definitions

1) definition 5, page 165, text book

let f be defined and bounded on [a,b] and let $P = \{x_i\}_{i=0}^{n}$ be a partition on [a,b], then defined $m_i(f)$ and $M_i(f)$ by

$m_i(f) = \inf\{\ f(x) : x \in [x_{i-1}, x_i]\ \}$

$M_i(f) = \sup\{\ f(x) : x \in [x_{i-1}, x_i]\ \}$

when only one function is involved, just say m_i and M_i

then $\sum_{i=1}^{n} m_i \nabla x_i$ = lower sum = $L(f,P)$

$\sum_{i=1}^{n} M_i \nabla x_i$ = upper sum = $U(f,P)$

then if $\sum_{i=1}^{n} f(c_i) \nabla x_i$ is any Riemann sum determined by f and P then

$$L(f,P) \le \sum_{i=1}^{n} f(c_i) \nabla x_i \le U(f,P)$$

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2)

definition 6, page 167 book

if f is bounded on [a,b], define

lower integral of f on [a,b] $= \underline{\int_a^b} f = \sup\{L(f,p) : P \text{ partition on } [a,b]\}$

upper integral of f on [a,b] $= \overline{\int_a^b} f = \inf\{U(f,p) : P \text{ partition on } [a,b]\}$

or, $\underline{\int_a^b} f = \sup_P L(f,P)$

$\overline{\int_a^b} f = \inf_P U(f,P)$

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4) theorm 26.1, page 181, text book

if f is continuous on interval [a,b] ⇒ f is RI on [a,b]

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5) theorm 26.2, page 182, text

if f is monotone on [a,b] ⇒ f is RI on [a,b]

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6) theorm 26.9, page 187, text

if f is IR over [a,b] and g is continuous on [c,d] so that $f(I) \subset \subseteq [c,d]$ ⇒ g o f is RI on I.

Lemma 27.1, page 188, text

1. f is RI on [a,b]
2. F is continuous on [a,b]
3. $F' = f$ on (a,b)

⇒ every patition P of [a,b] has Riemann sum = F(b)-F(a)

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7) theorm 27.2, page 189, text

THE FUNDEMENTAL THEOREM OF CALCULUS

1. $f(t)$ is RI on [a,b]
2. F is continuous on [a,b] with $F' = f$ on (a,b)

⇒ $\int_a^b f(x)\, dx = F(b) - F(a)$

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8)theorm 27.3, page 189, text

(the integral is a continuous function)

1. $f(t)$ is RI on interval I containing a

⇒ $F(x) = \int_a^x f(t)\, dt$ *is* continuouse on I

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9) theorm 27.4, page 190, text

1. f(t) is RI on I
2. a ∈ I
3. f(t) continouse at x_o

⇒ $F(x) = \int_a^x f(t)\, dt$ is differentiable at x_o and $F'(x_o) = f(x_o)$

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10) MEAN VALUE THEORMS

***** theorm 20.3, page 133, text

mean value theorm

1. f is continuous on [a,b]
2. f is differentiable on (a,b)

⇒ there is at least one value c ∈ (a,b) s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

this has geometric meaning, is that if you draw the line ab, then there is a point c between a,b such that a tangent to f at that point will be parallel to line ab.

****** theorm 22.1, page 150, text

cauchy mean value theorm

1. f, g are continuous on [a,b] and differentiable on (a,b)
2. assume that $g'(x) \neq 0$ for $a<x<b$

⇒ there exist some c ∈ (a,b) s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

***** theorm 25.7 , page 179, text book
MEAN VALUE THEORM FOR INTEGRALS
if f is continuous on [a,b], then there is some $c \in [a,b]$
so that $\int_a^b f(x)\, dx = f(c)(b-a)$

Bolzano-Weierstrass theorm for sequences
theorm 7.1, page 58, text
1. A is a bounded sequence
$\Rightarrow$ A has at least on convergent subsequence

