Physics 3041 (Spring 2021) Solutions to Homework Set 6

1. Problem 9.5.11. (40 points)

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) From the diagonal form of S_z, we know that its eigenvalues are $s_z = 1,\ 0,\ -1$ corresponding to eigenvectors

$$|s_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ |s_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ |s_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So the possible measured values for S_z are 1, 0, −1.

(b) For S_x,

$$\begin{vmatrix} -s_x & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -s_x & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -s_x \end{vmatrix} = -s_x(s_x^2 - \frac{1}{2}) - \frac{1}{\sqrt{2}}\frac{(-s_x)}{\sqrt{2}} = s_x(1-s_x^2) = s_x(1-s_x)(1+s_x)$$

$$\Rightarrow s_x = 1,\ 0,\ -1.$$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} -a_1 + b_1/\sqrt{2} \\ (a_1+c_1)/\sqrt{2} - b_1 \\ b_1/\sqrt{2} - c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |s_x = 1\rangle = a_1\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_2/\sqrt{2} \\ (a_2+c_2)/\sqrt{2} \\ b_2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |s_x = 0\rangle = a_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_3 + b_3/\sqrt{2} \\ (a_3+c_3)/\sqrt{2} + b_3 \\ b_3/\sqrt{2} + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |s_x = -1\rangle = a_3\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

For S_y,

$$\begin{vmatrix} -s_y & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -s_y & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -s_y \end{vmatrix} = -s_y(s_y^2 - \frac{1}{2}) + \frac{i}{\sqrt{2}}\frac{(-is_y)}{\sqrt{2}} = s_y(1 - s_y^2) = s_y(1 - s_y)(1 + s_y)$$

$$\Rightarrow s_y = 1,\ 0,\ -1.$$

So for both S_x and S_y, the possible measured values are 1, 0, and -1.

(c) After measuring the largest possible value of $s_x = 1$, the state vector is

$$|s_x = 1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(d) If S_z is measured, the possible values are 1, 0, and -1 with probabilities of $(1/2)^2 = 1/4$, $(1/\sqrt{2})^2 = 1/2$, and $(1/2)^2 = 1/4$, respectively.

If the largest possible value of $s_z = 1$ is measured, the state vector becomes

$$|s_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Because $|s_z = 1\rangle$ differs from $|s_x = 1\rangle$, the probability of measuring $s_x = 1$ is

$$|\langle s_x = 1|s_z = 1\rangle|^2 = \left|\frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right|^2 = \frac{1}{4}.$$

(e) From

$$\begin{aligned} S^2 &= S_x^2 + S_y^2 + S_z^2 \\ &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \end{aligned}$$

the measured S^2 value is always 2 for any state vector because $S^2 = 2I$.

(f) From the matrix representation and the results in (a) and (b), S_x, S_y, and S_z are Hermitian operators with non-degenerate eigenvalues. So if S_z commutes with S_x or S_y, they would share the same eigenvectors and be diagonal in the corresponding eigenbasis. However, because S_x and S_y are not diagonal in the basis where S_z is diagonal, we conclude S_z does not commute with either S_x or S_y.

Although we did not solve for the eigenvectors of S_y, it is clear that they are distinct from those of S_x because S_x differs from S_y in the structure of matrix elements but both operators have the same eigenvalues. So S_x and S_y do not commute, either.

On the other hand, S^2 commutes with S_x, S_y, and S_z. Therefore, the maximum number of commuting operators is 2, which corresponds to S^2 and any one of the other three (i.e., S_x, or S_y, or S_z).

(g) From

$$|V\rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \langle V|V\rangle = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = 14,$$

the normalized state vector is

$$|V'\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{\sqrt{14}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{\sqrt{14}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So the probabilities for measuring $s_z = 1$, 0, and -1 are 1/14, 2/7, and 9/14, respectively. The statistical average of the measured values is $\langle s_z\rangle = 1\times(1/14)+0\times(2/7)+(-1)\times(9/14) = -4/7$, which is the same as

$$\langle V'|S_z|V'\rangle = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = -\frac{4}{7}.$$

