6 What is the relation between variance and power for a random signal $x(t)$?

Variance is the sum of the total average normalized power and the DC power.

$$\sigma_x^2 = \overbrace{E\left[x^2(t)\right]}^{\text{total Power}} + \overbrace{E\left[x(t)\right]^2}^{\text{DC power}}$$

For the a signal whose mean is zero,

$$\sigma_x^2 = \overbrace{E\left[x^2(t)\right]}^{\text{total Power}}$$

How to find average, power, PEP, effective value (or the RMS) of a periodic function?

Let $x(t)$ be a periodic function, of period T, then

$$\text{average of } x(t) = \langle x(t) \rangle = \frac{1}{T}\int_0^T x(t)\,dt$$

The average power is

$$p_{av} = \left\langle x^2(t) \right\rangle = \frac{1}{T}\int_0^T |x(t)|^2\,dt$$

Effective value, or the RMS value is

$$x_{rms}(t) = \sqrt{\langle x^2(t) \rangle} = \sqrt{p_{av}} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}$$

For example, for $x(t) = \cos(x), \langle x(t) \rangle = 0, P_{av} = \frac{1}{2}, x_{rms}(t) = 0.707$

To find PEP (which is the peak envelope power), find the complex envelope $\tilde{x}(t)$, then find the average power of it. i.e.

$$PEP = \frac{1}{2}\tilde{x}_{\max}^2(t)$$

7 How to derive the Phase and Frequency modulation signals?

For any bandpass signal, we can write it as

$$x(t) = \operatorname{Re}\left(\tilde{x}(t)\,e^{j\omega_c t}\right)$$

Where $\tilde{x}(t)$ is the complex envelope of $x(t)$. For PM and FM, the baseband modulated signal, $\tilde{x}(t)$ has the form $A_c e^{j\theta(t)}$ Hence the above becomes

$$\begin{aligned} x(t) &= \operatorname{Re}\left(A_c e^{j\theta(t)} e^{j\omega_c t}\right) \\ &= A_c\left(\cos\omega_c t \cos\theta(t) - \sin\omega_c t \sin\theta(t)\right) \end{aligned}$$

But $\cos(A+B) = \cos A \cos B - \sin A \sin B$, hence the above becomes

$$x(t) = \cos(\omega_c t + \theta(t)) \tag{1}$$

The above is the general form for PM and FM. Now, for PM, $\theta(t) = k_p m(t)$ and for FM, $\theta(t) = k_f \int_0^t m(t_1)\, dt_1$. Hence, substituting in (1) we obtain

$$x_{FM}(t) = \cos\left(\omega_c t + k_f \int_0^t m(t_1)\, dt_1\right)$$

and

$$x_{PM}(t) = \cos(\omega_c t + k_p m(t))$$

Short sheet
443 CSUF

I) Amplitude Modulation

a) • AM wave $s_{AM}(t) = A_c[1 + K_a m(t)]\cos w_c t$

• modulation index $\mu = \frac{A_{max} - A_{min}}{A_{max} + A_{min}}$, where A_{max} is the max. of envelope

b) DSB-SC $S(t) = A_c m(t) \cos w_c t$

c) SSB $s(t) = \frac{A_c}{2} m(t)\cos w_c t \mp \frac{A_c}{2}\hat{m}(t)\sin w_c t$

where (−) for USB and (+) for LSB

$\hat{m}(t) = H.T[m(t)] = m(t) \circledast \frac{1}{\pi t}$ or

$\hat{M}(f) = -j\,\mathrm{sgn}(f) M(f)$

II) PM wave;

$s(t) = A_c \cos(w_c t + K_p m(t))$

III) FM wave:

• $s(t) = A_c \cos[2\pi f_c t + 2\pi K_f \int_0^t m(x)\,dx]$ (1)

• If $m(t)$ is a sine or cosine wave for example

if $m(t) = A_m \cos W_m t$ then eq (1) becomes single tone modulated signal;

• $s(t) = A_c \cos[2\pi f_c t + \beta \sin W_m t]$, where

• $\beta = \frac{\Delta f}{f_m} = \frac{K_f \cdot A_m}{f_m}$, β is modulation index

• $\Delta f = K_f A_m$ is the freq. deviation

• $f_i(t) = f_c + K_f m(t)$ inst. freq.

• $\theta_i(t) = 2\pi \int_0^t f_i(t_1)\,dt_1$ or $f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$

$\theta_i(t)$ is the inst. phase.

IV) Narrow band noise $n(t)$: Note if $E\{n(t)\} = 0 \Rightarrow E\{n_I\} = E\{n_Q\} = 0$

• $n(t) = n_I(t)\cos w_c t - n_Q(t)\sin w_c t$

• $S_{N_I}(f) = S_{N_Q}(f) = [S_N(f - f_c) + S_N(f + f_c)]\,\mathrm{rect}(\frac{f}{2B})$

where these are p.s.d of the narrowband noise and its in-phase and quadrature components

