\ln(2) - xe^{\ln(2)} = 0$$

$$= x^2\cancel{\ln(2)} + \cancel{2x} - \ln(2) - 2 + x^2y + xe^y - x^2\cancel{\ln(2)} - \cancel{2x} = 0$$

$$\phi(x,y) = x^2y + xe^y = \ln(2) + 2$$

3. Alternate solution method from first principles

$(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0 \;;\; y(1) = \ln(2)$

$\phi(x,y) = c \Rightarrow d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$

$M(x,y) = (2xy + e^y)$ $\qquad$ $N(x,y) = (x^2 + xe^y)$

$\frac{\partial M}{\partial y} = (2x + e^y)$ $\qquad$ $\frac{\partial N}{\partial x} = (2x + e^y)$ $\Big\}$ Eqs are exact

$\frac{\partial \phi}{\partial x} = M(x,y) = (2xy + e^y) \Rightarrow \phi(x,y) = \int (2xy + e^y)\,dx \Big|_{\text{hold } y \text{ const.}}$

$= x^2y + xe^y + h(y)$

now $\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}(x^2y + xe^y + h(y)) = x^2 + xe^y + \frac{dh(y)}{dy}$ $\qquad$ note $\frac{\partial \phi}{\partial y}$ is also equal to $N(x,y) = (x^2 + xe^y)$

$\therefore \cancel{(x^2 + xe^y)} = \cancel{x^2 + xe^y} + \frac{dh(y)}{dy}$

$\frac{dh(y)}{dy} = 0 \Rightarrow \int dh(y) = 0 \Rightarrow h(y) = c$

thus $\phi(x,y) = x^2y + xe^y + c = c_1$

$\Rightarrow \phi(x,y) = x^2y + xe^y = c_2$ $\qquad$ what is c_2?

now $y(1) = \ln(2)$ $\quad$ so $x_0 = 1 \quad y_0 = \ln(2)$

$c_2 = (1)^2 \ln(2) + (1)e^{\ln(2)}$

$= \ln(2) + 2$

$\phi(x,y) = x^2y + xe^y = \ln(2) + 2$

4. $e^y + (xe^y - 1)y' = 0 \; ; \; y(5) = 0$

$e^y\,dx + (xe^y - 1)\,dy = 0$

$M(x,y) = e^y \qquad \frac{\partial M}{\partial y} = e^y$

$N(x,y) = xe^y - 1 \qquad \frac{\partial N}{\partial x} = e^y$ } Eq. is exact.

$f(x,y) = \int_{x_0}^{x} M(x', y_0)\,dx' + \int_{y_0}^{y} N(x, y')\,dy' = c$

$= \int_{x_0}^{x} e^{y_0}\,dx' + \int_{y_0}^{y} (xe^{y'} - 1)\,dy' = c$

$= \int_{5}^{x} e^{y_0}\,dx' + \int_{0}^{y} (xe^{y'} - 1)\,dy' = c$

$= e^{y_0} x' \Big|_5^x + (xe^{y'} - y')\Big|_0^y = c$

$= e^{y_0}(x - 5) + (xe^y - y - x) = c$

$e^{y_0}x - x$

$= xe^y - y - 5 = c$

what is c? Use initial condition

$((5)e^0 - 0 - 5) = c \Rightarrow 5 - 5 = c \quad \therefore c = 0$

$xe^y - y - 5 = 0$

$\therefore \; xe^y - y = 5$

5. find the general solution of $(3x-y)dx+(3y+x)dy=0$

$$(3x-y)dx+(3y+x)dy=0 \Rightarrow 3x\,dx-y\,dx+3y\,dy+x\,dy=0$$
$$\Rightarrow 3x\,dx+3y\,dy-y\,dx+x\,dy=0$$
$$\Rightarrow \tfrac{3}{2}d(x^2+y^2)-y\,dx+x\,dy=0$$

let's try $(x^2+y^2)^P$ as integrating factor

$$\overbrace{(3x-y)(x^2+y^2)^P}^{M}dx+\overbrace{(3y+x)(x^2+y^2)^P}^{N}dy=0$$

$$\frac{\partial M}{\partial y}=(-1)(x^2+y^2)^P+(3x-y)(P)(2y)(x^2+y^2)^{(P-1)}$$

let's equate them.

