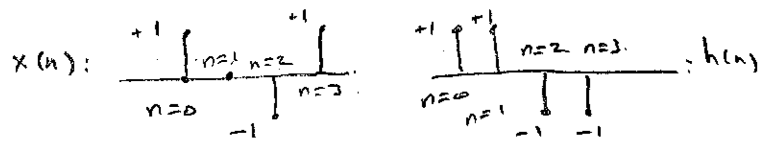
EE 3015 Midterm 1 – Oct 10th 2018 **Name & ID** Solution

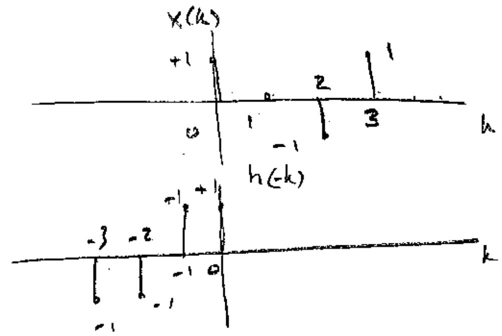
(Duration 50 min)
Calculators allowed

Problem 1. (25 pts.) Given the discrete time system with input x[n] and impulse response h[n] obtain the output sequence y[n] by applying discrete convolution.

x[n] = [1 , 0 , -1 , 1] , h[n] = [1 , 1 , -1 , -1] both sequences are positive and start with n = 0 position.
Show your work below.



$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)$$



$n<0$ No overlap $y(n)=0$

$n=0$ $y(0) = (1)(1) = 1$

$n=1$ $y(1) = (1)(1) + (1)(0) = 1$

$n=2$ $y(2) = (-1)(1) + (0)(1) + (-1)(1) = -1-1 = -2$

$n=3$ $y(3) = (-1)(1) + (0)(-1) + (1)(-1) + (1)(1) = -1-1+1 = -1$

$n=4$ $y(4) = (-1)(0) + (-1)(-1) + (1)(1) = 1+1 = 2$

$n=5$ $y(5) = (-1)(-1) + (-1)(1) = 1-1 = 0$

$n=6$ $y(6) = (-1)(1) = -1$

$\therefore$ $y(n) = [1, 1, -2, -1, 2, 0, -1]$

Problem 2. (35 pts.) The impulse response of a LTI system is given as

h(t) = u(t) - 2 u(t - 1) + u(t − 2)

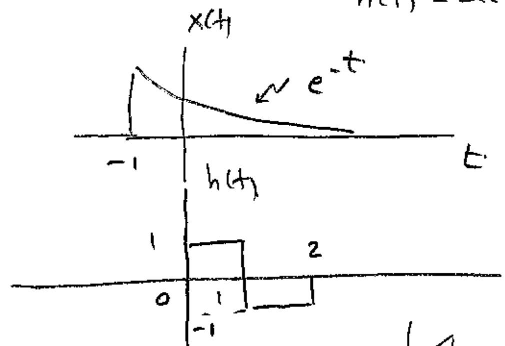
Where u(t) is a unit step signal. Determine the output of this system y (t) where it's input x(t) is given as

x(t) = e ^ (-t) . u(t + 1)

$x(t) = e^{-t}\, u(t+1)$

$h(t) = u(t) - 2u(t-1) + u(t-2)$

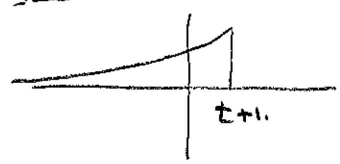
$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$



Region 1.

$-\infty < t+1 < 0$ No overlap

$y(t) = 0$



Region 2:

$1 > t+1 > 0 \Rightarrow 0 > t > -1$

$$y(t) = \int_0^{t+1} e^{-(t-\tau)}\, d\tau = e - e^{-t}$$

Region 3.