• The envelope of $n(t)$ is $a(t) = \sqrt{n_I^2 + n_Q^2}$

Table A11.1 Summary of Properties of the Fourier Transform

Property	Mathematical Description
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t)g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\, dt = G(0)$
7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)\, df$
8. Differentiation in the time domain	$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{G(0)}{2} \delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$
11. Multiplication in the time domain	$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda)\, d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)\, d\tau \rightleftharpoons G_1(f)G_2(f)$

instant. phase

$A_c \cos(\theta_i(t))$

phase deviation

$\theta_i(t) = \omega_c t + \phi(t)$

$\sin(t - \frac{\pi}{2}) = -\cos t$

$\cos(t - \frac{\pi}{2}) = \sin(t)$

$\frac{d\theta_i}{dt} = \omega_i(t) = \omega_c + \frac{d\phi}{dt}$

instant. Frequency

Frequency deviation

so inst. phase $= \omega_c t +$ phase deviation

inst. Freq $= \omega_c +$ freq. deviation

Table A11.4 Trigonometric Identities

$\exp(\pm j\theta) = \cos\theta \pm j\sin\theta$

$\cos\theta = \frac{1}{2}[\exp(j\theta) + \exp(-j\theta)]$

$\sin\theta = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$

$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)]$

$2\sin\theta\cos\theta = \sin(2\theta)$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$

$\cos\alpha\sin\beta = -\frac{1}{2}\sin(\alpha - \beta) + \frac{1}{2}\sin(\alpha + \beta)$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix};\ W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2}-\frac{i\sqrt{3}}{2} & -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\ 1 & -\frac{1}{2}+\frac{i\sqrt{3}}{2} & -\frac{1}{2}-\frac{i\sqrt{3}}{2} \end{bmatrix};\ W_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So if $x[n] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $X(k) = x[n]W_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$

To find IDFT, do $x[n] = \frac{W^*}{4} X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\Big[\ \Big]$

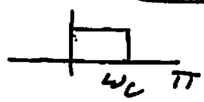
$$\boxed{W_N^k} = e^{-j\frac{2\pi k}{N}}$$

Properties of DFT: $X(k)$ periodic, period $= N$, Linear: $a_1x_1 + a_2x_2 \xleftrightarrow{DFT} a_1X_1 + a_2X_2$

$x_p(n) = \sum_{\ell=-\infty}^{\infty} x[n-\ell N]$.

$$\boxed{X_1 X_2 = \text{circular Convolution of } x_1, x_2}$$

ideal low pass



$$\Rightarrow h(n) = \begin{cases} \frac{\omega_c}{\pi} & n=0 \\ \frac{\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n} & n \neq 0 \end{cases}$$

LTI is Causal if specified by difference equation $y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

. To do Circular Convolution, do same as Linear convolution, just shift right!

input signal

$\Omega = 2\pi F$ → cycle/sec.

$\omega = 2\pi f n$

continuous $x(t)$ | energy | $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF$ | discrete $x[n]$

power → periodic(T)

CTFS (Fourier Series)

$$c(k) = \frac{1}{T}\int_T x(t) e^{-j\omega_0 k t} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 k t}$$

$$\boxed{P_x = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2}$$

aperiodic (Transform)

$$X(\Omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$x(t) = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty} X(\Omega) e^{j\omega t} d\Omega$$

if $x(t)$ real then $c(-k) = c^*(k)$

periodic T.

DTFS

$$C[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

$$x[n] = \sum_{k=0}^{N-1} c[k] e^{j\frac{2\pi k}{N}n}$$

$$E_N = \sum_{n=0}^{N-1} |x[n]|^2 = N\sum_{k=0}^{N-1} |c_k|^2$$

aperiodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$E_x = \sum |x(n)|^2 = \frac{1}{2\pi}\int |X(\omega)|^2 d\omega$$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n} \qquad k = 0,1,\dots N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\left(\frac{2\pi k}{N}\right)n} \qquad n = 0,1,\dots N-1$$

properties of Z transform

$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}}$ Roc $|z|>1$

$u(-n) \xleftrightarrow{z} \frac{1}{1-z}$ Roc $|z|<1$

$n x(n) \xleftrightarrow{z} -z\frac{dX(z)}{dz}$

$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$

$a^n x(n) \xleftrightarrow{z} X(a^{-1}z)$

$\boxed{x(-n) = X(z^{-1})}$

so find $X(z)$ for $x(n)$ and replace z by z^{-1}

$x^*(n) \xleftrightarrow{z} X^*(z^*)$

If $y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$

Then $H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$

Correlation

$$r_{x_1x_2}(\ell) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-\ell) \xleftrightarrow{z} X_1(z) X_2(z^{-1})$$

$r_{x_1x_2}(\ell) = x_1(\ell) * x_2(-\ell)$ (so Correlation is Convolution but without flipping)

Time		Frequency
Real, even	$\longleftrightarrow$	Real, even
Real, odd	$\longleftrightarrow$	Imaginary, odd
Im, even	$\longleftrightarrow$	Imaginary, even
Im, odd	$\longleftrightarrow$	Real, odd

$\delta(n) \xleftrightarrow{z} 1$ All z

$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}}$ $|z|>1$

$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$ $|z|>a$

$$\sum_{n=0}^{\infty} (a)^n = \frac{1}{1-a} \qquad |a|<1$$

$$\sum_{n=0}^{N} (a)^n = \frac{1-a^{N+1}}{1-a} \text{ or } \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \quad a \neq 1$$

Symmetry relationships for $X(\omega)$

$x^*(n) \longleftrightarrow X^*(-\omega)$

$x^*(-n) \longleftrightarrow X^*(\omega)$

If $x[n]$ is real Then:

$X(\omega) = X^*(-\omega)$

$|X(\omega)| = |X(-\omega)|$

convolution

$$y(n) = \sum_{k=-\infty}^{k=\infty} x(k)\, h(n-k)$$

IIR $\Rightarrow y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

FIR = all zero $\Rightarrow y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$

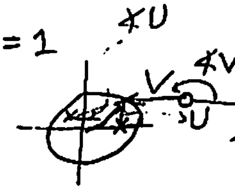
$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

1.