$$\frac{\partial N}{\partial x}=(x^2+y^2)^P+(3y+x)(P)(2x)(x^2+y^2)^{(P-1)}$$

$$-(x^2+y^2)^P+(3x-y)(P)(2y)(x^2+y^2)^{(P-1)}=(x^2+y^2)^P+(3y+x)(P)(2x)(x^2+y^2)^{(P-1)}$$
$$-x^2-y^2+(3x-y)(P)(2y)=x^2+y^2+(3y+x)(P)(2x)$$
$$-x^2-y^2+\cancel{6xyp}-2y^2p=x^2+y^2+\cancel{6xyp}+2x^2p$$
$$-(x^2+y^2)-2y^2p=x^2+y^2+2x^2p$$
$$-2(x^2+y^2)=2p(x^2+y^2) \Rightarrow -2=2p \Rightarrow p=-1$$

$$\therefore \mu(x,y)=\frac{1}{(x^2+y^2)}$$

our D.E. becomes $\underbrace{\frac{(3x-y)}{(x^2+y^2)}}_{\frac{d\phi}{dx}}dx+\underbrace{\frac{(3y+x)}{(x^2+y^2)}}_{\frac{d\phi}{dy}}dy=0$

① $$\frac{d\phi}{dx}=\frac{3x-y}{(x^2+y^2)}=\frac{3x}{(x^2+y^2)}-\frac{y}{(x^2+y^2)}$$

$$\int d\phi=\int\frac{3x}{(x^2+y^2)}dx-y\int\frac{dx}{(x^2+y^2)} \Rightarrow \phi=\frac{3}{2}(x^2+y^2)-y\int\frac{dx}{(x^2+y^2)}+g(y)$$

$$\Rightarrow \phi=\frac{3}{2}(x^2+y^2)-y\int\frac{dx}{x^2(1+(\frac{y}{x})^2)}+g(y)$$

let $t=\frac{y}{x} \Rightarrow dt=\frac{-y}{x^2}dx \Rightarrow dx=-\frac{x^2}{y}dt$

$$\phi = \frac{3}{2}(x^2+y^2) - y\int \frac{1}{(1+t^2)}\left(\frac{1}{x^2}\right)\left(-\frac{x^2}{y}\,dt\right) + g(y)$$

$$= \frac{3}{2}(x^2+y^2) + \int \frac{dt}{(1+t^2)} + g(y) = \frac{3}{2}(x^2+y^2) + \tan^{-1}(t) + g(y)$$

$$= \frac{3}{2}(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + g(y)$$

② $$\frac{d\phi}{dy} = \frac{(3y+x)}{(x^2+y^2)} = \frac{3y}{(x^2+y^2)} + \frac{x}{(x^2+y^2)}$$

$$\int d\phi = \int \frac{3y}{(x^2+y^2)}\,dy + x\int \frac{dy}{(x^2+y^2)} \Rightarrow \phi = \frac{3}{2}\ln(x^2+y^2) + x\int \frac{dy}{x^2(1+(\frac{y}{x})^2)} + f(x)$$

let $t = \frac{y}{x} \Rightarrow dt = \frac{1}{x}\,dy \Rightarrow dy = x\,dt$

$$\phi = \frac{3}{2}\ln(x^2+y^2) + \left(\frac{1}{x}\right)\int \frac{x\,dt}{(1+t^2)} + f(x) = \frac{3}{2}\ln(x^2+y^2) + \underbrace{\int \frac{1}{1+t^2}}_{\tan^{-1}(t)} + f(x)$$

$$= \frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + f(x)$$

let's set ~~them~~ two ϕ's equal to each other.

$$\frac{3}{2}(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + g(y) = \frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + f(x)$$

$$g(y) = f(x) = c_1$$

$$\therefore \phi = \frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) + c_1 = c$$

$$\frac{3}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) = c_2$$

which is the same as $3\ln(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = c_3$

we can also write the solution as

$$\frac{3}{2}\ln(x^2+y^2) - \tan^{-1}\left(\frac{x}{y}\right) = c_2$$

or $3\ln(x^2+y^2) - 2\tan^{-1}\left(\frac{x}{y}\right) = c_3$

5. Alternative way to find the solution of $(3x-y)\,dx + (3y+x)\,dy = 0$

$$(3x-y)\,dx + (3y+x)\,dy = 0 \Rightarrow (3x-y)\,dx = -(3y+x)\,dy$$

$$\frac{-(3x-y)}{3y+x} = \frac{dy}{dx} \Rightarrow \frac{-(3-\frac{y}{x})}{3\frac{y}{x}+1} = \frac{dy}{dx}$$ note eq. is a homogeneous eq.