$2 > t+1 > 1 \Rightarrow 1 > t > 0$

$$y(t) = \int_0^1 1 \cdot e^{-(t-\tau)}\, d\tau + \int_1^{t+1} (-1) \cdot e^{-(t-\tau)}\, d\tau = 2e^{(1-t)} - e^{-t} - e$$

Region 4

$\infty > t+1 > 2 \Rightarrow \infty > t > 1$

$$y(t) = \int_0^1 e^{-(t-\tau)} (1) \cdot d\tau + \int_1^2 (-1) \cdot e^{-(t-\tau)}\, d\tau$$

$$= 2e^{(1-t)} - e^{-t} - e^{(2-t)}$$

$$\therefore \quad y(t) = \begin{cases} 0 & -\infty < t < -1 \\ e - e^{-t} & -1 < t < 0 \\ 2e^{(1-t)} - e^{-t} - e & 0 < t < 1 \\ 2e^{(1-t)} - e^{-t} - e^{(2-t)} & 1 < t < \infty. \end{cases}$$

Problem 3.(40 pts.) A discrete periodic sequence is given as x[n]:

x [n] = 2 sin (3πn/8 + π/2) + cos (πn/4 + π/3) obtain the following :

(a) (10 pts) Find the fundamental frequency and period of this signal.
(b) (15 pts) The Fourier series coefficients for x(n) sequence.
(c) (15 pts) If x(n) = cos (πn/4 + π/3) is an input to a system with frequency response H(e ^jΩ) where

H (e^jΩ) = (1 - e^(-jΩ))/ (2 + e^(-j2Ω)) obtain the expression for output y(n).

$$H(e^{j\Omega}) = \frac{1 - e^{-j\Omega}}{2 + e^{-j2\Omega}}$$

$x(n) \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow y(n)$

$|H(e^{j\Omega})| =$

$$x(n) = 2\sin(3\pi n/8 + \pi/2) + \cos(\pi n/4 + \pi/3)$$

$\Omega_1 = \frac{3\pi}{8}$ $\Omega_2 = \pi/4$ Fundamental freq $\Omega_0 = \frac{2\pi}{N}$.

(a) $\Omega_1 = \frac{3(2\pi)}{8(2)}$ $\Omega_2 = \frac{3(2\pi)}{16}$ $\Rightarrow$ $N = 16$ period

$\therefore$ $\Omega_0 = \frac{2\pi}{16} = \pi/8$ Fundamental freq.

(b) $x(n) = \sum_{k=-\infty}^{\infty} X(k) \cdot e^{jk\Omega_0 n}$
$$= \frac{2}{2j}\left(e^{j(3\Omega_0 n + \pi/2)} - e^{-j(3\Omega_0 n + \pi/2)}\right) + \frac{1}{2}\left(e^{j(2\Omega_0 n + \pi/3)} + e^{-j(2\Omega_0 n + \pi/3)}\right)$$

$$\therefore \quad X(k) = \begin{cases} \frac{e^{-j\pi/2}}{j} = 1 & k = -3 \\ \frac{e^{-j\pi/3}}{2} & k = -2 \\ \frac{e^{j\pi/3}}{2} & k = +2 \\ \frac{e^{j\pi/2}}{j} = 1 & k = +3 \end{cases}$$

$$\frac{1 - \cos(\Omega) + j\sin\Omega}{2 - \cos(2\Omega) + j\sin(2\Omega)} = H(e^{j\Omega})$$

$$|H(e^{j\Omega})| = \frac{\sqrt{(1-\cos(\Omega))^2 + \sin^2\Omega}}{\sqrt{(2-\cos(2\Omega))^2 + \sin^2(2\Omega)}}$$

at $\Omega = \pi/4$.

(c) $y(n) = |H(e^{j\Omega})|_{\Omega=\pi/4} \cdot \cos\left(\frac{\pi}{4} n + \pi/3 + \measuredangle H(e^{j\Omega})\big|_{\Omega=\pi/4}\right)$

$$\measuredangle H(e^{j\Omega}) = \tan^{-1}\left(\frac{\sin\Omega}{1-\cos(\Omega)}\right) - \tan^{-1}\left(\frac{\sin(2\Omega)}{2-\cos(2\Omega)}\right)$$

$$|H(e^{j\Omega})|_{\Omega=\pi/4} = \frac{|1 - e^{j\pi/4}|}{|2 - e^{-j\pi/2}|} = \sqrt{\frac{(1-\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2}{3}} =$$

$$\measuredangle H(e^{j\Omega})\big|_{\Omega=\pi/4} = \tan^{-1}\left(\frac{\sin(\pi/4)}{1-\cos(\pi/4)}\right) - \tan^{-1}\left(\frac{\sin(\pi/2)}{2-\cos(\pi/2)}\right)$$