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1-z_k z^{-1})}{\prod_{k=1}^{N}(1-p_k z^{-1})} = \frac{(z-z_1)(z-z_2)\cdots}{(z-p_1)(z-p_2)\cdots}$$

b_0 selected such that $|H(\omega_0)|=1$

$$|H(\omega)|=|b_0|\frac{|V_1(\omega)||V_2(\omega)|\cdots|V_M(\omega)|}{|U_1(\omega)||U_2(\omega)|\cdots|U_N(\omega)|}$$



$$\boxed{\measuredangle H(\omega)=\measuredangle b_0+\omega(N-M)+\measuredangle V_1+\measuredangle V_2+\cdots-[\measuredangle U_1+\measuredangle U_2+\cdots]}$$

$$|H(\omega)|^2 = H(\omega)H^*(\omega)$$
$$= H(\omega)H(-\omega)$$
Real a_k, b_k.
$$= H(z)H(z^{-1})\big|_{z=e^{j\omega}}$$

Properties of DFT $X(N-k)=X^*(k)=X(-k)$ for real $x[n]$, $|X[N-k]|=|X[k]|$, $\measuredangle X(N-k)=-\measuredangle X[k]$

if $x[n]$ real & even, then $X[k]=\sum_{0}^{N-1} x[n]\cos\frac{2\pi k}{N}n$, $x[n]=\frac{1}{N}\sum X(k)\cos\frac{2\pi k}{N}n$

if $x[n]$ is real & odd, then $X[k]=\sum_{0}^{N-1}\left(-j\sin\frac{2\pi k}{N}n\right)$, $x[n]=j\frac{1}{N}\sum X(k)\sin\frac{2\pi k}{N}n$.

$x[N-n]=X[N-k]$, $x((n-\ell))_N \longleftrightarrow X(k)e^{-j\frac{2\pi k}{N}\ell}$; $x[n]e^{j\frac{2\pi \ell}{N}n} \longleftrightarrow X((k-\ell))_N$

$x^*[n]=X^*[N-k]$

$\tilde{r}_{xy}(\ell)=x(\ell)\,Ⓝ\,y^*(-\ell)$ ← circular correlation $x_1 x_2 \longleftrightarrow \frac{1}{N}X_1 Ⓝ X_2$

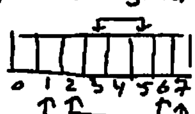
given $x[n]$ of length L, $h[n]$ of length M, How to know Length of DFT? $N=L+M-1$

so if x has $L=4$, Length $h[n]=3$, then we need $N=6$

note we can $X_1 X_2$ to find X_3, Then IDFT to find response of system. like Linear convolution, as long as we padd sequences.

$$\int_0^\infty e^{-t(1+j2\pi F)}dt=\frac{1}{1+j2\pi F}$$

if we are given some points of DFT, we can find rest using properties $\boxed{X(N-k)=X^*(k)=X(-k)}$



$N=8$. $X(6)=X(8-2)=X^*(2)$

for DFT when input is sin/cos, use

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k-1)} = N\delta(k-1)$$
$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k+1)} = N\delta(k+1)$$

$$\cos\alpha=\frac{e^{j\alpha}+e^{-j\alpha}}{2}$$
$$\sin\alpha=\frac{e^{j\alpha}-e^{-j\alpha}}{2j}$$

so if $x(n)=\cos\frac{2\pi}{N}n \rightarrow DFT=\frac{N}{2}[\delta(k-1)+\delta(k+1)]$

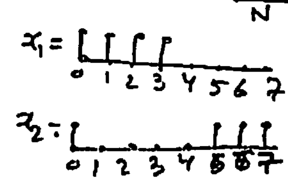
$x(n)=\sin\frac{2\pi}{N}n \rightarrow DFT=\frac{N}{2j}[\delta(k-1)-\delta(k+1)]$

if $X(k)=\frac{N^2}{4j}[\delta(k-1)+\delta(k+1)] \Rightarrow x(n)=\frac{N}{2}\sin\left(\frac{2\pi}{N}n\right)$

DFT if given sequence $x(n)$ and asked to find its Energy do

$E=\sum_{n=0}^{N-1}x(n)x^*(n)$. Example. $x(n)=\cos\frac{2\pi k}{N}n \Rightarrow x(n)x^*(n)=\frac{1}{4}\left(2+e^{+j\frac{4\pi kn}{N}}+e^{-j\frac{4\pi kn}{N}}\right)$

so $E=\sum_{n=0}^{N-1}x(n)x^*(n)=\frac{1}{4}2N=\frac{N}{2}$.



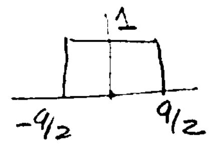
$x_2=x_1((n-5))_8 \Rightarrow X_2=X_1 e^{-j\frac{2\pi 5k}{N}}$ delay