let $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$ now to substitute

$$u + x\frac{du}{dx} = \frac{-(3-u)}{3u+1} \Rightarrow x\frac{du}{dx} = \frac{u-3}{3u+1} - u = \frac{u-3-u(3u+1)}{3u+1}$$

$$x\frac{du}{dx} = \frac{u-3-3u^2-u}{3u+1} = \frac{-3(u^2+1)}{3u+1}$$

$$du\,\frac{(3u+1)}{3(u^2+1)} = -\frac{dx}{x} \Rightarrow \frac{1}{3}\int \frac{3u}{1+u^2}\,du + \frac{1}{3}\int \frac{du}{1+u^2} = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln(1+u^2) + \frac{1}{3}\tan^{-1}(u) = -\ln(x) + C$$

$$3\ln(1+u^2) + 2\tan^{-1}(u) = -6\ln(x) + 6C$$

$$3\left(\ln(1+u^2) + 2\ln(x)\right) + 2\tan^{-1}(u) = C_1$$

$$3\left(\ln(1+u^2) + \ln(x^2)\right) + 2\tan^{-1}(u) = C_1$$ recall $u = \frac{y}{x}$

$$3\left(\ln\left(1+\left(\frac{y}{x}\right)^2\right) + \ln(x^2)\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = C_1$$

$$3\ln\left(\left(1+\left(\frac{y}{x}\right)^2\right)x^2\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = C_1$$

$$3\ln(y^2+x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = C_1$$

6. Solve the initial-valued problem: $2xy + 3y' = 0$; $y(0) = 4$

$2xy + 3y' = 0 \Rightarrow \underbrace{2xy}_{M}\,dx + \underbrace{3}_{N}\,dy = 0$ let's check if Exact $\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 0$ Not Exact

multiply by $\mu(x,y) = y^a e^{bx^2}$

$\underbrace{2xy^{(a+1)}e^{bx^2}} + 3y^a e^{bx^2} = 0 \qquad \frac{\partial M}{\partial y} = (a+1)y^a(2xe^{bx^2}) \qquad \frac{\partial N}{\partial x} = 3y^a(2xbe^{bx^2})$

set them equal: $(a+1)y^a(2xe^{bx^2}) = 3b\,y^a(2xe^{bx^2}) \Rightarrow (a+1) = 3b$ $\quad a = 0$, $b = 1/3$

$\mu(x,y) = e^{\frac{x^2}{3}}$

$\underbrace{2xye^{\frac{x^2}{3}}}_{M}\,dx + \underbrace{3e^{\frac{x^2}{3}}}_{N}\,dy = 0$ let's check $\frac{\partial M}{\partial y} = 2xe^{\frac{x^2}{3}} \quad \frac{\partial N}{\partial x} = 3(\frac{2}{3}xe^{\frac{x^2}{3}}) = 2xe^{\frac{x^2}{3}}$

it is now exact

now $M = \frac{\partial \phi}{\partial x} = 2xye^{x^2/3} \Rightarrow \int d\phi = y\int 2xe^{x^2/3}dx \Rightarrow \phi = 3ye^{x^2/3} + g(y)$

$N = \frac{\partial \phi}{\partial y} = 3e^{x^2/3} \Rightarrow \int d\phi = 3e^{x^2/3}\int dy \Rightarrow \phi = 3ye^{x^2/3} + f(x)$

equate results: $3ye^{x^2/3} + g(y) = 3ye^{x^2/3} + f(x) \Rightarrow g(y) = f(x) = \text{constant} = c$

$3ye^{x^2/3} + c = 0 \Rightarrow y(x) = c_1 e^{-x^2/3}$ now to find c_1

$y(0) = 4 = c_1 e^{-0} \Rightarrow c_1 = 4.$

$y(x) = 4e^{-x^2/3}$

6 Alternative method $2xy + 3y' = 0$; $y(0) = 4$

$2xy + 3y' = 0 \Rightarrow \frac{dy}{dx} + \frac{2}{3}xy = 0$ first order eq. need to find integrating factor.

$I_f(x) = e^{\int p(x)dx} = e^{\frac{2}{3}\int x\,dx} = e^{x^2/3}$

$e^{x^2/3}\frac{dy}{dx} + \frac{2}{3}xe^{x^2/3}y = 0 \Rightarrow \frac{d(y(x)e^{x^2/3})}{dx} = 0$

$\int d(y(x)e^{x^2/3}) = 0 \Rightarrow y(x)e^{x^2/3} = C$

$y(x) = Ce^{-x^2/3}$ now to find C.

