Final exam, ECE 3341 Stochastic processes, Northeastern Univ. Boston

Nasser M. Abbasi

1993 Compiled on November 16, 2018 at 11:39am [public]

Contents

1 problem 1

part a

$$\begin{aligned}\mu_Y(n) &= E[Y_n] \\ &= E\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} E[X_i]\end{aligned}$$

but $E[X_i] = 1 \cdot P\{X_i = 1\} + 0 \cdot P\{X_i = 0\} = p$

$$\begin{aligned}\mu_Y(n) &= \sum_{i=1}^{n} p \\ &= np\end{aligned}$$

so

$$\boxed{\mu_Y(n) = \text{np}}$$

I'll find now a general expressing for $E[Y_m Y_n]$ that I need to use in this problem.

$$\begin{aligned}Y_m Y_n &= \left(\sum_{j=1}^{m} X_j\right)\left(\sum_{i=1}^{n} X_i\right) \\ &= (X_1 + X_2 + \cdots + X_m)(X_1 + X_2 + \cdots + X_n) \\ &= X_1X_1 + X_1X_2 + \cdots + X_1X_n \\ &+ X_2X_1 + X_2X_2 + \cdots + X_2X_n \\ &+ X_3X_1 + X_3X_2 + \cdots + X_3X_n \\ &+ \cdots \\ &+ X_mX_1 + X_mX_2 + \cdots + X_mX_n\end{aligned}$$

so, there are m rows, and n columns. also note that $E[X_i X_i] = E\left[X_i^2\right] = 0 \cdot (1-p) + 1^2 \cdot p = p$

and since $X_1, X_2, X_3, \cdots$ are all independent with each others. then $E\left[X_i X_j\right] = E\left[X_i\right] E\left[X_j\right] = p \cdot p = p^2$

now, if $m < n$ then there are m pairs of $X_i X_i$ and there are $(m \cdot (n-1))$.

if $n < m$, then are n pairs of $X_i X_i$ and there are $(n \cdot (m-1))$.

so, the general case is then

$$\boxed{\mathrm{E}[Y_m Y_n] = \min(m,n)(\max(m,n)-1)\mathrm{p} + \min(m,n)\mathrm{p}^2}$$

i.e. if

$$\boxed{\mathrm{m}<\mathrm{n} \Rightarrow \mathrm{E}[Y_m Y_n] = \mathrm{m}(n-1)\mathrm{p} + \mathrm{mp}^2}$$

if

$$\boxed{\mathrm{m}>\mathrm{n} \Rightarrow \mathrm{E}[Y_m Y_n] = \mathrm{n}(m-1)\mathrm{p} + \mathrm{np}^2}$$

when

$$\boxed{\mathrm{m}=\mathrm{n} \Rightarrow \mathrm{E}[Y_n Y_n] = \mathrm{E}\left[Y_n^2\right] = \mathrm{n}(n-1)\mathrm{p} + \mathrm{np}^2}$$

now,

$$\begin{aligned} \sigma_Y^2(n) &= E\left[Y_n^2\right] - E^2\left[Y_n\right] \\ &= n(n-1)p + np^2 - (np)^2 \end{aligned}$$

so

$$\begin{gathered} \sigma_Y^2(n) = n(n-1)p + np^2 - n^2p^2 \\ = n(n-1)p + np^2(1-n) \\ = (n-1)\left(np - np^2\right) \\ = np(n-1)(1-p) \end{gathered}$$

so

$$\boxed{\sigma_Y^2(n) = \mathrm{np}(n-1)(1-p)}$$

part b

$$K_Y(m,n) = E\left[Y_m X_n^*\right] - \mu_Y(m)\mu_X(n)$$
$$= \min(m,n)(\max(m,n)-1)p + \min(m,n)p^2 - (mp)(np)$$

so

$$K_Y(m,n) = \min(m,n)(\max(m,n)-1)p + \min(m,n)p^2 - mnp$$

i.e.

$$m < n \Rightarrow K_Y(m,n) = m(n-1)p + mp^2 - mnp = mp(p-1)$$

$$n < m \Rightarrow K_Y(m,n) = n(m-1)p + np^2 - mnp = np(p-1)$$

so

$$\boxed{K_Y(m,n)=\min(m,n)p(p-1)}$$

part c

$$\sigma_A^2 = E\left[A^2\right] - E^2[A]$$
$$= E\left[(Y_m - Y_n)^2\right] - E^2[Y_m - Y_n]$$
$$= E\left[Y_m^2 + Y_n^2 - 2Y_mY_n\right] - (E[Y_m] - E[Y_n])^2$$
$$= E\left[Y_m^2\right] + E\left[Y_n^2\right] - 2E[Y_mY_n] - \left(E^2[Y_m] + E^2[Y_n] - 2E[Y_m]E[Y_n]\right)$$
$$= E\left[Y_m^2\right] + E\left[Y_n^2\right] - 2E[Y_mY_n] - E^2[Y_m] - E^2[Y_n] + 2E[Y_m]E[Y_n]$$
$$= \left(E\left[Y_m^2\right] - E^2[Y_m]\right) + \left(E\left[Y_n^2\right] - E^2[Y_n]\right) - 2E[Y_mY_n] + 2E[Y_m]E[Y_n]$$
$$= \sigma_y^2(m) + \sigma_y^2(n) - 2E[Y_mY_n] + 2E[Y_m]E[Y_n]$$

now, since X_i are all independent with each others, then $E[Y_mY_n] = E[Y_m]E[Y_n]$, only if $E[X_i]E[X_i] = E[X_iX_i]$

for all i. $E[X_i]E[X_i] = p^2$ and $E[X_iX_i] = p$, so Y_ma nd Y_n are not independent with each others even though X_i, X_j are. so the general expression becomes:

