my solution to practice exam midterm 1 2001

EE 3015
Signals and Systems

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Contents

0.1 Problem 1

Obtain impulse and step response for LTI described by (a) $h[n] = \left(\frac{1}{2}\right)^n u[n]$ (b) $h(t) = e^{-\frac{1}{2}t} u(t)$

solution

Part a Let $\underline{x[n] = \delta[n]}$, hence

$$\begin{aligned} y[n] &= \delta[n] \circledast h[n] \\ &= \sum_{k=-\infty}^{\infty} \delta[k] h[n-k] \end{aligned}$$

But $\delta[k] = 0$ for all k except when $k = 0$. Hence the above reduces to

$$\begin{aligned} y[n] &= h[n] \\ &= \left(\frac{1}{2}\right)^n u[n] \end{aligned}$$

Let $\underline{x[n] = u[n]}$, hence

$$\begin{aligned} y[n] &= u[n] \circledast h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \end{aligned}$$

Folding $x[-n]$, we see that for $n < 0$ then there no overlap with $h[n]$. Hence $y[n] = 0$ for $n < 0$. As $x[-n]$ is shifted to the right, then the convolution sum becomes

$$\begin{aligned} y[n] &= \sum_{k=0}^{n} h[k] \qquad n \geq 0 \\ &= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \end{aligned}$$

This is the partial sum, given by $\frac{a^{1+n}-1}{a-1}$ where $a = \frac{1}{2} < 1$

$$\begin{aligned} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k &= \frac{\left(\frac{1}{2}\right)^{1+n} - 1}{\frac{1}{2} - 1} \\ &= \frac{\left(\frac{1}{2}\right)^{1+n} - 1}{-\frac{1}{2}} \\ &= 2 - 2(2)^{-n-1} \\ &= 2 - 2^{-n} \end{aligned} \tag{2}$$

Therefore

$$y[n] = \begin{cases} 2 - 2^{-n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Part b Let $\underline{x(t) = \delta(t)}$, hence

$$\begin{aligned} y(t) &= u(t) \circledast h(t) \\ &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} \delta(t) h(t-\tau) d\tau \\ &= h(t) \\ &= e^{-0.5t} \end{aligned}$$

Let $x(t) = u(t)$, hence

$$
\begin{aligned}
y(t) &= u(t) \circledast h(t) \\
&= \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau
\end{aligned}
$$

Folding $u(-t)$, we see that for $t < 0$ then there no overlap with $h(\tau) = e^{-0.5\tau} u(\tau)$. Hence $y(t) = 0$ for $t < 0$. As $u[-n]$ is shifted to the right, then the convolution becomes

$$
\begin{aligned}
y[n] &= \int_0^t h(\tau) d\tau \qquad t > 0 \\
&= \int_0^t e^{-0.5\tau} d\tau \\
&= \left(\frac{e^{-0.5\tau}}{-0.5} \right)_0^t \\
&= -2\left(e^{-0.5t} - 1\right) \\
&= 2 - 2e^{-0.5t} \\
&= 2\left(1 - e^{-0.5t}\right)
\end{aligned}
$$

Hence

$$
y(t) = \begin{cases} 2\left(1 - e^{-0.5t}\right) & t \geq 0 \\ 0 & t < 0 \end{cases}
$$

0.2 Problem 2

Given the frequency response of LTI system $H(\Omega)$ for the following input signal, find the steady state expression of the output signal. (a) $x[n] = 2\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right)$ (b) $x[n] = 5\sin\left(\frac{\pi}{3}n + \frac{\pi}{8}\right)$

solution

Part a

$$
x[n] = 2\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right)
$$

To find the fundamental period, $\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right) = \cos\left(\frac{\pi}{6}(n+N) + \frac{\pi}{5}\right) = \cos\left(\left(\frac{\pi}{6}n + \frac{\pi}{5}\right) + \frac{\pi}{6}N\right)$. Hence need $\frac{\pi}{6}N = m2\pi$ or $\frac{m}{N} = \frac{1}{12}$. Hence. $N = 12$. Therefore

$$
\Omega_0 = \frac{2\pi}{12}
$$

And the input is $x[n] = 2\cos\left(\Omega_0 n + \frac{\pi}{5}\right)$. Hence the output is

$$
y[n] = 2\left|H(\Omega_0)\right| \cos\left(\Omega_0 n + \frac{\pi}{5} + \arg H(\Omega_0)\right)
$$

