1. Problem 9.7.3. (15 points)

(a) Using the orthonormal basis $\{|m\rangle \to \frac{e^{i2m\pi x/L}}{\sqrt{L}},\ m = 0,\ \pm 1,\ \pm 2,\ \cdots\}$, we can expand

$$f(x) = \begin{cases} 2xh/L, & 0 \le x \le L/2, \\ 2(L-x)h/L, & L/2 \le x \le L \to 2x'h/L,\ 0 \le x' \le L/2, \end{cases}$$

as follows. Note that a change of variable $x' = L - x$ is made to simplify the integration over $L/2 \le x \le L$.

$$f_{m=0} = \langle m=0|f\rangle = \frac{1}{\sqrt{L}}\int_0^L f(x)dx = \frac{1}{\sqrt{L}}\left[\int_0^{L/2}\frac{2xh}{L}dx + \int_0^{L/2}\frac{2x'h}{L}dx'\right] = \frac{2}{\sqrt{L}}\frac{h}{L}\left(\frac{L}{2}\right)^2 = \frac{h\sqrt{L}}{2},$$

$$\begin{aligned} f_{m\neq 0} = \langle m|f\rangle &= \frac{1}{\sqrt{L}}\int_0^L e^{-\frac{i2m\pi x}{L}} f(x)dx = \frac{2h}{L\sqrt{L}}\left[\int_0^{L/2} x e^{-\frac{i2m\pi x}{L}}dx + \int_0^{L/2} x' e^{-\frac{i2m\pi(L-x')}{L}}dx'\right] \\ &= \frac{2h}{L\sqrt{L}}\int_0^{L/2} x\left[e^{-\frac{i2m\pi x}{L}} + e^{\frac{i2m\pi x}{L}}\right]dx = \frac{4h}{L\sqrt{L}}\int_0^{L/2} x\cos\frac{2m\pi x}{L}dx \\ &= \frac{4h}{L\sqrt{L}}[(-1)^m - 1]\left(\frac{L}{2m\pi}\right)^2 = [(-1)^m - 1]\frac{h\sqrt{L}}{(m\pi)^2} = \begin{cases} -2h\sqrt{L}/(m\pi)^2, & m = \text{odd}, \\ 0, & m = \text{even}. \end{cases} \end{aligned}$$

$$\begin{aligned} f(x) = \sum_m \frac{f_m}{\sqrt{L}} e^{\frac{i2m\pi x}{L}} &= \frac{h}{2} - \frac{2h}{\pi^2}\sum_{m=\text{odd}}\frac{e^{\frac{i2m\pi x}{L}}}{m^2} = \frac{h}{2} - \frac{2h}{\pi^2}\sum_{n=0}^{\infty}\frac{e^{\frac{i2(2n+1)\pi x}{L}} + e^{\frac{-i2(2n+1)\pi x}{L}}}{(2n+1)^2} \\ &= \frac{h}{2} - \frac{4h}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{2(2n+1)\pi x}{L} \end{aligned}$$

(b) For $x = 0$,

$$f(x) = 0 = \frac{h}{2} - \frac{4h}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

2. Problem 9.7.8. (35 points)

(i) We use the orthonormal basis $\{|m=0\rangle \to \frac{1}{\sqrt{L}},\ |m,\alpha=1\rangle \to \sqrt{\frac{2}{L}}\cos\frac{2m\pi x}{L},\ |m,\alpha=2\rangle \to \sqrt{\frac{2}{L}}\sin\frac{2m\pi x}{L},\ m=1,\ 2,\ \cdots\}$ to expand $f(x),\ -L/2 \le x \le L/2$.

For $f(x)=e^{-|x|},\ -1\le x\le 1$, we use the orthonormal basis $\{|m=0\rangle \to \frac{1}{\sqrt{2}},\ |m,\alpha=1\rangle \to \cos m\pi x,\ |m,\alpha=2\rangle \to \sin m\pi x,\ m=1,\ 2,\ \cdots\}$ for $L=2$. Note that $f(-x)=f(x)$.

$$a_0 = \langle m=0|f\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} e^{-|x|}dx = \frac{2}{\sqrt{2}}\int_0^1 e^{-x}dx = (1-e^{-1})\sqrt{2},$$

$$a_m = \langle m,\alpha=1|f\rangle = \int_{-1}^{1} e^{-|x|}\cos m\pi x\, dx = 2\int_0^1 e^{-x}\cos m\pi x\, dx = \int_0^1 e^{-x}(e^{im\pi x}+e^{-im\pi x})dx$$

$$= \frac{1-e^{-1+im\pi}}{1-im\pi} + \frac{1-e^{-1-im\pi}}{1+im\pi} = \frac{2[1-(-1)^m e^{-1}]}{1+(m\pi)^2},$$

$$b_m = \langle m,\alpha=2|f\rangle = \int_{-1}^{1} e^{-|x|}\sin m\pi x\, dx = 0,$$