$\sigma_A^2 = np(n-1)(1-p) + mp(m-1)(1-p) - 2\left[\min(m,n)(\max(m,n)-1)p + \min(m,n)p^2\right] + 2nmp^2$

so

$$m < n \Rightarrow \sigma_A^2 = np(n-1)(1-p) + mp(m-1)(1-p) - 2\left[m(n-1)p + mp^2\right] + 2nmp^2$$
$$= n^2\left(p-p^2\right) + n\left(p^2-p\right) + m^2\left(p-p^2\right) + m\left(p-p^2\right) + 2nm\left(p^2-p\right)$$

and

$$n < m \Rightarrow \sigma_A^2 = np(n-1)(1-p) + mp(m-1)(1-p) - 2\left[n(m-1)p + np^2\right] + 2nmp^2$$
$$= n^2\left(p-p^2\right) + n\left(p-p^2\right) + m^2\left(p-p^2\right) + m\left(p^2-p\right) + 2nm\left(p^2-p\right)$$

and

$$n = m \Rightarrow \sigma_A^2 = 0$$

I can simplify this more by writing

$$\gamma = p - p^2$$

so

$$m < n \Rightarrow \sigma_A^2 = n^2\gamma - n\gamma + \gamma m^2 - \gamma m + 2\gamma nm$$

and

$$n < m \Rightarrow \sigma_A^2 = \gamma n^2 + \gamma n + \gamma m^2 + \gamma m + 2\gamma nm$$

so, finally

$$\boxed{m<n \Rightarrow \sigma_A^2 = \gamma\left(n^2 + m^2 + 2nm\right) - \gamma(n+m)}$$

and

$$\boxed{n<m \Rightarrow \sigma_A^2 = \gamma\left(n^2 + m^2 + 2nm\right) + \gamma(n+m)}$$

where

$$\boxed{\gamma = p - p^2}$$

2 problem 2

$$\begin{aligned}
R_X(l) &= 5\delta(l) \\
S_X(\omega) &= 5 \\
S_Y(\omega) &= 5
\end{aligned}$$

$$\begin{aligned}
R_{X,Y}(l) &= 2\delta(l) \\
S_{XY}(\omega) &= 2
\end{aligned}$$

$$\begin{aligned}
h_1(n) &= u(n+2) - u(n-3) = \{1, 1, \boxed{1}, 1, 1\} \\
H_1(j\omega) &= \frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}
\end{aligned}$$

$$\begin{aligned}
h_2(n) &= [2 - |n|]\, h_1(n) = \{1, \boxed{2}, 1\} \\
H_2(j\omega) &= 2(1 + \cos\omega) \\
h_3(n) &= \left(\tfrac{1}{2}\right)^{|n|} = \left\{\cdots, \tfrac{1}{4}, \tfrac{1}{2}, \boxed{1}, \tfrac{1}{2}, \tfrac{1}{4}, \cdots\right\} \\
H_3(j\omega) &= \frac{1-\left(\frac{1}{2}\right)^2}{1-2\frac{1}{2}\cos\omega+\left(\frac{1}{2}\right)^2} = \frac{\frac{3}{4}}{\frac{3}{2}-\cos\omega} = \frac{3}{6-4\cos\omega}
\end{aligned}$$

$$\begin{aligned}
R_U(l) &= R_X(l) * h_1(l) * h_3(l) * h_1^*(-l) * h_3^*(-l) \\
&\quad + \\
&\quad R_Y(l) * h_2(l) * h_3(l) * h_2^*(-l) * h_3^*(-l) \\
&\quad + \\
&\quad R_{XY}(l) * h_3(l) * h_3^*(-l)
\end{aligned}$$

$$\begin{aligned}
S_U(\omega) &= S_X(\omega)\,|H_1(j\omega)|^2\,|H_3(j\omega)|^2 \\
&\quad + \\
&\quad S_Y(\omega)\,|H_2(j\omega)|^2\,|H_3(j\omega)|^2 \\
&\quad + \\
&\quad S_{XY}(\omega)\,|H_3(j\omega)|^2
\end{aligned}$$

$$\begin{aligned}
&= 5\left|\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\right|^2 \left|\frac{3}{6-4\cos\omega}\right|^2 \\
&\quad + \\
&\quad 5\,|2(1+\cos\omega)|^2 \left|\frac{3}{6-4\cos\omega}\right|^2 \\
&\quad + \\
&\quad 2\left|\frac{3}{6-4\cos\omega}\right|^2
\end{aligned}$$

so

$$S_U(\omega) = 5\left|\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\right|^2 \left|\frac{3}{6-4\cos\omega}\right|^2 + 5\left|2(1+\cos\omega)\right|^2 \left|\frac{3}{6-4\cos\omega}\right|^2 + 2\left|\frac{3}{6-4\cos\omega}\right|^2$$

so

$$\boxed{S_U(\omega) = -\frac{9}{4}\frac{57+80\cos(\omega)+20\cos(3\omega)+40\cos(2\omega)+10\cos(4\omega)}{-2\cos(2\omega)+12\cos(\omega)-11}}$$

3 problem 3

part a

let the time average of X_n be $\widehat{M}$, where

$$\widehat{M} \equiv \frac{1}{N}\sum_{n=1}^{N} X_n \qquad\qquad 0 \le n < \infty$$

the mean of $\widehat{M}$ is the ensemble mean of process X_n, i.e.

$$E\left[\widehat{M}\right] = E[X_n] = \mu_X$$

so, if the variance of $\widehat{M}$ is small, then we can say that the time average of R.P. X_n converges to the ensemble average of X_n. that is, we say that X_n is ergodic in the mean.

so, the condition I need to look for is to see if the variance of $\widehat{M}$ goes to zero as N goes very large.

i.e. if

$$\lim_{N\nearrow\infty} \sigma^2_{\widehat{M}} \longrightarrow 0$$

then X_n is ergodic in the mean.

since $\widehat{M}$ is a random variable, the convergence above is in the mean square sense.

