my cheat sheet

EE 3015
Signals and Systems

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Contents

■ Fourier series. Periodic signals, Continuous time

Let $\omega_0 = \frac{2\pi}{T_0}$ be the fundamental frequency (rad/sec), and T_0 the fundamental period, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} dt$$

■ Fourier series. Periodic signals, Discrete time

Let $\Omega_0 = \frac{2\pi}{N}$ be the fundamental frequency (rad/sample), and N the fundamental period, then

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$$
$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\Omega_0 n}$$

■ Fourier transform. Non periodic signal, Continuous time.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t} d\omega$$
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t} dt$$

It is also possible to obtain a Fourier transform for periodic signal. For $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t}$ its Fourier transform becomes ($\omega_0 = \frac{2\pi}{T_0}$)

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

■ Fourier transform. Non periodic signal, Discrete time.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n} d\Omega$$
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

It is also possible to obtain a Fourier transform for periodic discrete signal, where $\Omega_0 = \frac{2\pi}{N}$

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$

■ When input to LTI system is $x(t) = e^{j\omega t}$ and system has impulse response $h(t)$ then output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)} d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} d\tau \\ &= e^{j\omega t} H(\omega) \end{aligned}$$

Where $H(\omega)$ is the Fourier transform of $h(t)$. In the above $e^{j\omega t}$ is called eigenfucntions of the system and $H(\omega)$ the eigenvalues.

■ If input $x(t) = a\cos(5\omega_0 t + \theta)$ and $H(\omega)$ is the Fourier transform of the system, then

$$y(t) = a\left|H(5\omega_0)\right| \cos(5\omega_0 t + \theta + \arg H(5\omega_0))$$

Same for discrete time.

■ Modulation. $y(t) = x(t)\, h(t)$ in CTFT becomes $Y(\omega) = \frac{1}{2\pi} X(\omega) \circledast H(\omega)$ where $X(\omega) \circledast H(\omega) = \int_{-\infty}^{\infty} X(z)\, H(\omega - z)\, dz$. Notice the extra $\frac{1}{2\pi}$ factor.

■ To find discrete period given a signal, write $x[n] = x[n+N]$ and then solve for N. See HW's.

■ $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ and $\sum_{n=N}^{\infty} a^n = \frac{a^N}{1-a}$ and $\sum_{n=0}^{N} a^n = \frac{a^{1+N}-1}{a-1}$, and $\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1}-a^{N_2+1}}{1-a}$

■ Fourier transform relations. $y(t) \Longleftrightarrow Y(\omega)$ then $y(at) \Longleftrightarrow \frac{1}{a} Y\left(\frac{\omega}{a}\right)$

■ Euler relations. $\cos x = \frac{e^{jx}+e^{-jx}}{2}, \sin x = \frac{e^{jx}-e^{-jx}}{2j}$

■ Circuit. Voltage cross resistor R is $V(t) = Ri(t)$. Voltage cross inductor L is $V(t) = L\frac{di}{dt}$ and current across capacitor C is $i(t) = C\frac{dV}{dt}$

■ Partial fractions.

$\frac{f(x)}{(x-a)(x-b)}$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{f(x)}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{f(x)}{(x-a)\left(x^2+bx+c\right)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
$\frac{f(x)}{(x-a)(x+d)^2}$	$\frac{A}{x-a} + \frac{B}{x+d} + \frac{C}{(x+d)^2}$
$\frac{f(x)}{(x+d)^2}$	$\frac{A}{x+d} + \frac{B}{(x+d)^2}$
$\frac{f(x)}{(x-a)\left(x^2-b^2\right)}$	$\frac{A}{x+d} + \frac{Bx+C}{x^2-b^2}$
$\frac{f(x)}{\left(x^2-a\right)\left(x^2-b\right)}$	$\frac{Ax+B}{x^2-a} + \frac{Cx+D}{x^2-b}$
$\frac{f(x)}{\left(x^2-a\right)^2}$	$\frac{Ax+B}{x^2-a} + \frac{Cx+D}{\left(x^2-a\right)^2}$

■ Parsevel's. For non-periodic cont. time: $\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$. For periodic cont. time : $\frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$. For discrete: $\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 \, d\Omega = \sum_{n=-\infty}^{\infty} |x[n]|^2$.