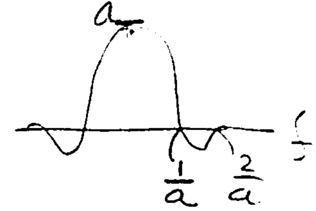
const spacing between

Least square method for design IIR & FIR

Properties of the Fourier Transform

Property	$f(t)$	$F(\omega)$
Linearity (Superposition)	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Time Shifting	$f(t - t_d)$	$e^{-j\omega t_d} F(\omega)$
Time Scaling	$f(ct)$	$\frac{1}{\lvert c\rvert} F\left(\frac{\omega}{c}\right)$
Symmetry (Duality)	$F(t)$	$2\pi f(-\omega)$
Time Reversal	$f(-t)$	$F(-\omega)$
Frequency Scaling	$f(t)e^{j\omega_c t}$	$F(\omega - \omega_c)$
Modulation	$f(t)cos(\omega_c t)$	$\frac{1}{2}F(\omega - \omega_c) + \frac{1}{2}F(\omega + \omega_c)$
Time Differentiation	$\frac{df(t)}{dt}$	$j\omega F(\omega)$
Frequency Differentiation	$tf(t)$	$j\frac{dF(\omega)}{d\omega}$
Conjugate	$f^*(t)$	$F^*(-\omega)$
Integration	$\int_{-\infty}^{t} f(\lambda)d\lambda$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$
Convolution	$\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$	$H(\omega)X(\omega)$
Multiplication	$f_1(t)f_2(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(v)F_2(\omega - v)dv$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2 d\omega$





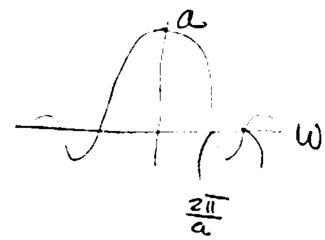
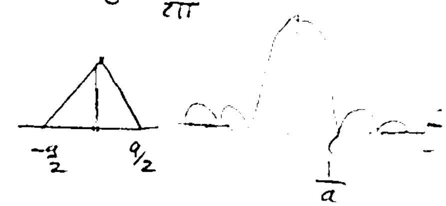


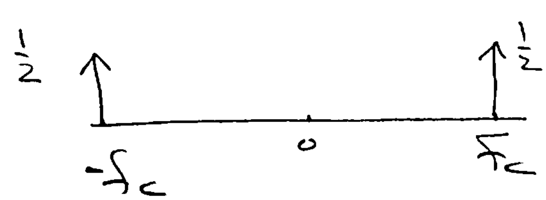
Table 16.2 Fourier Transform Pairs (a > 0)

$\omega = 2\pi f$

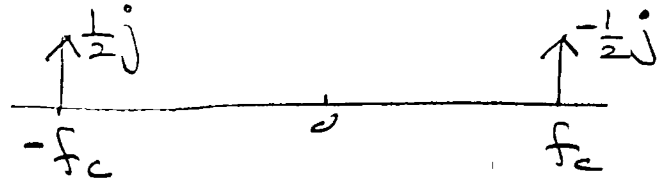
$f = \frac{\omega}{2\pi}$

$f(t)$	$F(\omega)$	$F(f)$
$\Pi\left(\frac{t}{a}\right) = rect\left(\frac{t}{a}\right)$	$a\,\text{sinc}\left(\frac{\omega a}{2\pi}\right)$	$a\,\text{sinc}(fa)$
$\Lambda\left(\frac{t}{a}\right) = tri\left(\frac{t}{a}\right)$	$a\,\text{sinc}^2\left(\frac{\omega a}{2\pi}\right)$	$a\,\text{sinc}^2(fa)$
$e^{-at}u(t)$	$\frac{1}{j\omega + a}$	$\frac{1}{j2\pi f + a}$
$e^{at}u(-t)$	$\frac{1}{-j\omega + a}$	$\frac{1}{-j2\pi f + a}$
$e^{-a\lvert t\rvert}$	$\frac{2a}{\omega^2 + a^2}$	$\frac{2a}{4\pi^2 f^2 + a^2}$
$e^{-at}u(t) - e^{at}u(-t)$	$\frac{-2j\omega}{\omega^2 + a^2}$	$\frac{-j4\pi f}{4\pi^2 f^2 + a^2}$
$\delta(t)$	1	1
1	$2\pi\delta(\omega)$	$\delta(f)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$tu(t)$	$\frac{\pi}{j\omega}\delta(\omega) + \frac{1}{(j\omega)^2}$	$\frac{1}{j4\pi f}\delta(f) + \frac{1}{(j2\pi f)^2}$
$te^{-at}u(t)$	$\frac{1}{(j\omega + a)^2}$	$\frac{1}{(j2\pi f + a)^2}$
$\cos(\omega_c t) = \cos(2\pi f_c t)$	$\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$