Now, I find expression to this condition:

$$\begin{aligned}\sigma^2_{\widehat{M}} &= E\left[\left|\widehat{M} - E\left[\widehat{M}\right]\right|^2\right] \\ &= E\left[\left|\widehat{M} - \mu_X\right|^2\right]\end{aligned}$$

but

$$\widehat{M} - \mu_X = \left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) - \mu_X$$

but

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} X_n &= \frac{1}{N}\left(X_1 + X_2 + X_3 + \cdots + X_N + (N\cdot\mu_X - N\cdot\mu_X)\right) \\ &= \frac{1}{N}\left((X_1-\mu_X) + (X_2-\mu_X) + \cdots + (X_N-\mu_X) + (N\cdot\mu_X)\right) \\ &= \frac{1}{N}\left((X_1-\mu_X) + (X_2-\mu_X) + \cdots + (X_N-\mu_X)\right) + \mu_X \\ &= \left(\frac{1}{N}\sum_{n=1}^{N} X_n - \mu_X\right) + \mu_X\end{aligned}$$

so, substitute the above in equation (2) we get:

$$\widehat{M} - \mu_X = \left(\frac{1}{N}\sum_{n=1}^{N} X_n - \mu_X \right) + \mu_X - \mu_X = \frac{1}{N}\sum_{n=1}^{N} X_n - \mu_X$$

so

$$\begin{aligned}
\sigma^2_{\widehat{M}} &= E\left[\left| \widehat{M} - \mu_X \right|^2 \right] \\
&= E\left[\left| \frac{1}{N}\sum_{n=1}^{N} X_n - \mu_X \right|^2 \right] \\
&= \frac{1}{N^2} E\left[\left| \sum_{n=1}^{N} X_n - \mu_X \right|^2 \right] \\
&= \frac{1}{N^2} E\left[\sum_{n_1=1, n_2=1}^{N} \left(X_{n_1} - \mu_X \right) \left(X_{n_2} - \mu_X \right)^* \right] \\
&= \frac{1}{N^2} \sum_{n_1, n_2=1}^{N} E\left[\left(X_{n_1} - \mu_X \right) \left(X^*_{n_2} - \mu_X \right) \right]
\end{aligned}$$

since M.S. limit and $E[\cdot]$ operator can commute. so:

$$\sigma^2_{\widehat{M}} = \frac{1}{N^2} \sum_{n_1, n_2=1}^{N} K_X\left(n_1 - n_2\right)$$

since the process is stationary.

so my condition can be stated as

$$\lim_{N->\infty} \sigma^2_{\widehat{M}} = \lim_{N->\infty} \frac{1}{N^2} \sum_{\substack{n_1=1 \\ n_2=1}}^{N} K_X\left(n_1 - n_2\right) \longrightarrow 0$$

so, if the above goes to zero in the limit as indicated, then one can say that X_n is M.S. ergodic in the mean.

This in addition to the condition stated above, that

$$\boxed{\mathrm{E}\left[\widehat{M}\right] \equiv \mathrm{E}\left[\frac{1}{N}\sum_{n} = 1^N X_n\right] = \mathrm{E}[X_n]}$$

To simplify the condition in equation (3) above:

I need to find the sum $\sum_{\substack{n_1=1 \\ n_2=1}}^{N} K_X\left[n_1 - n_2\right]$

fix $n_2 = 1$,then partial sum $= K_X[1-1] + K_X[2-1] + K_X[3-1] + \cdots + K_X[N-1]$

fix $n_2 = 2$,then partial sum $= K_X[1-2] + K_X[2-2] + K_X[3-2] + \cdots + K_X[N-2]$

fix $n_2 = 3$,then partial sum $= K_X[1-3] + K_X[2-3] + K_X[3-3] + \cdots + K_X[N-3]$

. . ..

fix $n_2 = N$,then partial sum $= K_X[1-N] + K_X[2-N] + K_X[3-N] + \cdots + K_X[N-N]$

so, the above total sum is

$(K_X[0] + K_X[1] + K_X[2] + \cdots + K_X[N-1]) + (K_X[-1] + K_X[0] + K_X[1] + \cdots + K_X[N-2]) + \ldots (K_X[1-N] + K_X[2-N] + K_X[3-N] + \cdots + K_X[0])$

so $\sum_{\substack{n_1=1\\n_2=1}}^{N} K_X[n_1-n_2] = N\cdot K_X[0] + (N-1)(K_X[1]+K_X[-1]) + (N-2)(K_X[-2]+K_X[2]) +$
$(N-3)(K_X[-3]+K_X[3]) + \cdots + (1)(K_X[-(N-1)]+K_X[N-1])$

so

$$\frac{1}{N^2}\sum_{\substack{n_1=1\\n_2=1}}^{N} K_X[n_1-n_2] = \frac{1}{N}\sum_{n=-N}^{N}\left(1-\frac{|n|}{N}\right)K_X[n]$$

4 problem 4

part a

$X(t)$, for $t>0$, takes in 2 values, $\{1,-1\}$, so

$$E[X(t)] = (1\cdot P\{X(t)=1\} + (-1)\cdot P\{X(t)=-1\}) = P\{X(t)=1\} - P\{X(t)=-1\} \quad (1)$$

but

$$P\{X(t)=1\} = P\left\{(-1)^{N(t)}=1\right\}$$

but $P\left\{(-1)^{N(t)}=1\right\}$ is the same as the probability that $N(t)$ takes in even values, because when $N(t)$ takes in even values, then $(-1)^{N(t)}$ will have value of 1.

so,$P\left\{(-1)^{N(t)}=1\right\} = P\{N(t)=\text{even values}\}$

but the probability that $N(t)$ takes in even values = $P\{N(t)=2\}+P\{N(t)=4\}+P\{N(t)=6\}+\cdots$ This is because since the times of arrivals are independent from each others in a poisson process.

then $P\left\{(-1)^{N(t)}=1\right\} = P\{N(t)=\text{even values}\} = P_t(2)+P_t(4)+P_t(6)+\cdots = \boxed{\sum_{n=0}^{\infty} P_t(2n)}$

Similarly,

$$P\{X(t)=-1\} = P\left\{(-1)^{N(t)}=-1\right\}$$

again, similar to above argument, $P\left\{(-1)^{N(t)}=-1\right\}$ is the same as the probability that $N(t)$ takes in odd values, because when $N(t)$ takes in odd values, then $(-1)^{N(t)}$ will have value of -1.