$y(0) = 4 = Ce^{-(0)} \Rightarrow 4 = C$

$y(x) = 4e^{-x^2/3}$

7. page 38, prob 20; solve the initial valued problem:

$3x^2y + y^3 + 2xy^2y' = 0; \; y(2) = 1$

$\underbrace{(3x^2y + y^3)}_{M}dx + \underbrace{2xy^2}_{N}dy = 0 \quad \frac{\partial M}{\partial y} = 3x^2 + 3y^2 \quad \frac{\partial N}{\partial x} = 2y^2$ Eq is not Exact.

multiply by $x^p y^q$ to make it exact

$x^p y^q (3x^2y + y^3) = 3x^{p+2}y^{q+1} + x^p y^{q+3} \; ; \; \frac{\partial M}{\partial y} = 3x^{p+2}(q+1)y^q + x^p(q+3)y^{q+2}$

$x^p y^q (2xy^2) = 2x^{p+1}y^{q+2} \; ; \; \frac{\partial N}{\partial x} = 2(p+1)x^p y^{q+2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} : 3x^{p+2}(q+1)y^q + x^p(q+3)y^{q+2} = 2(p+1)x^p y^{q+2}$

$3\cancel{x^p} x^2 (q+1)\cancel{y^{q+2}} y^{-2} + \cancel{x^p}(q+3)\cancel{y^{q+2}} = 2(p+1)\cancel{x^p}\cancel{y^{q+2}}$

$3x^2(q+1)y^{-2} + (q+3) = 2(p+1)$

thus we have $q+1 = 0$ and $q+3 = 2(p+1)$

$q = -1$ and $2 = 2(p+1) \Rightarrow 1 = p+1 \Rightarrow p = 0$

our integrating factor is $x^p y^q = y^{-1}$

$(3x^2y + y^3)y^{-1}dx + 2xy^2y^{-1}dy = 0 \Rightarrow (3x^2 + y^2)dx + 2xy\,dy = 0$

$f(x,y) = c \Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \; ; \; \frac{\partial f}{\partial y} = 2xy \; ; \; \underbrace{\frac{\partial f}{\partial x} = (3x^2 + y^2)}$

$\frac{\partial f}{\partial y} = 2xy \Rightarrow f(x,y) = \int 2xy\,dy = xy^2 + h(x)$

now $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^2 + h(x)) = y^2 + \frac{dh}{dx} = (3x^2 + y^2) \quad \therefore \frac{dh}{dx} = 3x^2$

$\therefore \int dh(x) = \int 3x^2 dx \Rightarrow h(x) = x^3 + k$

$f(x,y) = xy^2 + x^3 + k = c \Rightarrow x^3 + xy^2 = c_1$ $\quad y = 1$ when $x = 2$

$(2)^3 + (2)(1)^2 = c_1 \Rightarrow 8 + 2 = c_1 \Rightarrow c_1 = 10$

$\therefore x^3 + xy^2 = 10$

8. Find the general solution of the Eq. $x^3 y' = x^2 y - y^3$

$x^3 y' = x^2 y - y^3 \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^3$ let $x \to tx$ and $y \to ty$

substitution gives $\Rightarrow \left(\frac{t}{t}\right)\frac{dy}{dx} = \left(\frac{t}{t}\right)\left(\frac{y}{x}\right) - \left(\frac{t}{t}\right)^3\left(\frac{y}{x}\right)^3 \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^3$

note the Eq. is a homogeneous Eq.

let $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$ and $u = \frac{y}{x}$, substitution gives

$\cancel{u} + x\frac{du}{dx} = \cancel{u} - u^3 \Rightarrow x\frac{du}{dx} = -u^3 \Rightarrow -u^{-3}du = \frac{dx}{x}$

$\frac{u^{-2}}{2} = \ln(x) + c \Rightarrow u^{-2} = 2\ln(x) + 2c \Rightarrow u^2 = \frac{1}{\ln(x^2) + c_1}$

$\left(\frac{y}{x}\right)^2 = \frac{1}{\ln(x^2)+c_1} \Rightarrow y^2 = \frac{x^2}{\ln(x^2)+c_1}$

$y(x) = \frac{x}{\sqrt{\ln(x^2)+c_1}}$

9. page 45, prob 17; find the general solution to: $y' = \frac{3x - y - 9}{x + y + 1}$

This Eq. is of the Form $y' = F\left(\frac{ax + by + c}{dx + ey + r}\right)$. In its current form, the Eq is inhomogeneous. We need to make a change of variables to put it in the form of a homogeneous Eq.