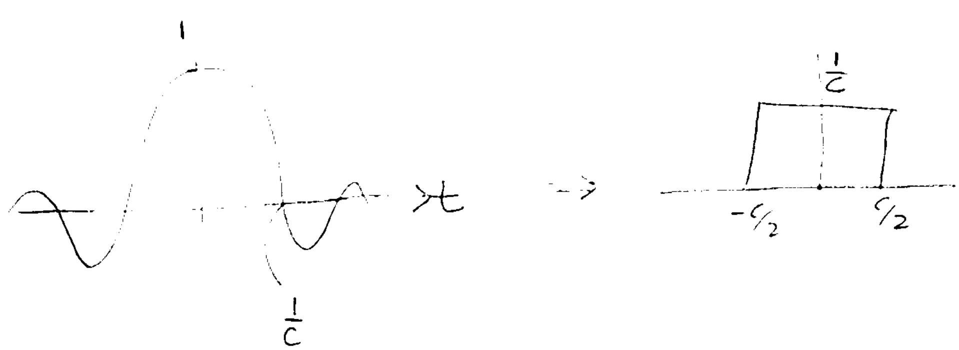




$\sin 2\pi f_c t$



$\sin(\omega_c t) = \sin(2\pi f_c t)$	$-j\pi[\delta(\omega-\omega_c)-\delta(\omega+\omega_c)]$	$\frac{-j}{2}[\delta(f-f_c)-\delta(f+f_c)]$
$e^{-at}u(t)\cos(\omega_c t)$	$\frac{j\omega+a}{(j\omega+a)^2+\omega_c^2}$	$\frac{j2\pi f+a}{(j2\pi f+a)^2+(2\pi f_c)^2}$
$e^{-at}u(t)\sin(\omega_c t)$	$\frac{\omega_c}{(j\omega+a)^2+\omega_c^2}$	$\frac{2\pi f_c}{(j2\pi f+a)^2+(2\pi f_c)^2}$
$sgn(t)$	$\frac{2}{j\omega}$	$\frac{1}{j\pi f}$
$\mathrm{sinc}(ct)$	$\frac{1}{c}rect\left(\frac{\omega}{2\pi c}\right)$	$\frac{1}{c}rect\left(\frac{f}{c}\right)$
$\mathrm{sinc}^2(ct)$	$\frac{1}{c}tri\left(\frac{\omega}{2\pi c}\right)$	$\frac{1}{c}tri\left(\frac{f}{c}\right)$
$\cos\left(\frac{\pi t}{a}\right)rect\left(\frac{t}{a}\right)$	$\frac{2a}{\pi}\frac{\cos\left(\frac{\omega a}{2}\right)}{1-\left(\frac{\omega a}{\pi}\right)^2}$	$\frac{2a}{\pi}\frac{\cos(\pi af)}{1-(2af)^2}$
$\frac{1}{2}\left[1+\cos\left(\frac{\pi t}{a}\right)\right]rect\left(\frac{t}{2a}\right)$	$a\frac{\sin(\omega a)}{\omega a\left[1-\left(\frac{\omega a}{\pi}\right)^2\right]}$	$a\frac{\sin(2\pi fa)}{2\pi fa\left[1-(2af)^2\right]}$

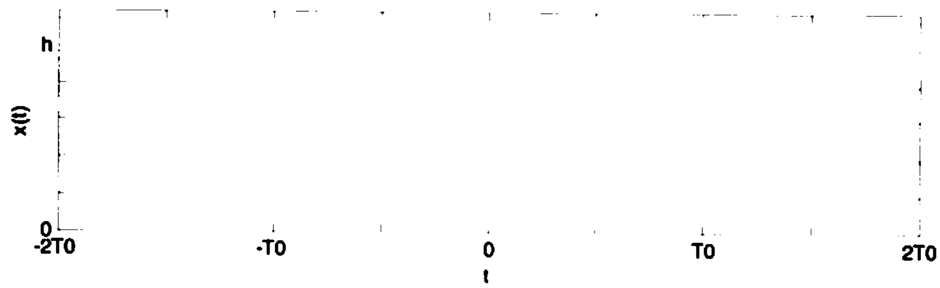


$\sin(\pi ct) = \pi K$

$t = \frac{k}{c}$

FOURIER SERIES REPRESENTATION OF COMMON SIGNALS

Rectangular Pulse Train



τ = pulse width ($-\tau/2$ to $\tau/2$)
d = duty cycle = τ/T_0.
ω_0 = fundamental frequency = $2\pi/T_0$
sinc(x) = sin(πx)/(πx)

$$X_n = \frac{ha}{T_0}\,\mathrm{sinc}(nd) = h\,d\,\mathrm{sinc}(nd) = \begin{cases} hd, & n = 0 \\ h\dfrac{\sin(n\pi d)}{n\pi}, & n \neq 0 \end{cases}$$

$$x(t) = hd + \sum_{n=1}^{\infty} 2hd\,\mathrm{sinc}(nd)\cos(n\omega_0 t)$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

$$c_0 = hd = \frac{h\tau}{T_0}, \quad c_n = |2hd\,\mathrm{sinc}(nd)|, \quad \theta_n = \begin{cases} \pi, & 2hd\,\mathrm{sinc}(nd) < 0 \\ 0, & \mathit{otherwise} \end{cases}$$

If $\tau = T_0/2$, d = 1/2, and the equations given above becomes

$$X_n = \frac{h}{2}\mathrm{sinc}\left(\frac{\mathrm{n}}{2}\right) = \begin{cases} \dfrac{h}{2}, & n = 0 \\ h\dfrac{\sin\left(\dfrac{\mathrm{n}\pi}{2}\right)}{n\pi}, & n \neq 0 \end{cases}$$

$$x(t) = \frac{h}{2} + \sum_{n=1}^{\infty} h\,\mathrm{sinc}\left(\frac{\mathrm{n}}{2}\right)\cos(n\omega_0 t)$$

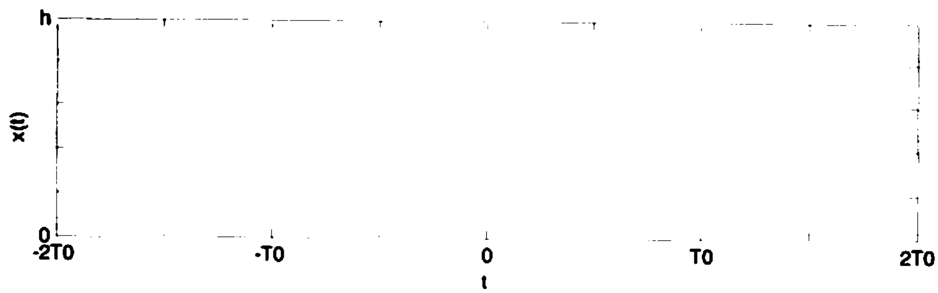
$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

$$c_0 = \frac{h}{2}, \quad c_n = \left| h\,\mathrm{sinc}\left(\frac{n}{2}\right) \right|, \quad \theta_n = \begin{cases} \pi, & h\,\mathrm{sinc}\left(\frac{n}{2}\right) < 0 \\ 0, & otherwise \end{cases}$$