so $P\left\{(-1)^{N(t)}=1\right\} = P\{N(t)=\text{odd values}\}$

but the probability that $N(t)$ takes in odd values = $P\{N(t)=1\}+P\{N(t)=3\}+P\{N(t)=5\}+\cdots$

then $P\left\{(-1)^{N(t)}=1\right\} = P\{N(t)=\text{odd values}\} = P_t(1)+P_t(3)+P_t(5)+\cdots = \boxed{\sum_n = 1^{\infty}\mathrm{P}_t(2n-1)}$

so, substituting in equation 1 above, we see

$$E[X(t)] = P\{X(t)=1\} - P\{X(t)=-1\} = \sum_{n=0}^{\infty} P_t(2n) - \sum_{n=1}^{\infty} P_t(2n-1) \quad (1)$$

but

$$\begin{aligned}\sum_{n=0}^{\infty} P_t(2n) &= P_t(0)+P_t(2)+P_t(4)+\cdots\\ &= \frac{(\lambda t)^0}{0!}e^{-\lambda t} + \frac{(\lambda t)^2}{2!}e^{-\lambda t} + \frac{(\lambda t)^4}{4!}e^{-\lambda t} + \cdots\\ &= e^{-\lambda t}\left(\frac{(\lambda t)^0}{0!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right)\\ &= e^{-\lambda t}\cosh\lambda t\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} P_t(2n-1) &= P_t(1) + P_t(3) + P_t(5) + \cdots \\ &= \frac{(\lambda t)^1}{1!} e^{-\lambda t} + \frac{(\lambda t)^3}{3!} e^{-\lambda t} + \frac{(\lambda t)^5}{5!} e^{-\lambda t} + \cdots \\ &= e^{-\lambda t}\left(\frac{(\lambda t)^1}{1!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots\right) \\ &= e^{-\lambda t} \sinh \lambda t\end{aligned}$$

so, equation 2 above becomes

$$\begin{aligned}\mu_X(t) &= \sum_{n=0}^{\infty} P_t(2n) - \sum_{n=1}^{\infty} P_t(2n-1) \\ &= e^{-\lambda t} \cosh \lambda t - e^{-\lambda t} \sinh \lambda t \\ &= e^{-\lambda t} (\cosh \lambda t - \sinh \lambda t) \qquad (3)\end{aligned}$$

now, $e^{-x} = \cosh x - \sinh x$ so let $y \equiv -\lambda t$ so

$$e^{-\lambda t} = \cosh \lambda t - \sinh \lambda t$$

we see immediately that equation (3) becomes

$$\boxed{\mu_X(t) = e^{-\lambda t} e^{-\lambda t} = e^{-2\lambda t}} \qquad t > 0$$

part b

first, let $t_1 - t_2 = \tau > 0$. now

$$\begin{aligned}R_X(t_1, t_2) &= E[X(t_1) X(t_2)] \\ &= (1) \cdot P\{X(t_1) = 1, X(t_2) = 1\} \\ &\quad + (-1) \cdot P\{X(t_1) = -1, X(t_2) = 1\} \\ &\quad + (-1) \cdot P\{X(t_1) = 1, X(t_2) = -1\} \\ &\quad + (1) P\{X(t_1) = -1, X(t_2) = -1\} \qquad (5)\end{aligned}$$

now,using the relation that $P\{A \mid B\} = \frac{P\{A,B\}}{P\{B\}}$, then

$$\begin{aligned}P\{X(t_1) = 1, X(t_2) = 1\} &= P\{X(t_1) = 1 \mid X(t_2) = 1\} \cdot P\{X(t_2) = 1\} \\ &= P\left\{(-1)^{N(t_1)} = 1 \mid (-1)^{N(t_2)} = 1\right\} \cdot P\left\{(-1)^{N(t_2)} = 1\right\} \\ &= P\{N(t_1) = even \mid N(t_2) = even\} \cdot P\{N(t_2) = even\} \qquad (6)\end{aligned}$$

now,when $X(t_2) = 1$,then for $X(t_1)$ to have value of 1, means that even number of points are between $t_2$and t_1, where the point, is the point of time when $X(t)$ switches between 1,-1.

so $P\{X(t_1) = 1 \mid X(t_2) = 1\} = P\{\text{there is even number of points between } t_2 \text{and } t_1\}$

But from part a, we find that $P\{\text{there is even number of points between 0 and } t\}$ =probability that $X(t)$ takes in a value of 1 at time t.

this means that probability that $X(t)$ takes in a value of 1 at time t is the same as talking about the probability that there are even number of points between 0 and t.

so, now I can say that $P\{\text{there is even number of points between 0 and } t\} = \sum_{n=0}^{\infty} P_t(2n) = e^{-\lambda t}\cosh\lambda t$

when $t_1 - t_2 = \tau \geq 0$, I can write the above by replacing t with τ as

$P\{\text{there is even number of points between } t_1 \text{ and } t_2\} = \sum_{n=0}^{\infty} P_{t_1-t_2}(2n) = e^{-\lambda\tau}\cosh\lambda\tau$

in other words,

$$P\{X(t_1) = 1 \mid X(t_2) = 1\} = e^{-\lambda\tau}\cosh\lambda\tau$$

and , from part a, we know that

$P\{X(t_2) = 1\} = P\{\text{there is even number of points between 0 and } t_2\} = e^{-\lambda t_2}\cosh\lambda t_2$

$$P\{X(t_2) = 1\} = e^{-\lambda t_2}\cosh\lambda t_2$$

so, substitute the above 2 relations in equation (6) gives:

$$\boxed{P\{X(t_1) = 1, X(t_2) = 1\} = e^{-}\lambda\tau\cosh\lambda\tau\ e^{-}\lambda t_2\cosh\lambda t_2} \qquad (7)$$

similarly,

$$P\{X(t_1) = -1, X(t_2) = 1\} = P\{X(t_1) = -1 \mid X(t_2) = 1\} \cdot P\{X(t_2) = 1\}$$