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{m=1}^{\infty} a_m \cos m\pi x + \sum_{m=1}^{\infty} b_m \sin m\pi x = 1-e^{-1} + 2\sum_{m=1}^{\infty}\frac{1-(-1)^m e^{-1}}{1+(m\pi)^2}\cos m\pi x$$

(ii) $f(x)=\cosh x,\ -1\le x\le 1$. Note that $f(-x)=f(x)$.

$$a_0 = \langle m=0|f\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1}\cosh x\, dx = \frac{2}{\sqrt{2}}\int_0^1 \cosh x\, dx = (\sinh 1)\sqrt{2},$$

$$a_m = \langle m,\alpha=1|f\rangle = \int_{-1}^{1}\cosh x\cos m\pi x\, dx = 2\int_0^1 \cosh x \cos m\pi x\, dx$$

$$= \frac{1}{2}\int_0^1 (e^x+e^{-x})(e^{im\pi x}+e^{-im\pi x})dx$$

$$= \frac{1}{2}\left(\frac{e^{1+im\pi}-1}{1+im\pi} + \frac{e^{1-im\pi}-1}{1-im\pi} + \frac{1-e^{-1+im\pi}}{1-im\pi} + \frac{1-e^{-1-im\pi}}{1+im\pi}\right)$$

$$= \frac{(-1)^m(e-e^{-1})}{1+(m\pi)^2} = \frac{2(-1)^m(\sinh 1)}{1+(m\pi)^2},$$

$$b_m = \langle m,\alpha=2|f\rangle = \int_{-1}^{1}\cosh x \sin m\pi x\, dx = 0,$$

$$f(x) = \sinh 1 + \sum_{m=1}^{\infty}\frac{2(-1)^m(\sinh 1)}{1+(m\pi)^2}\cos m\pi x = (\sinh 1)\left[1 + 2\sum_{m=1}^{\infty}\frac{(-1)^m\cos m\pi x}{1+(m\pi)^2}\right]$$

(iii) For $f(x)=e^x,\ -\pi<x<\pi$, we use the orthonormal basis $\{|m=0\rangle \to \frac{1}{\sqrt{2\pi}},\ |m,\alpha=1\rangle \to \frac{\cos mx}{\sqrt{\pi}},\ |m,\alpha=2\rangle \to \frac{\sin mx}{\sqrt{\pi}},\ m=1,\ 2,\ \cdots\}$ for $L=2\pi$. Note that to enforce $f(-\pi)=f(\pi)$, we must allow two discontinuities at $x=\pm\pi$, which does not affect the convergence of the expansion within the interval $-\pi < x < \pi$.

$$
\begin{aligned}
a_0 &= \langle m=0|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^x\,dx = \frac{e^\pi - e^{-\pi}}{\sqrt{2\pi}} = (\sinh\pi)\sqrt{\frac{2}{\pi}}\,,\\
a_m &= \langle m,\alpha=1|f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} e^x \cos mx\,dx = \frac{1}{2\sqrt{\pi}}\int_{-\pi}^{\pi} e^x(e^{imx}+e^{-imx})dx\\
&= \frac{1}{2\sqrt{\pi}}\left[\frac{e^{(1+im)\pi}-e^{-(1+im)\pi}}{1+im}+\frac{e^{(1-im)\pi}-e^{-(1-im)\pi}}{1-im}\right]\\
&= \frac{(-1)^m(e^\pi-e^{-\pi})}{(1+m^2)\sqrt{\pi}} = \frac{2(-1)^m(\sinh\pi)}{(1+m^2)\sqrt{\pi}},\\
b_m &= \langle m,\alpha=2|f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} e^x \sin mx\,dx = \frac{1}{2i\sqrt{\pi}}\int_{-\pi}^{\pi} e^x(e^{imx}-e^{-imx})dx\\
&= \frac{1}{2i\sqrt{\pi}}\left[\frac{e^{(1+im)\pi}-e^{-(1+im)\pi}}{1+im}-\frac{e^{(1-im)\pi}-e^{-(1-im)\pi}}{1-im}\right]\\
&= -\frac{m(-1)^m(e^\pi-e^{-\pi})}{(1+m^2)\sqrt{\pi}} = -\frac{2m(-1)^m(\sinh\pi)}{(1+m^2)\sqrt{\pi}},\\
f(x) &= \frac{\sinh\pi}{\pi} + \sum_{m=1}^{\infty}\frac{2(-1)^m(\sinh\pi)}{(1+m^2)\pi}\cos mx - \sum_{m=1}^{\infty}\frac{2m(-1)^m(\sinh\pi)}{(1+m^2)\pi}\sin mx\\
&= \frac{\sinh\pi}{\pi}\left[1+2\sum_{m=1}^{\infty}\frac{(-1)^m(\cos mx - m\sin mx)}{1+m^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
\sinh x &= \frac{e^x-e^{-x}}{2} = \frac{f(x)-f(-x)}{2} = \frac{\sinh\pi}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m(-2m\sin mx)}{1+m^2}\\
&= \frac{2\sinh\pi}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}m\sin mx}{1+m^2}\\
\cosh x &= \frac{e^x+e^{-x}}{2} = \frac{f(x)+f(-x)}{2} = \frac{\sinh\pi}{\pi}\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m(2\cos mx)}{1+m^2}\right]\\
&= \frac{\sinh\pi}{\pi}\left[1+2\sum_{m=1}^{\infty}\frac{(-1)^m\cos mx}{1+m^2}\right]
\end{aligned}
$$