Rolle's Theorm
Theorm 20.2, page 132, text
1. f is continuous on [a,b]
2. f is differentiable on (a,b)
3. $f(a) = f(b) = 0$

$\Rightarrow$ there exist some value $c \in (a,b)$ s.t. $f'(c) = 0$

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pointwise convergence of sequence of functions
defintion 1, page 201, text
sequence of functions $\{f_n(x)\}$ *converges pointwise to a limit function* $f(x)$ on the set S, if for EACH $xo \in S$, the sequence of constants $\{f_n(xo)\}$ converges to $\{f(xo)\}$.
this means that for each $x0 \in S$ and for each $\varepsilon>0$, there is some $N(xo,\varepsilon)$ so that $|f_n(xo)-f(xo)| < \varepsilon$ for all $n>N(xo,\varepsilon)$

Defintion 2, page 201, text uniform convergence
sequence of functions $\{f_n(x)\}$ converges uniformly to the limit function $f(x)$ on set S if for every $\varepsilon>0$, there is some $N(\varepsilon)$ so that $|f_n(x)-f(x)| < \varepsilon$ for all $n>N(\varepsilon)$ and for all $x \in S$

prof.'s definition:
$f_n \to f$ uniformly on S if $\underset{x \in s}{\text{lub}} |f_n(x)-f(x)| \to 0$ as $n \to \infty$

In words, pointwise convergence:
find the limit function f, i.e. given $f_n(x)$, see what happens when $n \to \infty$, then after you determined f, see if for each x, the limit $n \to \infty$ of $f_n(x) \to f(x)$, this should be true for each x. the above then means, that for each x, we generate a sequence see if that sequence go in the limit for the value of the limit function f at that point.

for pointwise convergence, just find the limit, that will be the pointwise convergence, if it exist.

for uniform convergence, we again first find the limit f as above. then we generate ONE sequence say M.

to build M, we do this, for each $n=1,2..\infty$, find what is the largest value of $f_n(x)$, i.e. over the entire range of $f_n(x)$, there will be a max. point, say $f\_max_n(x\_max)$, then find $f\_max_n(x\_max)-f(x\_max)$ i.e for each n, the function $f_n(x)$ will attain some max at some x, find the difference between f_n at this max x, and between the limit f at this max x.
do this for each n.
see if the sequence generated go to ZERO.

<u>to test for uniform convergence:</u>
two methods:

1) requires knowing the limit f:
   - find f
   - find where $f_n(x)$ is max, i.e. find x where $f_n(x)$ is max
   - find $M_n = |f_n(x\_max)-f|$
   - see if $M_n \to 0$ as $n \to \infty$
2) does not require knowing the limit f :
   M-test of Weierstrass.
   - see if you can find $M_n \geq |f_n(x)|$
   - if $\sum M_n$ converges, then $\{f_n\}$ converges uniformly

   proof: *if* $\sum M_n$, *then for arbitrary* $\epsilon>0$,

   $$\left|\sum_{i=n}^{m} f_i(x)\right| \leq \sum_{i=n}^{m} M_i \leq \epsilon \quad \text{(for all } x\text{)}$$

   NOTE: if M-test fails, that does not mean it is not uniform convergent. use method 1 to make sure, which means the need to find the limit.

Lemma 29.1, page 202, text
Let the sequence of functions $\{f_n(x)\}$ converge pointwise to $f(x)$ on the set S. Choose $xo \in S$ and a sequence $\{x_n\}$ so that $x_n \in S$ for all n.
if $\lim_{n\to\infty} x_n = xo$, and $\lim_{n\to\infty} f_n(x_n) \neq f(xo)$, then $\{f_n(x)\}$ *does not converge uniformly on S.*

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theorm 30.1 , page 208, text
(Uniform convergence and continuity)
1. each function f_n is continuous on set S
2. sequence $\{f_n(x)\}$ converges uniformly to f on S

$\Rightarrow$ f is continouse on S

corollary 30.2, page 209, text

1. each function f_n is continuous on set S for every n
2. sequence $\{f_n(x)\}$ converges pointwise to f on S
3. f is not continuous