The homogeneous form is $y' = F\left(\frac{ax + by}{dx + cy}\right) \Rightarrow y' = F\left(\frac{a + b\frac{y}{x}}{d + e\,y/x}\right) = F\left(\frac{y}{x}\right)$

$$\left.\begin{aligned} \text{let } x &= X - h\,;\ dx = dX \\ y &= Y - k\,;\ dy = dY \end{aligned}\right)\quad y' = F\left(\frac{a(X - h) + b(Y - k) + c}{d(X - h) + e(Y - k) + r}\right)$$

$$= F\left(\frac{aX - ah + bY - bk + c}{dX - dh + eY - ek + r}\right) = F\left(\frac{aX + bY - (ah + bk) + c}{dX + eY - (dh + ek) + r}\right)$$

we need to choose h and k such that $-(ah + bk) + c = 0$ and $-(dh + ek) + r = 0$.

we need to solve for h and k interms of a, b, c, d, e and r.

$$\left.\begin{aligned} ah + bk = c &\Rightarrow h = \frac{c}{a} - \frac{b}{a}k \\ dh + ek = r &\Rightarrow h = \frac{r}{d} - \frac{e}{d}k \end{aligned}\right\} \to \frac{c}{a} - \frac{b}{a}k = \frac{r}{d} - \frac{e}{d}k \Rightarrow \left(\frac{c}{a} - \frac{r}{d}\right) = \left(\frac{b}{a} - \frac{e}{d}\right)k$$

$$\Rightarrow k = \left(\frac{ar - cd}{ae - db}\right)$$

and

$$h = \left(\frac{ce - rb}{ae - bd}\right)$$

From our original starting Eq. $a = 3,\ b = -1,\ c = -9$

$d = 1,\ e = 1,\ r = 1$

$$\therefore k = \frac{(3)(1) - (-9)(1)}{(3)(1) - (1)(-1)} = \frac{3 + 9}{3 + 1} = \frac{12}{4} = 3 \Rightarrow k = 3 \qquad \therefore x = X - 3$$

$$h = \frac{(-9)(1) - (1)(-1)}{(3)(1) - (-1)(1)} = \frac{-9 + 1}{3 + 1} = \frac{-8}{4} = -2 \Rightarrow h = -2 \qquad x = X + 2$$

Now having found h and k, we can proceed to solve the problem.

$$\frac{dy}{dx} = \frac{dY}{dX} = F\left(\frac{aX + bY}{dX + eY}\right) = F\left(\frac{a + b\,Y/X}{d + e\,Y/X}\right) \quad \text{homogeneous form}$$

$$\frac{dY}{dX} = \frac{3X - Y}{X + Y} = \frac{3 - Y/X}{1 + Y/X}$$

let $Y = u(X)\,X \qquad u(X) = \frac{Y}{X}$

$$\frac{dY}{dX} = \frac{du(X)}{dX}X + u(X)$$

Substituting for $\bar{Y}$ and $\frac{d\bar{Y}}{d\bar{X}}$

$$\frac{du(\bar{X})}{d\bar{X}}\bar{X} + u(\bar{X}) = \frac{3-u(\bar{X})}{1+u(\bar{X})} \Rightarrow \frac{du(\bar{X})}{d\bar{X}}\bar{X} = \frac{3-u(\bar{X})}{1+u(\bar{X})} - u(\bar{X})$$

$$\frac{d\bar{X}}{\bar{X}} = \frac{du(\bar{X})}{\frac{3-u(\bar{X})}{1+u(\bar{X})} - u(\bar{X})}$$ variables are separated

$$= \frac{du(\bar{X})}{\frac{3-u(\bar{X})-u(\bar{X})-u^2(\bar{X})}{1+u(\bar{X})}} = \frac{(1+u(\bar{X}))\,du(\bar{X})}{3-2u(\bar{X})-u^2(\bar{X})}$$

$$\int \frac{d\bar{X}}{\bar{X}} = -\int \frac{(u(\bar{X})+1)}{u^2(\bar{X})+2u(\bar{X})-3}\,du(\bar{X})$$