Part b

$$
x[n] = 5\sin\left(\frac{\pi}{3}n + \frac{\pi}{8}\right)
$$

To find the fundamental period, $\sin\left(\frac{\pi}{3}n + \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{3}(n+N) + \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{3}n + \frac{\pi}{8} + \frac{\pi}{3}N\right)$. Hence need $\frac{\pi}{3}N = m2\pi$ or $\frac{m}{N} = \frac{1}{6}$. Hence. $N = 6$. Therefore

$$
\Omega_0 = \frac{2\pi}{6}
$$

And the input is $x[n] = 5\sin\left(\Omega_0 n + \frac{\pi}{8}\right)$. Hence the output is

$$
y[n] = 5\left|H(\Omega_0)\right| \sin\left(\Omega_0 n + \frac{\pi}{8} + \arg H(\Omega_0)\right)
$$

0.3 Problem 3

Compute Fourier series coeff. for the following signals. (a) $x[n] = 2\sin\left(\frac{\pi}{3}n + \frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right)$. (b) $x(t) = e^{j2\pi t} + e^{j3\pi t}$

solution

Part a For discrete periodic signal, the Fourier series coeff. a_k is given by

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{1}$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{2}$$

In this problem

$$x[n] = 2\sin\left(\frac{\pi}{3}n + \frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right)$$

To find the common fundamental period. $\sin\left(\frac{\pi}{3}n + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{3}(n+N) + \frac{\pi}{2}\right) = \sin\left(\left(\frac{\pi}{3}n + \frac{\pi}{2}\right) + \frac{\pi}{3}N\right)$. Hence $\frac{\pi}{3}N = m2\pi$ or $\frac{m}{N} = \frac{1}{6}$. hence $N = 6$ for first signal. For second signal $\cos\left(\frac{\pi}{6}n + \frac{\pi}{5}\right)$ we obtain $\frac{\pi}{6}N = m2\pi$ or $\frac{m}{N} = \frac{1}{12}$ or $N = 12$. hence the least common multiplier between 6 and 12 is $N = 12$. Therefore

$$\Omega_0 = \frac{2\pi}{12}$$

Hence (2) becomes

$$\begin{aligned} a_k &= \frac{1}{12}\sum_{n=0}^{11} x[n]\, e^{-jk\left(\frac{2\pi}{12}\right)n} \\ &= \frac{1}{12}\sum_{n=0}^{11} x[n]\, e^{-jk\Omega_0 n} \end{aligned}$$

But instead of using the above formula, an easier way is to rewrite $x[n]$ using Euler relation and use (1) to read off a_k directly from the result. Writing $x[n]$ in terms of the fundamental frequency Ω_0 gives

$$\begin{aligned} x[n] &= 2\sin\left(2\Omega_0 n + \frac{\pi}{2}\right) + 3\cos\left(\Omega_0 n + \frac{\pi}{5}\right) \\ &= 2\left(\frac{e^{j\left(2\Omega_0 n + \frac{\pi}{2}\right)} - e^{-j\left(2\Omega_0 n + \frac{\pi}{2}\right)}}{2j}\right) + 3\left(\frac{e^{j\left(\Omega_0 n + \frac{\pi}{5}\right)} + e^{-j\left(\Omega_0 n + \frac{\pi}{5}\right)}}{2}\right) \\ &= \frac{2}{2j}\left(e^{j\frac{\pi}{2}} e^{j2\Omega_0 n} - e^{-j\frac{\pi}{2}} e^{-j2\Omega_0 n}\right) + \frac{3}{2}\left(e^{j\frac{\pi}{5}} e^{j\Omega_0 n} + e^{-j\frac{\pi}{5}} e^{-j\Omega_0 n}\right) \\ &= \frac{1}{j}e^{j\frac{\pi}{2}} e^{j2\Omega_0 n} - \frac{1}{j}e^{-j\frac{\pi}{2}} e^{-j2\Omega_0 n} + \frac{3}{2}e^{j\frac{\pi}{5}} e^{j\Omega_0 n} + \frac{3}{2}e^{-j\frac{\pi}{5}} e^{-j\Omega_0 n} \end{aligned}$$

Now we can read the Fourier coefficients by comparing the above to Eq(1).