Note that $\cosh(-\pi) = \cosh\pi$, so we expect that its expansion is continuous at $x = \pm\pi$. We obtain

$$
\begin{aligned}
\cosh\pi &= \frac{\sinh\pi}{\pi}\left[1+2\sum_{m=1}^{\infty}\frac{(-1)^m\cos m\pi}{1+m^2}\right] = \frac{\sinh\pi}{\pi}\left(1+2\sum_{m=1}^{\infty}\frac{1}{1+m^2}\right)\\
\Rightarrow \sum_{m=1}^{\infty}\frac{1}{1+m^2} &= \frac{\pi\cosh\pi}{2\sinh\pi}-\frac{1}{2}\,.
\end{aligned}
$$

Clearly, the expansion of $\sinh x$ is discontinuous at $x = \pm\pi$ as it gives zero. Similarly, the expansion of e^x is also discontinuous at $x = \pm\pi$, as it gives $\cosh\pi$, i.e., the average of the actual values at $x = \pm\pi$.

3. Perform appropriate integration to show the following results regarding the Dirac delta function (25 points):

$$\delta(ax) = \delta(x)/|a|, \text{ where } a \text{ is a real number},$$
$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|df/dx|_{x_i}}, \text{ where } x_i \text{ satisfies } f(x_i)=0,$$
$$\frac{d}{dx}\delta(x-x') = \delta(x-x')\frac{d}{dx'}\,.$$

Consider $a > 0$ and let $y = ax$. We have

$$\int_{-\epsilon}^{\epsilon} \delta(ax)dx = \frac{1}{a}\int_{-a\epsilon}^{a\epsilon} \delta(y)dy = \frac{1}{a} = \int_{-\epsilon}^{\epsilon} \frac{\delta(x)}{a}dx \Rightarrow \delta(ax) = \delta(x)/a.$$

For $a < 0$, we have $\delta(ax) = \delta(-|a|x) = \delta(|a|x) = \delta(x)/|a|$, where the last equality follows from the result for $a > 0$. Therefore, $\delta(ax) = \delta(x)/|a|$ in general.

Consider Taylor expansion near a specific root x_i of $f(x)$:

$$f(x) = f(x_i) + f'(x_i)(x-x_i) + \cdots = f'(x_i)(x-x_i) + \cdots\,,$$
$$\Rightarrow \int_{x_i-\epsilon}^{x_i+\epsilon} \delta(f(x))dx = \int_{x_i-\epsilon}^{x_i+\epsilon} \delta(f'(x_i)(x-x_i))dx = \frac{1}{|f'(x_i)|},$$

where $f'(x_i) = (df/dx)_{x_i}$ and the last equality follows from $\delta(ax) = \delta(x)/|a|$. Including all roots of $f(x)$, we have

$$\sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} \delta(f(x))dx = \sum_i \frac{1}{|f'(x_i)|} = \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} \frac{\delta(x-x_i)}{|f'(x_i)|}dx \Rightarrow \delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}\,.$$