$\Rightarrow$ $\{f_n(x)\}$ does not converge uniformly

Uniform Convergence and Integration

Theorm 30.3, page 211, text

1. sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ on [a,b]
2. each $f_n(x)$ is RI on [a,b]

$\Rightarrow$ $f(x)$ in RI on [a,b]

and $\int_a^b f(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

Corollary 30.3, page 213, text

1. *$f_n(x)$ is RI on [a,b] for each $n \in N$*
2. $\lim_{n\to\infty} f_n(x) = f(x)$
3. $\lim_{n\to\infty} \int_a^b f_n(x)\,dx \neq \int_a^b f(x)\,dx$

$\Rightarrow$ *$\{f_n(x)\}$ does not converge unformly to $f(x)$ on [a,b]*

a uniformaly convergent series of continuous functions can be integrated term by term.

a convergent series can be differentiated term by term, provided that the functions of the series have continuous derivatives and that the series of derivatives is itself is uniformaly convergent also.

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1. *$f_n \to f$ uniformly on S*

$\Rightarrow$ *fn-> f point wise*

Stone-Weierstrass Theorm

(class notes)

every continuous function can be approximated as closely as you want by a polynomial

if f is a continuous on [a,b], then f is the uniform limit of a sequence of polynomials.

Defintion (2.7) class notes

the sequence $\{f_n\}$ on S is uniformly cauchy if for $\varepsilon>0$ there exist N such that if $(m,n)>N \Rightarrow \text{lub } |f_m-f_n| < \varepsilon$

<u>POWER SERIES</u>

a power series is defined as

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{or when } c=0 \Rightarrow \sum_{n=0}^{\infty} a_n x^n$$

ABSOLUTE CONVERGENCE

if series $\sum_{n=1}^{\infty} x_n$ *converges, then* $\sum_{n=1}^{\infty} x_n$ *converges too.*

this is another test for convergence of a series, used for series with mixed signes. note that if this test fails, this does not mean that $\sum_{n=1}^{\infty} x_n$ diverges, but it means no conclusion can be made.

SERIES OF CONSTANTS:

nesessary condition for convergence if $\sum a_k$ *converges then*

$$\lim_{n\to\infty} a_k = 0$$

other tests avaliable are: integral test, ratio test, comparison test, the alternating series test , root-test , limit comparison test, all (?) of these require that the terms of the series by >0.

ratio-test: series must be of positive terms.

$$\text{let } p = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \qquad \text{if } p>1 \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges}$$

if $p<1$ it converges

if $p=1$ we cant decide

when using ratio-test to see where power series radius of convergence is, use absolute values on a_n, a_{n+1}

root-test: series must be of positive terms

let $p = \lim_{n\to\infty} \sqrt[k]{a_k}$, same results from p as the ratio test

integral test: *if* $a_n>0$ *for all* $n=1,2,..,$ *and if* $f(x)$ *is a* continuous decreasing *funvtion defined on* $[1,\infty)$ *so that* $f(n)=a_n$ *for each* $n=1,2,..$ *then*

$$\sum a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \quad \text{converges}$$

comparison test: *let* $\sum a_n$ *and* $\sum b_n$ *be infinite series with* $0<a_n\le b_n$ *for all* $n=1,2,..$ *then*

i) if $\sum b_n$ *converges then so does* $\sum a_n$

ii) if $\sum a_n$ *diverges, then so does* $\sum b_n$

limit comparison test: *let* $\sum a_n$ *and* $\sum b_n$ *be series with* $a_n>0$, $b_n>0$ *for all* $n=1,2,...$ *then if* $p = \lim_{n\to\infty} \frac{a_n}{b_n}$ *exits and* $p \neq 0$, *then either both series converge or both diverge. this way we can use this test to check on one series if we know if the other series converges or diverges.*

misc. theory on convergence: *let N be positive integer, the series*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Longleftrightarrow \sum_{n=K}^{\infty} a_n \quad \text{for all integer } N>1$$