(h) The probabilities for measuring $s_x = 1$, 0, and -1 are

$$|\langle s_x = 1|V'\rangle|^2 = \frac{1}{4\times 14} \left| \begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right|^2 = \frac{(1+2\sqrt{2}+3)^2}{56} = \frac{3+2\sqrt{2}}{7}$$

$$|\langle s_x = 0|V'\rangle|^2 = \frac{1}{2\times 14} \left| \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right|^2 = \frac{(1-3)^2}{28} = \frac{1}{7}$$

$$|\langle s_x = -1|V'\rangle|^2 = \frac{1}{4\times 14} \left| \begin{bmatrix} 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right|^2 = \frac{(1-2\sqrt{2}+3)^2}{56} = \frac{3-2\sqrt{2}}{7},$$

respectively. The statistical average of the measured values is $\langle s_x\rangle = 1 \times (3+2\sqrt{2})/7 + 0 \times (1/7) + (-1) \times (3-2\sqrt{2})/7 = 4\sqrt{2}/7$, which is the same as

$$\langle V'|S_s|V'\rangle = \frac{1}{14\sqrt{2}} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{14\sqrt{2}} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \frac{4\sqrt{2}}{7}.$$

2. Prove the following results on the commutators: $[A, B+C] = [A, B] + [A, C]$, $[A+B, C] = [A, C] + [B, C]$, $[A, BC] = B[A, C] + [A, B]C$, $[AB, C] = A[B, C] + [A, C]B$. (10 points)

$$\begin{aligned}
[A, B+C] &= A(B+C) - (B+C)A = AB + AC - BA - CA = [A, B] + [A, C] \\
[A+B, C] &= (A+B)C - C(A+B) = AC + BC - CA - CB = [A, C] + [B, C] \\
[A, BC] &= ABC - BCA = ABC - BAC + BAC - BCA = [A, B]C + B[A, C] \\
[AB, C] &= ABC - CAB = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B
\end{aligned}$$

3. Follow the discussion of $s_+ = s_x + is_y$ for the electron spin to derive the matrix representation of $s_- = s_x - is_y$. (20 points)

$$\begin{aligned}
[s_z, s_-] &= [s_z, s_x - is_y] = [s_z, s_x] - i[s_z, s_y] = i\hbar s_y - i(-i\hbar s_x) = -\hbar(s_x - is_y) = -\hbar s_- \\
[s_z, s_-] &= s_z s_- - s_- s_z = -\hbar s_- \Rightarrow s_z s_- = s_- s_z - \hbar s_- \\
s_z|1\rangle &= \frac{\hbar}{2}|1\rangle \Rightarrow s_z s_-|1\rangle = (s_- s_z - \hbar s_-)|1\rangle = (s_-\frac{\hbar}{2} - \hbar s_-)|1\rangle = -\frac{\hbar}{2}s_-|1\rangle \\
s_z|2\rangle &= -\frac{\hbar}{2}|2\rangle \Rightarrow s_-|1\rangle = c|2\rangle \\
(s_-|1\rangle)^\dagger &= \langle 1|s_-^\dagger = \langle 1|(s_x - is_y)^\dagger = \langle 1|(s_x^\dagger + is_y^\dagger) = \langle 1|(s_x + is_y) = \langle 1|s_+ = c^*\langle 2|
\end{aligned}$$

$$\langle 1|s_+s_-|1\rangle = c^*c\langle 2|2\rangle = |c|^2$$

$$\langle 1|s_+s_-|1\rangle = \langle 1|(s_x + is_y)(s_x - is_y)|1\rangle = \langle 1|s_x^2 + s_y^2 - i(s_x s_y - s_y s_x)|1\rangle = \langle 1|s^2 - s_z^2 - i(i\hbar s_z)|1\rangle$$

$$= \langle 1|s^2 - s_z^2 + \hbar s_z|1\rangle = \frac{3}{4}\hbar^2 - \frac{\hbar}{2} \times \frac{\hbar}{2} + \frac{\hbar^2}{2} = \hbar^2 = |c|^2$$

pick $c = \hbar \Rightarrow s_-|1\rangle = \hbar|2\rangle$, $\langle 1|s_-|1\rangle = \langle 1|\hbar|2\rangle = 0$, $\langle 2|s_-|1\rangle = \langle 2|\hbar|2\rangle = \hbar$

$$s_z|2\rangle = -\frac{\hbar}{2}|2\rangle \Rightarrow s_z s_-|2\rangle = (s_- s_z - \hbar s_-)|2\rangle = [s_-(-\frac{\hbar}{2}) - \hbar s_-]|2\rangle = -\frac{3\hbar}{2}s_-|2\rangle$$