$$\Rightarrow \ln(\bar{X}) = -\frac{1}{2}\ln\left(u^2(\bar{X})+2u(\bar{X})-3\right) + \ln(c)$$

$$2\ln(\bar{X}) + \ln\left(u^2(\bar{X})+2u(\bar{X})-3\right) = \ln(c_1)$$

$$\ln\left((\bar{X}^2)(u^2(\bar{X})+2u(\bar{X})-3)\right) = \ln(c_1)$$

$$\Rightarrow \bar{X}^2(u^2(\bar{X})+2u(\bar{X})-3) = c_1 \qquad \text{now } u(\bar{X}) = \frac{\bar{Y}}{\bar{X}} \quad \text{and} \quad \begin{array}{l}\bar{X} = x+h = x-2\\ \bar{Y} = y+k = y+3\end{array}$$

$$\bar{X}^2\left(\left(\frac{\bar{Y}}{\bar{X}}\right)^2+2\left(\frac{\bar{Y}}{\bar{X}}\right)-3\right) = c_1 \Rightarrow \bar{Y}^2+2\bar{Y}\bar{X}-3\bar{X}^2$$

$$\Rightarrow (y+3)^2+2(y+3)(x-2)-3(x-2)^2 = c_1$$

$$\Rightarrow y^2+6y+9+2(xy+3x-2y-6)-3(x^2-4x+4) = c_1$$

$$y^2+6y+9+2xy+6x-4y-12-3x^2+12x-12 = c_1$$

$$y^2+2y+2xy+18x-3x^2-15 = c_1$$

$$y^2+2y+2xy+18x-3x^2 = c_2$$

This problem can also be solved as an exact Eq.

$$\frac{dy}{dx} = \frac{3x-y-9}{x+y+1} \Rightarrow \underbrace{(x+y+1)}_{N} dy - \underbrace{(3x-y-9)}_{M} dx = 0$$

$$\frac{\partial M}{\partial y} = 1 \; ; \; \frac{\partial N}{\partial x} = 1 \qquad \text{Eq. is exact.}$$

$$f(x,y) = C \; : \; df = \underbrace{\frac{\partial f}{\partial x}}_{M(x,y)} dx + \underbrace{\frac{\partial f}{\partial y}}_{N(x,y)} dy = 0$$

$$\frac{\partial f}{\partial x} = -3x+y+9 \Rightarrow df = (-3x+y+9)dx \Rightarrow \int df = \int (-3x+y+9)\,dx$$

$f(x,y) = -\frac{3}{2}x^2 + xy + 9x + h(y)$ take $\frac{\partial f}{\partial y}$ of this Eq. and set equal to $N(x,y)$

$$\frac{\partial f}{\partial y} = 0 + x + 0 + \frac{\partial h(y)}{\partial y} = N(x,y) = x+y+1$$

$$\Rightarrow x + \frac{\partial h(y)}{\partial y} = x+y+1 \Rightarrow \frac{\partial h(y)}{\partial y} = y+1$$

$$\Rightarrow \int d(h(y)) = \int (y+1)\,dy \Rightarrow h(y) = \frac{y^2}{2} + y + k$$

$$\therefore f(x,y) = -\frac{3}{2}x^2 + xy + 9x + \frac{y^2}{2} + y + k = C$$

$$-\frac{3}{2}x^2 + xy + 9x + \frac{y^2}{2} + y = C - k = C_1$$

now let's multiply through by 2

$$-3x^2 + 2xy + 18x + y^2 + 2y = C_2 \qquad \text{where } C_2 = 2C_1$$

same as solution on previous page

10. Find the general solution: $(3y^2 - x^2)dx = 2xy\,dy$

$\underbrace{(3y^2-x^2)}_{M(x,y)}dx = \underbrace{2xy}_{N(x,y)}dy$ let's check to see it is homogeneous

$$\frac{M(x,y)}{N(x,y)} = \frac{(3y^2-x^2)}{2xy} = \frac{(3(\frac{y}{x})^2-1)}{2(\frac{y}{x})} = \frac{(3u^2-1)}{2u}$$

Let $u = \frac{y}{x} \Rightarrow y(x) = u(x)x$ Eq. is homogeneous

$\frac{dy}{dx} = x\frac{du}{dx} + u \Rightarrow dy = x\,du + u\,dx$

$(3y^2-x^2)dx = 2xy\,dy \Rightarrow (3(\frac{y}{x})^2-1)dx - 2(\frac{y}{x})dy = 0$