but again $P\{X(t_1) = -1 \mid X(t_2) = 1\} \equiv P\{\text{there is odd number of points between } t_1 \text{ and } t_2\}$

but $P\{\text{there is odd number of points between 0 and } t\} = \sum_{n=1}^{\infty} P_t(2n-1) = e^{-\lambda t}\sinh\lambda t$

so this means that the $P\{there\ is\ odd\ number\ of\ points\ between\ t_1\ and\ t_2\} = \sum_{n=1}^{\infty} P_{t_1-t_2}(2n-1) = e^{-\lambda\tau}\sinh\lambda\tau$

and $P\{X(t_2) = 1\} = P\{\text{there is even number of points between 0 and } t_2\} = \sum_{n=0}^{\infty} P_{t_2}(2n) = e^{-\lambda t_2}\lambda t_2$

so,

$P\{X(t_1) = -1, X(t_2) = 1\} = P\{N(t_1) = odd \mid N(t_2) = even\} \cdot P\{N(t_2) = even\} = e^{-\lambda\tau}\sinh\lambda\tau e^{-\lambda t_2}\cosh\lambda t_2$

i.e.

$$\boxed{P\{X(t_1) = -1, X(t_2) = 1\} = e^{-}\lambda\tau\sinh\lambda\tau e^{-}\lambda t_2\cosh\lambda t_2} \qquad (8)$$

similarly, i find

$$\boxed{P\{X(t_1) = 1, X(t_2) = -1\} = e^{-}\lambda\tau\sinh\lambda\tau e^{-}\lambda t_2\sinh\lambda t_2} \qquad (9)$$

and finally

$$\boxed{P\{X(t_1) = -1, X(t_2) = -1\} = e^{-}\lambda\tau\cosh\lambda\tau e^{-}\lambda t_2\sinh\lambda t_2} \qquad (10)$$

so, from equation (5), substitute in it equations 7,8,9,10, I get

$$
\begin{aligned}
R_X(t_1,t_2) &= e^{-\lambda\tau}\cosh\lambda\tau\ e^{-\lambda t_2}\cosh\lambda t_2 \\
&-e^{-\lambda\tau}\sinh\lambda\tau e^{-\lambda t_2}\cosh\lambda t_2 \\
&-e^{-\lambda\tau}\sinh\lambda\tau e^{-\lambda t_2}\sinh\lambda t_2 \\
&+e^{-\lambda\tau}\cosh\lambda\tau e^{-\lambda t_2}\sinh\lambda t_2
\end{aligned}
$$

so,

$$
\begin{aligned}
R_X(t_1,t_2) = e^{-\lambda\tau}e^{-\lambda t_2}(\cosh\lambda\tau\cosh\lambda t_2 - \sinh\lambda\tau\cosh\lambda t_2 - \sinh\lambda\tau\sinh\lambda t_2 + \cosh\lambda\tau\sinh\lambda t_2) \\
= e^{-\lambda\tau}e^{-\lambda t_2}(\cosh\lambda\tau(\cosh\lambda t_2+\sinh\lambda t_2) - \sinh\lambda\tau(\cosh\lambda t_2+\sinh\lambda t_2)) \qquad (11)
\end{aligned}
$$

but,

$$
\begin{aligned}
e^x &= \cosh x + \sinh x \\
e^{-x} &= \cosh x - \sinh x
\end{aligned}
$$

so, equation (11) becomes

$$
R_X(t_1,t_2) = e^{-\lambda\tau}e^{-\lambda t_2}\left(\cosh\lambda\tau\left(e^{\lambda t_2}\right) - \sinh\lambda\tau\left(e^{\lambda t_2}\right)\right) = e^{-\lambda\tau}(\cosh\lambda\tau - \sinh\lambda\tau) = e^{-\lambda\tau}e^{-\lambda\tau} = e^{-2\lambda\tau}
$$

i.e. for $t_1 > t_2 \geq 0$,and $\tau = t_1 - t_2$,

$$
\boxed{R_X(t_1,t_2) = e^{-}2\lambda(t_1-t_2)}
$$

similarly, one can let $t_2 > t_1 > 0$,and $\tau = t_2 - t_1$ and that would lead to

$$
\boxed{R_X(t_2,t_1) = e^{-}2\lambda(t_2-t_1)}
$$

so, from the above we see that

$$
\boxed{R_X(t_1,t_2)=e^{-}2\lambda\,|t_1-t_2| \qquad t_1,t_2\geq 0}
$$

part c

since $\mu_X(t)$ is a function of t, then $X(t)$ is a non-stationary process, so $X(t)$ is M.S. continuous at time t iff $R_X(t_1,t_2)$ is continuous at time $t_1=t_2\equiv t$.

so $R_X(t,t)=e^{-2\lambda|t-t|}=1$

so $X(t)$ is M.S. continuous.

R.P. $X(t)$ has M.S. derivative at time t iff $R_X(t_1,t_2)$ has a second order mixed derivative when $t_1=t_2\equiv t$.

$$\frac{\partial R_X(t_1,t_2)}{\partial t_1}=\frac{\partial}{\partial t_1}\left(e^{2\lambda(t_1-t_2)}u(-(t_1-t_2)+e^{-2\lambda(t_1-t_2)}u(t_1-t_2)\right)\begin{cases}\frac{1}{2\lambda}e^{2\lambda(t_1-t_2)} & t_2>t_1\\ -\frac{1}{2\lambda}e^{-2\lambda(t_1-t_2)} & t_1>t_2\end{cases}$$

and

$$\frac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}=\frac{\partial}{\partial t_2}\begin{cases}\frac{1}{2\lambda}e^{2\lambda(t_1-t_2)} & t_2>t_1\\ -\frac{1}{2\lambda}e^{-2\lambda(t_1-t_2)} & t_1>t_2\end{cases}=\begin{cases}-\frac{1}{4\lambda^2}e^{2\lambda(t_1-t_2)} & t_2>t_1\\ -\frac{1}{4\lambda^2}e^{2\lambda(t_1-t_2)} & t_1>t_2\end{cases}$$

at the line $t_1=t_2$, i.e. $\tau=0$ we get

$$\frac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}=-\frac{1}{4\lambda^2}$$

so

$$\lim_{\tau\searrow 0}\left(-\frac{1}{4\lambda^2}\right)=\left(-\frac{1}{4\lambda^2}\right)$$

so, the limit exist, so $X(t)$ is M.S. diferetiable.