The above result appears to imply that $s_-|2\rangle$ is an eigenstate of s_z with an eigenvalue of $-3\hbar/2$, which is in conflict with experiments. So the only logical result is

$$s_-|2\rangle = 0|2\rangle \Rightarrow \langle 1|s_-|2\rangle = 0,\ \langle 2|s_-|2\rangle = 0.$$

Finally, we obtain

$$s_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

4. Problem 9.6.2, and find the solutions for $x_1(t)$ and $x_2(t)$ with the initial conditions $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = v_1$ and $\dot{x}_2(0) = v_2$. (30 points)

$$m\ddot{x}_1 = -kx_1 + 2k(x_2 - x_1) = -3kx_1 + 2kx_2,\ \ddot{x}_1 = -\frac{3k}{m}x_1 + \frac{2k}{m}x_2$$
$$m\ddot{x}_2 = -2k(x_2 - x_1) - kx_2 = -3kx_2 + 2kx_1,\ \ddot{x}_2 = -\frac{3k}{m}x_2 + \frac{2k}{m}x_1$$
$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3k/m & 2k/m \\ 2k/m & -3k/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{vmatrix} -3k/m - \lambda & 2k/m \\ 2k/m & -3k/m - \lambda \end{vmatrix} = \left(-\frac{3k}{m} - \lambda\right)^2 - \left(\frac{2k}{m}\right)^2 = 0 \Rightarrow \lambda_I = -\frac{k}{m},\ \lambda_{II} = -\frac{5k}{m}$$
$$\begin{bmatrix} -2k/m & 2k/m \\ 2k/m & -2k/m \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \frac{2k}{m}\begin{bmatrix} -a_1 + b_1 \\ a_1 - b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow |I\rangle = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{vmatrix} 2k/m & 2k/m \\ 2k/m & 2k/m \end{vmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \frac{2k}{m}\begin{bmatrix} a_2 + b_2 \\ a_2 + b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow |II\rangle = a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|x(t)\rangle = x_I(t)|I\rangle + x_{II}(t)|II\rangle$$
$$\Rightarrow \frac{d^2}{dt^2}\begin{bmatrix} x_I \\ x_{II} \end{bmatrix} = \begin{bmatrix} -\frac{k}{m} & 0 \\ 0 & -\frac{5k}{m} \end{bmatrix}\begin{bmatrix} x_I \\ x_{II} \end{bmatrix} = \begin{bmatrix} -(k/m)x_I \\ -(5k/m)x_{II} \end{bmatrix} = \begin{bmatrix} -\omega_I^2 x_I \\ -\omega_{II}^2 x_{II} \end{bmatrix}$$

So the normal modes are

$$x_I(t) = \langle I|x(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{x_1(t) + x_2(t)}{\sqrt{2}},$$
$$x_{II}(t) = \langle II|x(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{x_1(t) - x_2(t)}{\sqrt{2}},$$

with eigenfrequencies $\omega_I = \sqrt{k/m}$ and $\omega_{II} = \sqrt{5k/m}$, respectively.

Applying the initial conditions $x_1(0) = x_2(0) = 0$, $\dot{x}_1(0) = v_1$, and $\dot{x}_2(0) = v_2$, we obtain $x_I(0) = x_{\mathrm{II}}(0) = 0$, $\dot{x}_I(0) = (v_1 + v_2)/\sqrt{2}$, $\dot{x}_{II}(0) = (v_1 - v_2)/\sqrt{2}$, and the solutions

$$\begin{bmatrix} x_I(t) \\ x_{II}(t) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} (v_1 + v_2)\omega_I^{-1}\sin\omega_I t \\ (v_1 - v_2)\omega_{II}^{-1}\sin\omega_{II} t \end{bmatrix}.$$

Going back to the original basis,

$$|x(t)\rangle = x_I(t)|I\rangle + x_{II}(t)|II\rangle = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{x_I(t)}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_{II}(t)}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} (v_1 + v_2)\omega_I^{-1}\sin\omega_I t + (v_1 - v_2)\omega_{II}^{-1}\sin\omega_{II} t \\ (v_1 + v_2)\omega_I^{-1}\sin\omega_I t - (v_1 - v_2)\omega_{II}^{-1}\sin\omega_{II} t \end{bmatrix}.$$