This gives for $k = 2, a_2 = \frac{1}{j}e^{j\frac{\pi}{2}}$ and for $k = -2, a_{-2} = -\frac{1}{j}e^{-j\frac{\pi}{2}}$ and for $k = 1, a_1 = \frac{3}{2}e^{j\frac{\pi}{5}}$ and for $k = -1, a_{-1} = \frac{3}{2}e^{-j\frac{\pi}{5}}$

$$\begin{aligned} a_1 &= \frac{3}{2}e^{j\frac{\pi}{5}} \\ a_{-1} &= \frac{3}{2}e^{-j\frac{\pi}{5}} \\ a_2 &= \frac{1}{j}e^{j\frac{\pi}{2}} \\ a_{-2} &= -\frac{1}{j}e^{-j\frac{\pi}{2}} \end{aligned}$$

But $e^{j\frac{\pi}{2}} = j\sin\frac{\pi}{2} = j$ and $e^{-j\frac{\pi}{2}} = -j\sin\frac{\pi}{2} = -j$. Hence the above becomes

$$\begin{aligned} a_1 &= \frac{3}{2}e^{j\frac{\pi}{5}} \\ a_{-1} &= \frac{3}{2}e^{-j\frac{\pi}{5}} \\ a_2 &= \frac{1}{j}j = 1 \\ a_{-2} &= -\frac{1}{j}\left(-j\right) = 1 \end{aligned}$$

And $a_k = 0$ for all other k.

Part b For continuos time periodic signal $x(t)$, the Fourier series coeff. a_k is given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt \end{aligned}$$

In this problem

$$x(t) = e^{j2\pi t} + e^{j3\pi t}$$

The period of $e^{j2\pi t}$ is 1 and the period of $e^{j3\pi t}$ is $\frac{2}{3}$. Hence least common multiplier is $T_0 = 2$ seconds. $\omega_0 = \frac{2\pi}{2} = \pi$ rad/sec. Both of the above terms can now be written

$$\begin{aligned} x(t) &= e^{j\frac{4\pi}{T_0}t} + e^{j\frac{6\pi}{T_0}t} \\ &= e^{j2\omega_0 t} + e^{j3\omega_0 t} \end{aligned}$$

Comparing the above to

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Shows that for $k = 2, a_k = 1$ and for $k = 3, a_k = 1$ and $a_k = 0$ for all other k.

0.4 Problem 4

Given the magnitude and phase profile of this filter, find impulse response.

solution

We are given $H(\Omega)$ and need to find $h[n]$. i.e. the inverse Fourier transform

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega)\, e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\Omega)|\, e^{j\arg H(\Omega)} e^{j\Omega n} d\Omega \end{aligned}$$

But $|H(\Omega)| = 1$ and $\arg H(\Omega) = -\Omega$ as given. The above reduces to

$$
\begin{aligned}
h[n] &= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j\Omega}e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j\Omega(1-n)}d\Omega \\
&= \left(\frac{1}{2\pi}\right)\frac{1}{-j(1-n)}\left[e^{-j\Omega(1-n)}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \left(\frac{1}{2\pi}\right)\frac{1}{-j(1-n)}\left(e^{-j\frac{\pi}{4}(1-n)} - e^{j\frac{\pi}{4}(1-n)}\right) \\
&= \frac{1}{\pi}\frac{1}{(1-n)}\left(\frac{e^{j\frac{\pi}{4}(1-n)} - e^{-j\frac{\pi}{4}(1-n)}}{2j}\right) \\
&= \frac{1}{\pi(1-n)}\sin\left(\frac{\pi}{4}(1-n)\right) \\
&= \frac{-1}{\pi(1-n)}\sin\left(\frac{\pi}{4}(n-1)\right) \\
&= \frac{1}{\pi(n-1)}\sin\left(\frac{\pi}{4}(n-1)\right)
\end{aligned}
$$