5 problem 5

$X(t)$ uncorrelated means $R_X(t_1,t_2)=0$ for $t_1\neq t_2$, in other words, $R_X(\tau)=0$ for $\tau\neq 0$.

also note that $X(t)$ and $N(t)$ are orthogonal since they are uncorrelated with zero-mean.

$$K_X(t_1,t_2)=\sigma_X^2(t_1)\ \delta(t_1-t_2)=e^{-|t_1|}\ \delta(t_1-t_2)$$

so, since $X(t)$ is a zero-mean process, then

$$R_X(t_1,t_2)=e^{-|t_1|}\ \delta(t_1-t_2)$$

let

$$h(t)=h_1(t)*h_2(t)$$

where L_i means the time variable of the operator L is t_i, and L^* is the adjoint operator whose impulse response is $h^*(t,\tau)$.

$$\begin{aligned}h(t)&=h_1(t)*h_2(t)\\ H(\omega)&=H_1(\omega)\ H_2(\omega)\\ &=\frac{1}{1+j\omega}\frac{2}{2+j\omega}\\ &=\frac{2}{1+j\omega}-\frac{2}{2+j\omega}\\ h(t)&=F^{-1}\left\{\frac{2}{1+j\omega}-\frac{2}{2+j\omega}\right\}\end{aligned}$$

so

$$\boxed{\text{h}(t)\text{=2}(e^{-}t-e^{-}2t)\text{u}(t)}$$

so

$$\begin{aligned} R_{XY}(t_1,t_2) &= L_2^*\{R_X(t_1,t_2)\} \\ &= \int_{-\infty}^{\infty} h^*(\alpha)R_X(t_1;t_2-\alpha)\ d\alpha \\ &= \int_{-\infty}^{\infty} 2\left(e^{-\alpha}-e^{-2\alpha}\right)u(\alpha)e^{-|t_1|}\ \delta(t_1-(t_2-\alpha))\ d\alpha \\ &= \int_{0}^{\infty} 2\left(e^{-\alpha}-e^{-2\alpha}\right)e^{-|t_1|}\ \delta(t_1-(t_2-\alpha))\ d\alpha \end{aligned}$$

when $t_2-\alpha=t_1 \Longrightarrow \alpha=t_2-t_1\ >0$ then the above integral has a value of

$$R_{XY}(t_1,t_2)=2\left(e^{-(t_2-t_1)}-e^{-2(t_2-t_1)}\right)e^{-|t_1|}u(t_2-t_1)$$

or

$$\boxed{\text{R}_X Y(t_1,t_2)\text{=2}(e^{-}(t_2-t_1)-e^{-}2(t_2-t_1))\text{e}^{-}|t_1|\ \text{u}(t_2-t_1)}$$

now, I find $R_{_1YY}$ due to contribution from R_{XY} and find $R_{_2YY}$ due to contribution from R_{NY} and add them to get final $R_{YY}=R_{_1YY}+R_{_2YY}$ (since $N\perp X$)

now, Find contribution due to R_{XY}

$$\begin{aligned} R_{_1YY}(t_1,t_2) &= L_1\{R_{XY}(t_1,t_2)\} \\ &= \int_{-\infty}^{\infty} h(\alpha)R_{XY}(t_1-\alpha;t_2)\ d\alpha \\ &= \int_{-\infty}^{\infty} 2\left(e^{-\alpha}-e^{-2\alpha}\right)u(\alpha)\ \left[2\left(e^{-(t_2-(t_1-\alpha))}-e^{-2(t_2-(t_1-\alpha))}\right)e^{-|t_1-\alpha|}u(t_2-(t_1-\alpha))\right]\ d\alpha \end{aligned}$$

the above integral is exist only for $\alpha>0$, else it is zero , so

$$R_{_1YY}(t_1,t_2) = \int_{0}^{\infty} 2\left(e^{-\alpha}-e^{-2\alpha}\right)\ \left[2\left(e^{-(t_2-(t_1-\alpha))}-e^{-2(t_2-(t_1-\alpha))}\right)e^{-|t_1-\alpha|}u(t_2-(t_1-\alpha))\right]\ d\alpha$$

now, when $t_2-(t_1-\alpha)>0 \Rightarrow t_2-t_1+\alpha>0 \Rightarrow \alpha>t_1-t_2>0 \Rightarrow t_1-t_2>0$

so $u(t_2-(t_1-\alpha))=u(t_1-t_2)$

then

$$R_{_1YY}(t_1,t_2) = \int_{0}^{\infty} 2\left(e^{-\alpha}-e^{-2\alpha}\right)\ \left[2\left(e^{-(t_2-(t_1-\alpha))}-e^{-2(t_2-(t_1-\alpha))}\right)e^{-|t_1-\alpha|}u(t_1-t_2)\right]\ d\alpha \quad (2)$$

now

$$e^{-|t_1-\alpha|}=e^{t_1-\alpha}u(-t_1+\alpha)+e^{-t_1+\alpha}u(t_1-\alpha)$$

so if $t_1<0$ then, since $\alpha>0$ then

$$\int_0^\infty e^{-|t_1-\alpha|}d\alpha=\int_0^\infty e^{t_1-\alpha}d\alpha$$

and, when $t_1 > 0$

$$\int_0^{\infty} e^{-|t_1-\alpha|} d\alpha = \int_0^{t_1} e^{-t_1+\alpha} d\alpha + \int_{t_1}^{\infty} e^{t_1-\alpha} d\alpha$$

