MATH 5525: HOMEWORK ASSIGNMENT 4

Date (May 8, 2020)

Problem 1. Exercise 6.6 of the textbook (page 81, second edition).

Hint: Apply theorems 6.5 and 6.6 of the textbook.

Solution. According to Theorem 6.6, we have that the exponents λ_1 and λ_2 satisfy

$$\lambda_1 + \lambda_2 = \frac{1}{2\pi}\int_0^{2\pi} \text{trace}A(t)\,dt\,(\text{mod}2\pi i).$$

(Note that we take $T = 2\pi$ since the period of the matrix coefficients is 2π.) Here

$$\text{trace}A(t) = \frac{1}{2} - \cos t + \frac{3}{2} + \sin t = 2 + \sin t - \cos t.$$

So,

$$\lambda_1 + \lambda_2 = 2.$$

Consequently at least one of the λ_1 or λ_2 is positive.

By Theorem 2.5, the fundamental matrix solution is

$$\Phi(t) = P(t)e^{Bt},$$

where λ_1, λ_2 are the eigenvalues of B. Therefore, at least one of the linearly independent solutions of the system (that is, a column of Φ) involves a positive exponential function of t, and therefore solutions of the given system become unbounded at $t \to \infty$.

Problem 2. Exercise 8.4 of the textbook (page 109, second edition).

Hint: Eliminating $(z-1)$ from equation (1) and equation (2) gives a system of two equations for x and y. Find a Lyapunov function for the latter system; call it $V_1(x, y)$. Likewise, eliminating x from equations (2) and (3) you get another system for the variables y and z. Find a Lyapunov function, $V_2(y, z)$. Then, show that $V(x, y, z) := V_1(x, y) + (V_2(y, z) - 1)^2$ is a Lyapunov function for the original system. Subsequent application of the Lyapunov theorem on stability (and/or asymptotic stability) gives the result.

Solution. An easy calculation shows that

$$V_1(x, y) = x^2 + 2y^2, \quad V_2(y, z) = y^2 + (z - 1)^2.$$

Let us start showing that

$$V(x, y, z) = V_1 + (V_2 - 1)^2 = x^2 + 2y^2 + (y^2 + z^2 - 2z)^2$$

is a Lyapunov function of the system. For this, we need to investigate two properties,

- $V(0,0,0) = 0$, which is satisfied.
- $V(x,y,z)$ is positive semidefinite. Indeed $V(x,y,z) \geq 0$. Note that $V(0,0,2) = 0$, so V is not positive definite.
- $\dot{V}$, the orbital derivative also denoted as $L_t V$, is negative semidefinite. Indeed, $\dot{V} = V_x \dot{x} + V_y \dot{y} + V_z \dot{z} = 0$. So, $\dot{V}$ is positive semidefinite.

By Theorem 8.1, the equilibruim solution $(0,0,0)$ is stable. However, we cannot conclude asymptotic stability since $\dot{V}$ is not negative definite.

Problem 3. Exercise 8.9 of the textbook (page 109, second edition).

Hint: Consider a quadratic function of the form $V(x,y) = ax^2 + by^2$. Choose particular values of a and b so that V is a Lyapunov function.

Solution. Let us determine a and b so that $V(xy) = ax^2 + by^2$ is a Lyapunov function.

1. First of all, note that $a > 0$ and $b > 0$ imply that $V(x,y) > 0$, if $(x,y) \neq (0,0)$ and $V(0,0) = 0$. So V is positive definite.
2. Calculate $\dot{V} = V_x \dot{x} + V_y \dot{y} = 2(a - \frac{b}{5})x^2 y^2 - ax^4 - by^4$. Choosing $a = \frac{b}{5}$ we see that $\dot{V} = -a(x^4 + 5y^4)$. So, it is negative definite.

Therefore, the equilibrium solution $(0,0)$ of the system is asymptotically stable. Indeed, we could find a Lyapunov function of the system with $\dot{V}$ negative definite.