■ Properties Fourier series. If $a_k = a_{-k}^*$ then $x(t)$ is real. If a_k is even, then $x(t)$ is even. For $x(t)$ real and odd, then a_k are pure imaginary and odd. i.e. $a_k = -a_{-k}$, and $a_0 = 0$.

■ More Fourier transform relations. Continuos time

$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
$x(t) e^{-j\omega_0 t}$	$X(\omega + \omega_0)$
$x(t) e^{j\omega_0 t}$	$X(\omega - \omega_0)$
$\frac{\sin(a\omega)}{\omega}$	Box from $t = -a \cdots a$

Discrete time

$u[n]$	$\frac{1}{1-e^{-j\Omega}}$
$u[n-1]$	$e^{-j\Omega} U(\Omega) = e^{-j\Omega} \frac{1}{1-e^{-j\Omega}}$
$a^n u[n]$	$\frac{1}{1-ae^{-j\Omega}}$
$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$

From above we see that unit delay in discrete time means multiplying by $e^{-j\Omega}$.

■ Difference equations. $y[n-1] \iff e^{-j\Omega} Y(\Omega)$. For example, given $y[n] - ay[n-1] = x[n]$ then applying DFT gives $Y(\Omega) - ae^{-j\Omega} Y(\Omega) = X(\Omega)$ or $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1-ae^{-j\Omega}}$. From tables, the inverse DFT of this is $a^n u[n]$. Need to know partial fractions sometimes. For example given $y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n]$ then

$$Y(\Omega) - \frac{3}{4} e^{-j\Omega} Y(\Omega) + \frac{1}{8} e^{-j2\Omega} Y(\Omega) = 2X(\Omega)$$

$$\begin{aligned} H(\Omega) &= \frac{Y(\Omega)}{X(\Omega)} \\ &= \frac{2}{\left(1 - \frac{3}{4} e^{-j\Omega} + \frac{1}{8} e^{-j2\Omega}\right)} \\ &= \frac{2}{\left(1 - \frac{1}{2} e^{-j\Omega}\right)\left(1 - \frac{1}{4} e^{-j\Omega}\right)} \end{aligned}$$

And using partial fractions gives $H(\Omega) = \frac{4}{1-\frac{1}{2}e^{-j\Omega}} - \frac{2}{1-\frac{1}{4}e^{-j\Omega}}$. Hence using above table gives

$$h[n] = \left(4\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n\right) u[n]$$

■ $|X(\omega)|^2$ may be interpreted as the energy density spectrum of $x(t)$. This means $\frac{1}{2\pi} |X(\omega)|^2 \, d\omega$ is amount of energy in $d\omega$ range of frequencies. i.e. between ω and $\omega + d\omega$. $|X(\omega)|$ is called the gain of the system and $\arg(H(\omega))$ is called the phase shift of the system. When $\arg(H(\omega))$ is linear function in ω then the effect in time domain is time shift. (delay).

■ z transforms $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$. If $x[n] \rightarrow X(z)$ then $x[n-1] \rightarrow z^{-1}X(z)$.

■ $\frac{\sin(ax)}{ax} = \operatorname{sinc}\left(\frac{ax}{\pi}\right)$ and $\frac{\sin(x)}{x} = \operatorname{sinc}\left(\frac{x}{\pi}\right)$. In class we use $\frac{\sin(\omega_c t)}{\pi t}$. This has FT as rectangle from $-\omega_c$ to ω_c and amplitude 1.

■ in digital, sampling rate is in hz, but units is samples per second and not cycles per second as with analog.