so , equation (1) can be written in 2 parts as

when $t_2 < t_1$ and $t_1 < 0$ then

$$R_{1YY}(t_1, t_2) = \int_0^{\infty} 2\left(e^{-\alpha} - e^{-2\alpha}\right) \left[2\left(e^{-(t_2-(t_1-\alpha))} - e^{-2(t_2-(t_1-\alpha))}\right) e^{t_1-\alpha}\right] d\alpha$$

$$R_{1YY}(t_1, t_2) = \boxed{\tfrac{1}{3}\mathrm{e}^2 t_1 - t_2 \text{-} \tfrac{1}{5}\mathrm{e}^3 t_1 - 2t_2}$$

when $t_2 < t_1$ and $t_1 > 0$ then

$$\begin{aligned} R_{1YY}(t_1, t_2) &= \int_0^{t_1} 2\left(e^{-\alpha} - e^{-2\alpha}\right) \left[2\left(e^{-(t_2-(t_1-\alpha))} - e^{-2(t_2-(t_1-\alpha))}\right) e^{-t_1+\alpha}\right] d\alpha \\ &\quad + \int_{t_1}^{\infty} 2\left(e^{-\alpha} - e^{-2\alpha}\right) \left[2\left(e^{-(t_2-(t_1-\alpha))} - e^{-2(t_2-(t_1-\alpha))}\right) e^{t_1-\alpha}\right] d\alpha \qquad (3) \\ &= \boxed{\text{-}\tfrac{8}{3}\mathrm{e}^{-} t_1 - t_2 + \mathrm{e}^{-} t_1 - 2t_2 + \mathrm{e}^{-} t_1 - t_2 \text{-} \tfrac{8}{15}\mathrm{e}^{-} 2t_1 - 2t_2 + 2\mathrm{e}^{-} t_2 \text{-} \tfrac{2}{3}\mathrm{e}^{-} 2t_2 + t_1} \end{aligned}$$

so, combine the above 2 expression in boxes, we get for when $t_2 < t_1$

$$\begin{aligned} R_{1YY}(t_1, t_2) &= \left(\tfrac{1}{3}e^{2t_1-t_2} - \tfrac{1}{5}e^{3t_1-2t_2}\right) u(-t_1) \\ &\quad + \left(-\tfrac{8}{3}e^{-t_1-t_2} + e^{-t_1-2t_2} + e^{-t_1-t_2} - \tfrac{8}{15}e^{-2t_1-2t_2} + 2e^{-t_2} - \tfrac{2}{3}e^{-2t_2+t_1}\right) u(t_1) \end{aligned}$$

part b

For white noise,

$$S_N(\omega) = \sigma_N^2 = 5$$

so

$$S_{NY}(\omega) = S_N(\omega) H_2^*(j\omega) = 5\frac{2}{2+j\omega} = \frac{10}{2+j\omega}$$

so

$$R_{NY}(\tau) = 10\ F^{-1}\{S_{NY}(\omega)\} = 10e^{-2\tau}\ u(\tau)$$

or we can write this by saying $\tau = t_1 - t_2$ then

$$\boxed{\mathrm{R}_N Y(t_1 - t_2)\text{=}10\mathrm{e}^{-}2\,(t_1 - t_2)\mathrm{u}(t_1 - t_2)}$$

Now,

$$S_{2YY}(\omega) = S_X(\omega) |H_2(j\omega)|^2$$

but

$$H_2(j\omega) = \frac{2}{2+j\omega}$$

so

$$|H_2(j\omega)|^2 = \frac{4}{4+\omega^2}$$

so

$$S_{2YY}(\omega) = 5\frac{4}{4+\omega^2}$$

so

$$R_{2YY}(\tau) = 5e^-2|\tau|$$

so, for $t_1 > t_2$ then, combine all results from part a and part b to get

$$\begin{aligned}
R_{YY}(t_1,t_2) &= R_{1YY}(t_1,t_2) + R_{2YY}(t_1,t_2) \\
&= \left(\tfrac{1}{3}e^{2t_1-t_2} - \tfrac{1}{5}e^{3t_1-2t_2}\right) u(-t_1)\, u(t_1-t_2) \\
&\quad + \left(-\tfrac{8}{3}e^{-t_1-t_2} + e^{-t_1-2t_2} + e^{-t_1-t_2} - \tfrac{8}{15}e^{-2t_1-2t_2} + 2e^{-t_2} - \tfrac{2}{3}e^{-2t_2+t_1}\right) u(t_1)\, u(t_1-t_2) \\
&\quad +5e^{-2|t_1-t_2|}
\end{aligned}$$

6 problem 6

since

$$K_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X\mu_X^*$$

then

$$\begin{aligned}
R_X(t_1,t_2) &= 25e^{-|t_1-t_2|} + 36 \\
R_X(\tau = t_1 - t_2) &= 25e^{-|\tau|} + 36
\end{aligned}$$

•$X(t)$ is strict sense stationary:

since the auto correlation function $R_X(t_1,t_2)$ is a function of (t_1-t_2) and since the mean is constant, then $X(t)$ is a WSS process. However to decide if it is SSS process, I need to determine if $X(t+T)$ has the same density function as $X(t)$ for any order. This I dont know from given information. so $\boxed{X(t) \text{ is not SSS processs based on what is given.}}$

•$X(t)$ has total average power DC of 36 watt:

find the power spectral:

$$\begin{aligned}
S_X(\omega) &= \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega t} d\tau \\
&= \int_{-\infty}^{\infty} \left(25e^{-|\tau|} + 36\right) e^{-j\omega\tau}\, d\tau \\
&= 36\int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau + 25\int_{-\infty}^{\infty} e^{-|\tau|}e^{-j\omega\tau}\, d\tau \\
&= 36\cdot 2\pi\delta(\omega) + 50\tfrac{1}{1+\omega^2}
\end{aligned}$$

so, let $\omega = 0$, total average DC power is $72\pi + 50 = 276.2$ watt

so, $\boxed{\text{the statment that } X(t) \text{ has total average DC power of 36 watt is NOT true.}}$