Let y(t) = x(t − T_0/2). Then,

$$Y_n = X_n e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}} = X_n e^{-jn\pi} = X_n \cos(n\pi) = \begin{cases} \frac{h}{2}, & n = 0 \\ h\dfrac{(-1)^n \sin\left(\frac{n\pi}{2}\right)}{n\pi}, & n \neq 0 \end{cases}$$

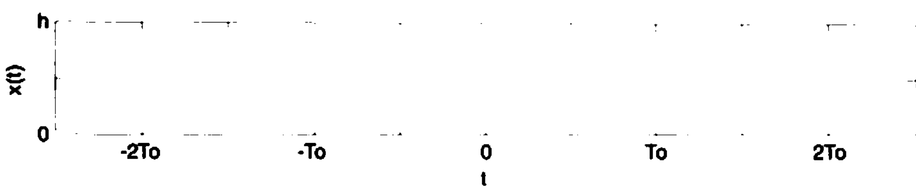
Triangular Pulse Train



τ = half of the base of the triangle ($-\tau \le t \le \tau$)
d = duty cycle = τ/T_0.
ω_0 = fundamental frequency = $2\pi/T_0$

$$X_n = hd\,\mathrm{sinc}^2(nd) = \frac{h\tau}{T_0}\mathrm{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right)$$

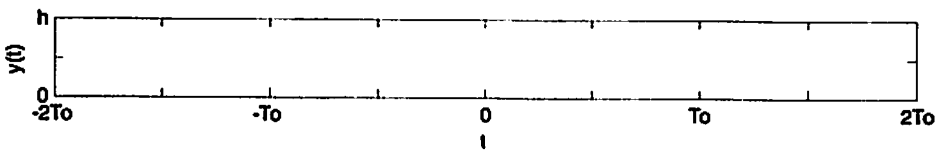
If $\tau = T_0/2$, then the pulse train looks like



and

$$X_n = \frac{h}{2}\text{sinc}^2\left(\frac{n}{2}\right) = \begin{cases} \frac{h}{2}, & n = 0 \\ 0, & n = even \\ \frac{2h}{n^2\pi^2}, & n = odd \end{cases}$$

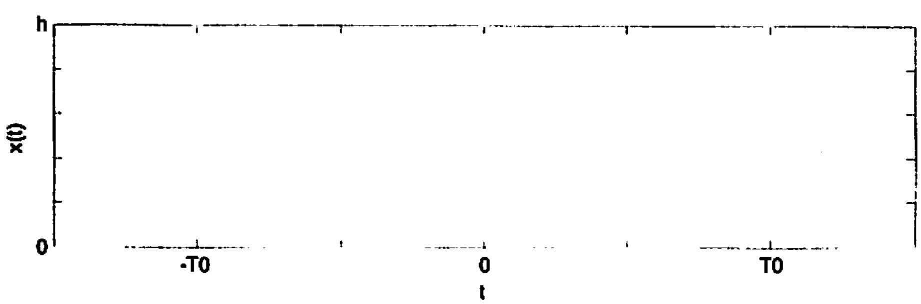
Let y(t) = x(t − T_0/2).



Then,

$$Y_n = X_n e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}} = X_n e^{-jn\pi} = X_n \cos(n\pi) = \begin{cases} \frac{h}{2}, & n = 0 \\ 0, & n = even \\ \frac{-2h}{n^2\pi^2}, & n = odd \end{cases}$$

Half-Wave Rectified Cosine

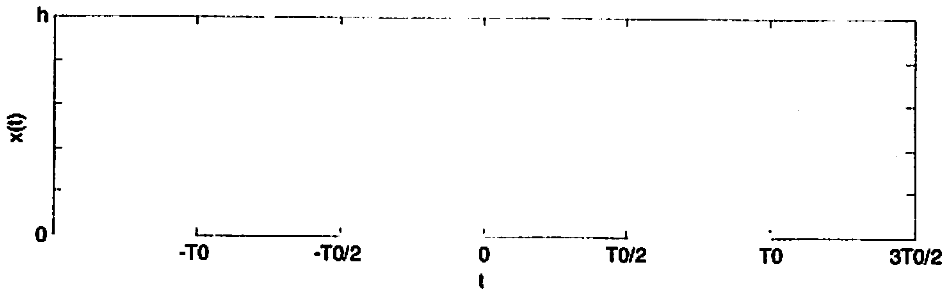


$$X_1 = \frac{h}{4}$$

$$X_{-1} = \frac{h}{4}$$

$$X_n = \frac{h}{\pi}\frac{\cos\left(\frac{n\pi}{2}\right)}{1-n^2}, \quad n \neq \pm 1$$