■

$$\Omega = \frac{\omega}{F_s}$$

where F_s is sampling rate in samples per second, and Ω is unnormalized digital frequency (radians per sample) and ω is analog frequency (radians per second). This can also be written as

$$\Omega = \omega T_s$$

where here T_s is seconds per sample (i.e. number of seconds to obtain one sample). Per sample is used to make the units come out OK.

■ Trig identities

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right) \\
\cos A \cos B &= \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right) \\
\sin A \sin B &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)
\end{aligned}$$

■ Group delay is given by $-\frac{d}{d\omega}(\arg(H(\omega)))$. For example, if $H(\omega) = \frac{1}{2+j\omega}$ then $\arg(H(\omega)) = -\arctan\left(\frac{\omega}{2}\right)$ which leads to group delay being $\frac{2}{4+\omega^2}$.

■ FT of $\cos(\omega_c t)$ has delta at $\pm\omega_c$ each of amplitude π. And FT of $\sin(\omega_c t)$ has delta at ω_c of amplitude $\frac{\pi}{j}$ and has delta at $-\omega_c$ of amplitude $\frac{-\pi}{j}$ and $\frac{\sin(\omega_c t)}{\pi t}$ has FT as rectangle of amplitude 1 and width from $-\omega_c$ to $+\omega_c$.

$$\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right) u[n] e^{-j\Omega n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right) e^{-j\Omega n}
\end{aligned}$$

But $\cos\left(\frac{\pi n}{2}\right) = \frac{1}{2}\left(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}\right)$ and the above becomes

$$
\begin{aligned}
X(\Omega) &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \frac{1}{2}\left(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}\right)e^{-j\Omega n} \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{j\frac{\pi n}{2}} e^{-j\Omega n} + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j\frac{\pi n}{2}} e^{-j\Omega n}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\left(\frac{\pi}{2}-\Omega\right)}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\left(-\frac{\pi}{2}-\Omega\right)}\right)^n\right)
\end{aligned}
$$

Since $\frac{1}{2}e^{j\left(\frac{\pi}{2}-\Omega\right)} < 1$ then we can use $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for both terms and the above becomes

$$
\begin{aligned}
X(\Omega) &= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{j\left(\frac{\pi}{2}-\Omega\right)}} + \frac{1}{1-\frac{1}{2}e^{j\left(-\frac{\pi}{2}-\Omega\right)}}\right) \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{j\frac{\pi}{2}}e^{-j\Omega}} + \frac{1}{1-\frac{1}{2}e^{-j\frac{\pi}{2}}e^{-j\Omega}}\right)
\end{aligned}
$$

But $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$ and the above becomes

$$
\begin{aligned}
X(\Omega) &= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}je^{-j\Omega}} + \frac{1}{1+\frac{1}{2}je^{-j\Omega}}\right) \\
&= \frac{1}{2}\left(\frac{1+\frac{1}{2}je^{-j\Omega}+1-\frac{1}{2}je^{-j\Omega}}{\left(1-\frac{1}{2}je^{-j\Omega}\right)\left(1+\frac{1}{2}je^{-j\Omega}\right)}\right) \\
&= \frac{1}{2}\left(\frac{2}{1+\frac{1}{2}je^{-j\Omega}-\frac{1}{2}je^{-j\Omega}-\frac{1}{4}j^2e^{-2j\Omega}}\right) \\
&= \frac{1}{1+\frac{1}{4}e^{-2j\Omega}}
\end{aligned}
$$

■ Z transforms

$u[n]$	Z
$a^n u[n]$	$\frac{1}{1-az^{-1}}$
$a^{n-1}u[n-1]$	$z^{-1}\frac{1}{1-az^{-1}}$
$a^{n-2}u[n-2]$	$z^{-2}\frac{1}{1-az^{-1}}$

If the ROC outside the out most pole, then right-handed signal. (Causal). If the ROC isinside the inner most pole, then left-handed signal (non causal).