$(3u^2-1)dx - 2u(x\,du + u\,dx) = 0$

$(3u^2-1)dx - 2ux\,du - 2u^2dx = 0$

$(3u^2-2u^2-1)dx - 2ux\,du = 0$

$(u^2-1)dx - 2ux\,du = 0$

$\frac{2u}{(u^2-1)} = \frac{dx}{x} \Rightarrow \ln(u^2-1) = \ln(x) + \ln(c) = \ln(xc)$

$u^2 - 1 = xc$ recall $u = y/x$

$(\frac{y}{x})^2 - 1 = xc \Rightarrow y^2 - x^2 = x^3c$

10. Find the general solution: $(2x^2 - y^2)\,dx + 3xy\,dy = 0$

let $x \to xt$ $M(2x^2 - y^2) \to M(2t^2x^2 - t^2y^2) = t^2 M(2x^2 - y^2)$
$y \to yt$ $N(3xy) \to N(3txyt) = t^2 N(3xy)$
Eq. is Homogeneous

let $y = ux \Rightarrow dy = u\,dx + x\,du$ substitution in the above Eq. gives

$$(2x^2 - u^2x^2)\,dx + 3x^2u\,(u\,dx + x\,du) = 0$$
$$(2x^2 - u^2x^2 + 3x^2u^2)\,dx + 3x^3u\,du = 0$$
$$(2x^2 + 2x^2u^2)\,dx = -3x^3u\,du$$
$$2x^2(1+u^2)\,dx = -3x^3u\,du$$

$$\frac{2x^2}{x^3}\,dx = -3\left(\frac{u}{1+u^2}\right)du$$

$$\int \frac{2}{x}\,dx = \int -3\left(\frac{u}{1+u^2}\right)du \Rightarrow 2\ln(x) = -\frac{3}{2}\ln(1+u^2) + c$$

$$-4\ln(x) = 3\ln(1+u^2) + c_1 \Rightarrow \ln(x^{-4}) = \ln((1+u^2)^3) + c_1$$

$$x^{-4}C_2 = (1+u^2)^3 \qquad \text{recall } u = \frac{y}{x}$$

$$x^{-4}C_2 = \left(1 + \frac{y^2}{x^2}\right)^3 \Rightarrow x^{-4}C_2 = (x^2+y^2)^3\left(\frac{1}{x^6}\right)$$

$$\Rightarrow x^2C_2 = (x^2+y^2)^3$$

11. Show that if one solution, say $y = u(x)$, of the Riccati equation $y' = P(x)y^2 + Q(x)y + R(x)$ is known, then the substitution $y = u + \frac{1}{z}$ will transform this equation into a linear first-order equation in the new variable z.

$$y' = P(x)y^2 + Q(x)y + R(x)$$

let $y(x) = u + \frac{1}{z}$

$$\frac{dy}{d(x)} = \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx}$$
$$= u' - \frac{1}{z^2} z'$$

substitute new variables into the above equation.

$$u' - \frac{1}{z^2} z' = P(x)\left(u^2 + 2\frac{u}{z} + \frac{1}{z^2}\right) + Q(x)\left(u + \frac{1}{z}\right) + R(x)$$

$$u' - \frac{1}{z^2} z' = P(x)u^2 + Q(x)u + R(x) + P(x)\left(2\frac{u}{z} + \frac{1}{z^2}\right) + Q(x)\frac{1}{z}$$

Note: since $u(x)$ is a solution to the differential equation, it satisfies the original D.E. $u' = P(x)u^2 + Q(x)u + R(x)$. Hence we are left with

$$-\frac{1}{z^2} z' = P(x)\left(2\frac{u}{z} + \frac{1}{z^2}\right) + Q(x)\frac{1}{z}$$
$$= P(x)\left(\frac{1}{z^2}\right) + \left(P(x)2u + Q(x)\right)\left(\frac{1}{z}\right)$$
$$= -P(x) - \left(2P(x)u(x) + Q(x)\right)z$$
$$\therefore\ z' + \left(2P(x)u(x) + Q(x)\right)z = -P(x)$$

where $P(x)$ and $Q(x)$ are functions from the original equation and $u(x)$ is the known solution.

This can also be written as

$$z' + H(x)z = -P(x) \qquad \text{where } H(x) = 2P(x)u(x) + Q(x)$$