Half-Wave Rectified Sine



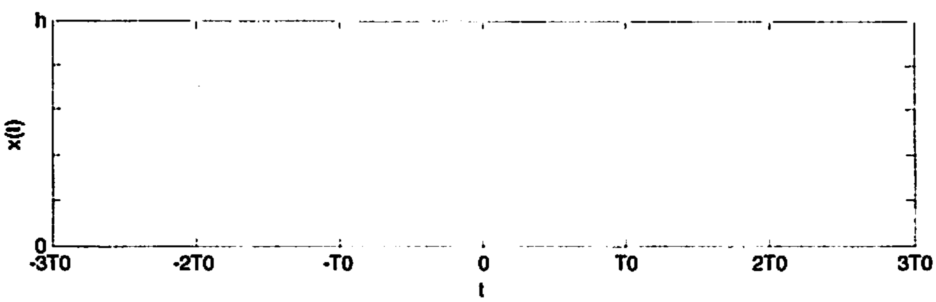
h = amplitude, $j = \sqrt{-1}$

$$X_1 = \frac{-jh}{4}$$

$$X_{-1} = \frac{jh}{4}$$

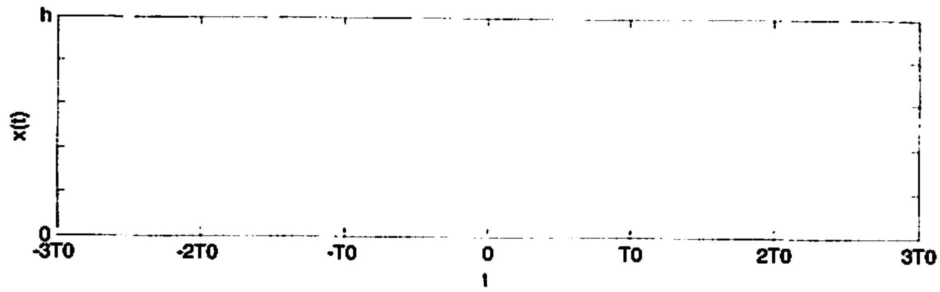
$$X_n = \frac{h}{\pi}\frac{\cos^2\left(\frac{n\pi}{2}\right)}{1-n^2} = \begin{cases} \frac{h}{\pi}, & n = 0 \\ 0, & n = \pm 3, \pm 5, \pm 7, ... \\ \frac{h}{\pi}\frac{1}{1-n^2}, & n = \pm 2, \pm 4, \pm 6, ... \end{cases}$$

Full-Wave Rectified Cosine



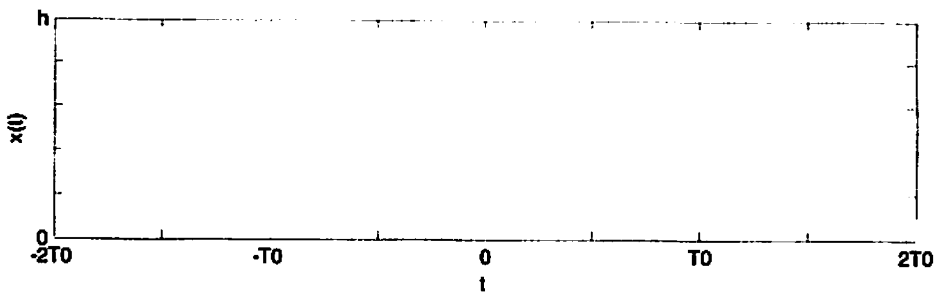
$$X_n = \frac{2h}{\pi}\frac{\cos(n\pi)}{1-4n^2} = \frac{2h}{\pi}\frac{(-1)^n}{1-4n^2}$$

Full-Wave Rectified Sine



$$X_n = \frac{2h}{\pi}\frac{\cos^2(n\pi)}{1-4n^2} = \frac{2h}{\pi}\frac{1}{1-4n^2}$$

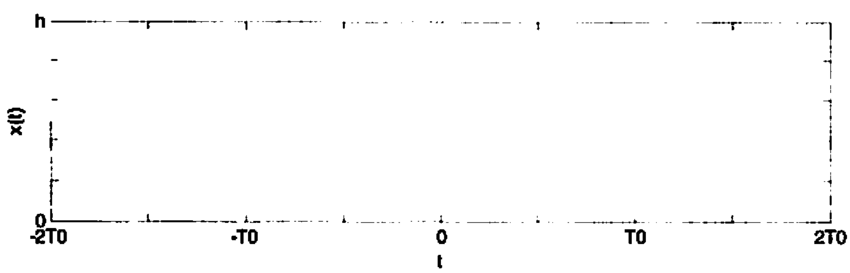
Sawtooth



$$X_0 = \frac{h}{2}$$

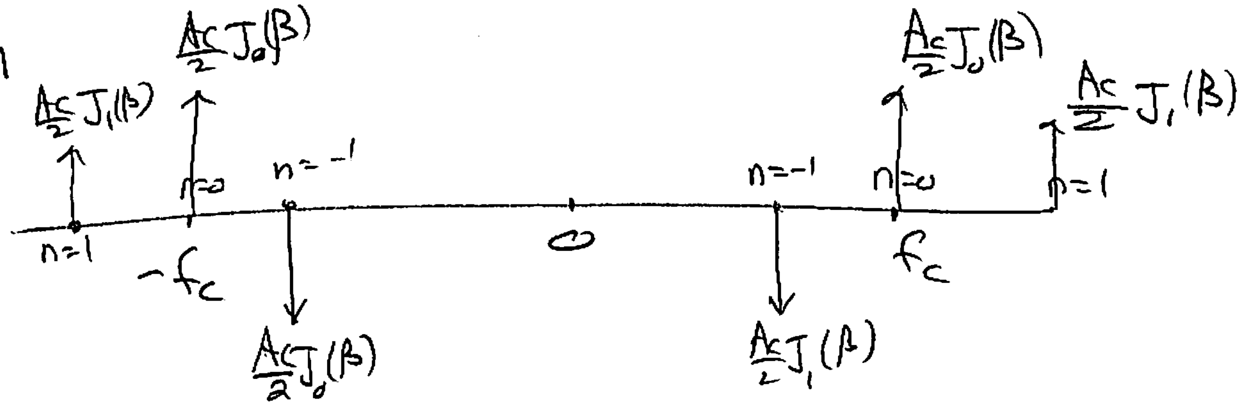
$$X_n = \frac{jh}{2\pi n}, \quad n \neq 0, \quad j = \sqrt{-1}$$