•$X(t)$ is M.S. ergodic in the mean:

a stationary R.P. is M.S. ergodic in the mean iff

$$\lim_{T\nearrow\infty} \frac{1}{2T}\int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) K_X(\tau)\, d\tau \longrightarrow 0$$

Fourier transform for triangular pulse$\left(1-\frac{|\tau|}{2T}\right)$ is $2T\left(\frac{\sin 2\pi fT}{2\pi fT}\right)^2$, using Parseval's theorem

$$
\begin{aligned}
\sigma_M^2 &= \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)K_X(\tau)\ d\tau \\
&= \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)25e^{-|\tau|}\ d\tau \\
&= \frac{1}{2T}\int_{-\infty}^{\infty}2T\left(\frac{\sin 2\pi fT}{2\pi fT}\right)^2 50\frac{1}{1+(2\pi f)^2}\ df \\
&= 50\int_{-\infty}^{\infty}\left(\frac{\sin 2\pi fT}{2\pi fT}\right)^2\frac{1}{1+(2\pi f)^2}\ df \\
\lim_{T\nearrow\infty}\sigma_M^2 &= 50\int_{-\infty}^{\infty}\lim_{T\nearrow\infty}\left(\frac{\sin 2\pi fT}{2\pi fT}\right)^2\frac{1}{1+(2\pi f)^2}\ df \\
&= 50\int_{-\infty}^{\infty}0\cdot df = 0
\end{aligned}
$$

so, $X(t)$ is M.S. ergodic in the mean.

•$X(t)$ has a periodic component:

A WSS process is a wide sense periodic if

$$\mu_X(t) = \mu_X(t+T) \quad \forall t$$

and the auto-covariance $K_X(t_1, t_2)$ is periodic.

the second condition above fails, so this is not a wide sense periodic function. This also Implies it is not M.S. period since M.S. periodicity is stronger than WS periodicity.

However, the question asks if $X(t)$ has at least one component of the process is periodic, Not if the process itself is periodic. It is possible that $X(t)$ has component that is periodic, but $X(t)$ not be periodic.

so I can't for certinity say that $X(t)$ has or not a periodic component.

•$X(t)$ has an AC power of 61 Watt:

total power = $\int_{-\infty}^{\infty} S_X(\omega)\ d\omega = \int_{-\infty}^{\infty} 36\cdot 2\pi\delta(\omega)+50\frac{1}{1+\omega^2}\ d\omega = 72\pi+50\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}\ d\omega = 72\pi+50\pi = 122\pi$ watt

but the DC power was found to be $(72\pi+50)$watt, so AC power$=122\pi-(72\pi+50) = 50\pi-50 = 107.07$ watt

so $X(t)$ do NOT have an AC power of 61 Watt.

•$X(t)$ has a variance of 25:

Variance $=\sigma_X^2(t) = K_X(t,t) \Rightarrow K_x(0) = 25e^0 = 25$

so $X(t)$ has a variance of 25 is True.

7 problem 7

$$R_x(\tau) = 3 + 2\exp\left(-4\tau^2\right)$$

part a

the power spectral density $S_X(\omega)$ is

$$S_X(\omega) = F\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j\omega\tau)\, d\tau$$

so

$$\begin{aligned} S_X(\omega) &= \int\limits_{-\infty}^{\infty} \left(3 + 2\exp\left(-4\tau^2\right)\right)\exp(-j\omega\tau)\, d\tau \\ &= \int\limits_{-\infty}^{\infty} 3\exp(-j\omega\tau)\, d\tau + 2\int\limits_{-\infty}^{\infty} \exp\left(-4\tau^2\right)\exp(-j\omega\tau)\, d\tau \\ &= 6\pi\, \delta(\omega) + \sqrt{\pi}\exp\left(-\tfrac{\omega^2}{16}\right) \end{aligned}$$

so

$$\boxed{S_X(\omega)=6\pi\ \delta(\omega)+\sqrt{\pi}\exp\left(-\tfrac{\omega^2}{16}\right)}$$

Part b

$$\begin{aligned} \text{total power} &= \int\limits_{-\infty}^{\infty} S_X(\omega)\ d\omega \\ &= \int\limits_{-\infty}^{\infty} \left(6\pi\, \delta(\omega) + \sqrt{\pi}\exp\left(-\tfrac{\omega^2}{16}\right)\right) d\omega \\ &= \int\limits_{-\infty}^{\infty} 6\pi\, \delta(\omega)\ d\omega + \int\limits_{-\infty}^{\infty} \sqrt{\pi}\exp\left(-\tfrac{\omega^2}{16}\right) d\omega \\ &= 6\pi + 4\pi \end{aligned}$$

$$\boxed{\text{total power}=10\pi}$$

now, power between $\frac{-1}{\sqrt{\pi}}$ and $\frac{1}{\sqrt{\pi}}$,call it p_1 , is given by

$$\begin{aligned} p_1 &= \int\limits_{-\frac{1}{\sqrt{\pi}}}^{\frac{1}{\sqrt{\pi}}} S_X(\omega)\ d\omega \\ &= \int\limits_{-\frac{1}{\sqrt{\pi}}}^{\frac{1}{\sqrt{\pi}}} 6\pi\, \delta(\omega)\ d\omega + \int\limits_{-\frac{1}{\sqrt{\pi}}}^{\frac{1}{\sqrt{\pi}}} \sqrt{\pi}\exp\left(-\tfrac{\omega^2}{16}\right) d\omega \\ &= 6\pi + 4\pi\ \text{erf}\left(\tfrac{1}{4\sqrt{\pi}}\right) \end{aligned}$$

where

$$\text{erf}(x) = \frac{2}{\pi}\int_0^x \exp\left(-t^2\right)\ dt$$

so, $\text{erf}\left(\frac{1}{4\sqrt{\pi}}\right)$ =erf(0.443) = 0.158

$$p_1 = 6\pi + 4\pi(0.158) = 20.84 \quad \text{Watt}$$

so fraction to total power is

$$\frac{p_1}{\text{total power}} = \frac{20.83}{10\pi} = 0.663 \Longrightarrow \boxed{\%\ 66.3}$$