Note that

$$\frac{d}{dx}\int_{x-\epsilon}^{x+\epsilon} \delta(x-x')f(x')dx' = \int_{x-\epsilon}^{x+\epsilon} \frac{d}{dx}\delta(x-x')f(x')dx' + [\delta(x-x')f(x')]_{x-\epsilon}^{x+\epsilon}$$
$$= \int_{x-\epsilon}^{x+\epsilon} \frac{d}{dx}\delta(x-x')f(x')dx',$$

where we have used $\delta(x-x') = 0$ for $x - x' \neq 0$.

$$\int_{x-\epsilon}^{x+\epsilon} \frac{d}{dx}\delta(x-x')f(x')dx' = \frac{d}{dx}\int_{x-\epsilon}^{x+\epsilon} \delta(x-x')f(x')dx' = \frac{d}{dx}f(x),$$
$$\int_{x-\epsilon}^{x+\epsilon} \delta(x-x')\frac{d}{dx'}f(x')dx' = \frac{d}{dx}f(x) \Rightarrow \frac{d}{dx}\delta(x-x') = \delta(x-x')\frac{d}{dx'}\,.$$

4. For each energy eigenstate of a particle of mass m in the infinitely-deep potential well between $x = 0$ and L, find the probability distribution of the possible results when the particle momentum is measured. (25 points)

The wavefunctions of the energy eigenstates with eigenvalues $E_n = n^2\pi^2\hbar^2/(2mL^2)$ for $n = 1, 2, \cdots$ are

$$\langle x|\psi_n\rangle = \psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, & 0 < x < L, \\ 0, & \text{elsewhere.} \end{cases}$$

The wave function of the momentum eigenstate with eigenvalue p is

$$\langle x|\phi_p\rangle = \phi_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}.$$

The probability amplitude for measuring momentum p is

$$\begin{aligned}
\langle\phi_p|\psi_n\rangle &= \int_{-\infty}^{\infty}\langle\phi_p|x\rangle\langle x|\psi_n\rangle dx = \int_{-\infty}^{\infty}\phi_p^*(x)\psi_n(x)dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{L}}\int_0^L e^{-ipx/\hbar}\sin\frac{n\pi x}{L}dx = \frac{1}{2i\sqrt{\pi\hbar L}}\int_0^L e^{-ipx/\hbar}(e^{in\pi x/L} - e^{-in\pi x/L})dx \\
&= \frac{1}{2i\sqrt{\pi\hbar L}}\left(\frac{e^{-ipL/\hbar+in\pi}-1}{-ip/\hbar+in\pi/L} - \frac{e^{-ipL/\hbar-in\pi}-1}{-ip/\hbar-in\pi/L}\right) \\
&= \frac{(-1)^n e^{-ipL/\hbar}-1}{2\sqrt{\pi\hbar L}}\left(\frac{1}{p/\hbar-n\pi/L} - \frac{1}{p/\hbar+n\pi/L}\right) \\
&= \frac{(-1)^n e^{-ipL/\hbar}-1}{2\sqrt{\pi\hbar L}}\left[\frac{2n\pi/L}{(p/\hbar)^2-(n\pi/L)^2}\right] = \frac{1-(-1)^n e^{-ipL/\hbar}}{\sqrt{\pi\hbar L}}\left[\frac{n\pi/L}{(n\pi/L)^2-(p/\hbar)^2}\right],
\end{aligned}$$

which corresponds to the probability distribution

$$\begin{aligned}
P_n(p) = |\langle\phi_p|\psi_n\rangle|^2 &= \left[\frac{n\pi/L}{(n\pi/L)^2-(p/\hbar)^2}\right]^2\frac{[1-(-1)^n e^{-ipL/\hbar}][1-(-1)^n e^{ipL/\hbar}]}{\pi\hbar L} \\
&= \left[\frac{n\pi/L}{(n\pi/L)^2-(p/\hbar)^2}\right]^2\frac{2[1-(-1)^n\cos(pL/\hbar)]}{\pi\hbar L} \\
&= \frac{2L}{\pi\hbar}\left[\frac{n\pi}{(n\pi)^2-(pL/\hbar)^2}\right]^2[1-\cos(n\pi-pL/\hbar)] \\
&= \frac{4L}{\pi\hbar}\left[\frac{n\pi}{(n\pi)^2-(pL/\hbar)^2}\right]^2\sin^2\left(\frac{n\pi}{2}-\frac{pL}{2\hbar}\right) \\
&= \frac{L}{\pi\hbar}\left(\frac{n\pi}{n\pi+pL/\hbar}\right)^2\frac{\sin^2[(n\pi-pL/\hbar)/2]}{[(n\pi-pL/\hbar)/2]^2}.
\end{aligned}$$

Note that $-\infty < p < \infty$.