Exponential Decay



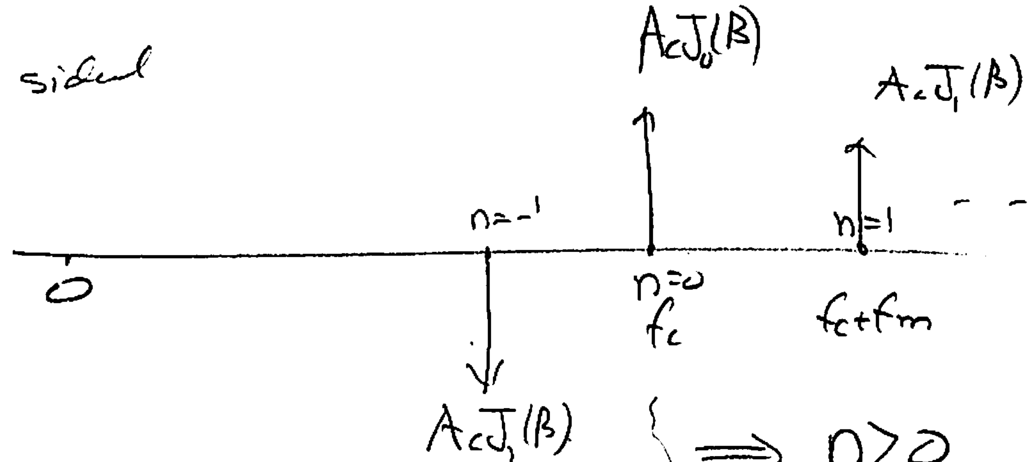
$$X_n = \frac{h}{T_0}\frac{1-e^{-aT_0}}{a + jn\frac{2\pi}{T_0}}$$

FM



$\Rightarrow$ 2 sided.

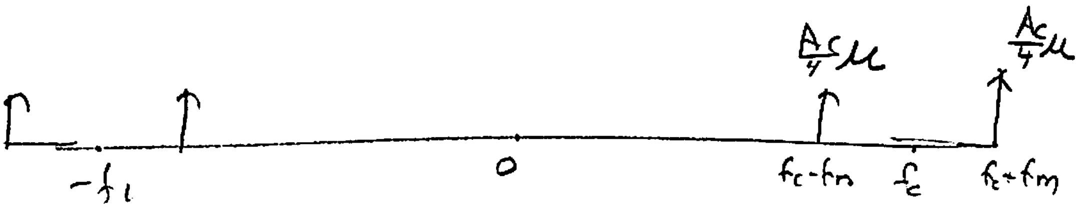
one sided



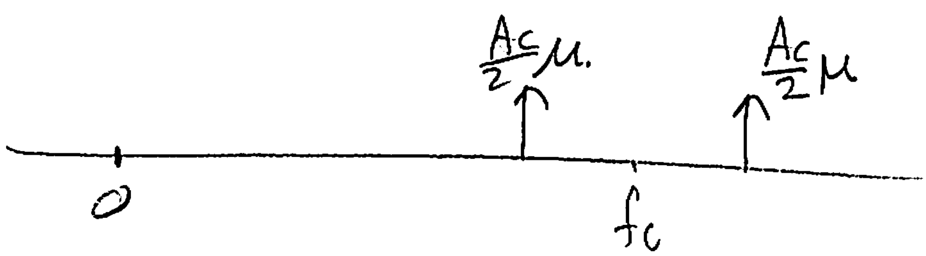
$\Rightarrow n>0$

AM

2 sided.



one sided



FM

$$FM(t) = A_c \cos(\overbrace{\omega_c t + \underbrace{k_f \int m(\lambda)d\lambda}_{\phi(t)\text{ phase deviation}}}^{\theta_i(t)})$$

Frequency deviation $= \frac{d}{dt}\phi(t)$

$= k_f\, m(t)$

max frequency deviation $= \max|k_f m(t)|$

$\Delta\omega = k_f \max|m(t)|$

$\omega_i(t)$ = instant. frequency

$$= \frac{d}{dt}\theta_i(t) = \omega_c + \overbrace{k_f m(t)}^{\phi'(t)}$$

PM

$$PM(t) = A_c \cos(\overbrace{\omega_c t + \underbrace{k_p m(t)}_{\phi(t)\text{ = phase deviation}}}^{\theta_i(t)})$$

Frequency deviation: $\frac{d}{dt}\phi(t) = k_p \frac{d}{dt}m(t)$.

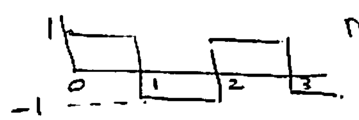
max frequency deviation: $k_p \max\left|\frac{dm}{dt}\right|$

$$\omega_i(t) = \frac{d}{dt}\theta_i(t) = \omega_c + k_p\frac{dm}{dt}$$

to plot FM(t), do

Find $\omega_i(t) = \omega_c + k_f\, m(t)$.

example.



$\omega_i(t) = \omega_c + k_f\, m(t)$