my solution to practice exam midterm 1 2018

EE 3015
Signals and Systems

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Contents

0.1 Problem 1

Given the discrete time system with input $x[n]$ and impulse response $h[n]$ obtain the output sequence $y[n]$ by applying discrete convolution.

$x[n] = [1, 0, -1, 1], h[n] = [1, 1, -1, -1]$. Both sequences are positive and start with $n = 0$ position.

Solution

$$y[n] = x[n] \circledast h[n]$$

Folding $x[-n]$. When $n = 0, y[0] = 1$. When $n = 1$, then $y[n] = (0)(1) + (1)(1) = 1$. When $n = 2, y[n] = (-1)(1) + (0)(1) + (1)(-1) = -2$. When $n = 3, y[n] = (1)(1) + (-1)(1) + (0)(-1) + (1)(-1) = 1 - 1 - 1 = -1$. When $n = 4, y[n] = 1 + 1 = 2$. When $n = 5, y[n] = -1 + 1 = 0$, when $n = 6, y[6] = -1$. When $n > 6, y[n] = 0$.

Hence

$$y[n] = [1, 1, -2, -1, 2, 0, -1]$$

0.2 Problem 2

The impulse response of LTI system is given by $h(t) = u(t) - 2u(t-1) + u(t-2)$ where $u(t)$ is unit step signal. Determine the output of this $y(t)$ where its input $x(t)$ is given as $x(t) = e^{-t}u(t+1)$.

Solution

By folding $x(t)$. See key solution. Used same method.

0.3 Problem 3

A discrete periodic sequence is given as $x[n] = 2\sin\left(\frac{3\pi}{8}n + \frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$. (a) Find fundamental frequency of this signal. (b) Fourier series coefficients for $x[n]$. (c) if $x[n] = \cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$ is an input to system with frequency response $H(\Omega) = \frac{1-e^{-j\Omega}}{2+e^{-2j\Omega}}$, obtain expression for $y[n]$

Solution

Part a For $\sin\left(\frac{3\pi}{8}n + \frac{\pi}{2}\right)$, we need $\frac{3}{8}\pi N = m2\pi$ or $\frac{m}{N} = \frac{3}{16}$. Since relatively prime, hence $N = 16$. For $\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$ we need $\frac{\pi}{4}N = m2\pi$ or $\frac{m}{N} = \frac{1}{8}$. Hence $N = 8$. The least common multiplier is 16. Hence fundamental period is $N = 16$. Therefore $\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}$.

Part b Since input is periodic, then

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 n} \tag{1}$$

By writing the input, using Euler relation, we can compare the input to the above and read off a_k. First we rewrite the input using common Ω_0 as

$$x[n] = 2\sin\left(3\Omega_0 n + \frac{\pi}{2}\right) + \cos\left(2\Omega_0 n + \frac{\pi}{3}\right)$$

Hence

$$\begin{aligned} x[n] &= 2\left(\frac{e^{j\left(3\Omega_0 n + \frac{\pi}{2}\right)} - e^{-j\left(3\Omega_0 n + \frac{\pi}{2}\right)}}{2j}\right) + \frac{e^{j\left(2\Omega_0 n + \frac{\pi}{3}\right)} + e^{-j\left(2\Omega_0 n + \frac{\pi}{3}\right)}}{2} \\ &= \frac{1}{j}e^{j\left(3\Omega_0 n + \frac{\pi}{2}\right)} - \frac{1}{j}e^{-j\left(3\Omega_0 n + \frac{\pi}{2}\right)} + \frac{1}{2}e^{j\left(2\Omega_0 n + \frac{\pi}{3}\right)} + \frac{1}{2}e^{-j\left(2\Omega_0 n + \frac{\pi}{3}\right)} \\ &= \frac{1}{j}e^{j\frac{\pi}{2}}e^{j3\Omega_0 n} - \frac{1}{j}e^{-j\frac{\pi}{2}}e^{-j3\Omega_0 n} + \frac{1}{2}e^{j\frac{\pi}{3}}e^{j2\Omega_0 n} + \frac{1}{2}e^{-j\frac{\pi}{3}}e^{-j2\Omega_0 n} \end{aligned}$$

But $e^{j\frac{\pi}{2}} = j\sin\frac{\pi}{2} = j$ and $e^{-j\frac{\pi}{2}} = -j\sin\frac{\pi}{2} = -j$ and $e^{j\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right) = \frac{1}{2}\sqrt{3}j + \frac{1}{2}$ and $e^{-j\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) - j\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{3}j$. Hence the above simplifies to

$$x[n] = e^{j3\Omega_0 n} + e^{-j3\Omega_0 n} + \frac{1}{4}\left(1+\sqrt{3}j\right)e^{j2\Omega_0 n} - \frac{1}{4}\left(1-\sqrt{3}j\right)e^{-j2\Omega_0 n} \tag{2}$$

Comparing (2) to (1) shows that

$$\begin{aligned}
a_3 &= 1 \\
a_{-3} &= 1 \\
a_2 &= \frac{1}{2}e^{j\frac{\pi}{3}} = \frac{1}{4}\left(1+\sqrt{3}j\right) \\
a_{-2} &= \frac{1}{2}e^{-j\frac{\pi}{3}} = -\frac{1}{4}\left(1-\sqrt{3}j\right)
\end{aligned}$$

Part c The output is

$$y[n] = |H(\Omega)|_{\Omega=\frac{\pi}{4}} \cos\left(\frac{\pi}{4}n + \frac{\pi}{3} + \arg H(\Omega)_{\Omega=\frac{\pi}{4}}\right) \tag{1}$$

But

$$\begin{aligned}
|H(\Omega)| &= \left|\frac{1-e^{-j\Omega}}{2+e^{-2j\Omega}}\right| \\
&= \frac{\left|1-e^{-j\Omega}\right|}{\left|2+e^{-2j\Omega}\right|} \\
&= \frac{\sqrt{(1-\cos\Omega)^2+\sin^2\Omega}}{\sqrt{(2+\cos 2\Omega)^2+\sin^2(2\Omega)}}
\end{aligned}$$

When $\Omega = \frac{\pi}{4}$ the above becomes

$$\begin{aligned}
\left|H\left(\frac{\pi}{4}\right)\right| &= \frac{\sqrt{\left(1-\cos\left(\frac{\pi}{4}\right)\right)^2+\sin^2\left(\frac{\pi}{4}\right)}}{\sqrt{\left(2+\cos\left(\frac{\pi}{2}\right)\right)^2+\sin^2\left(\frac{\pi}{2}\right)}} \\
&= \frac{\sqrt{\left(1-\cos\left(\frac{\pi}{4}\right)\right)^2+\sin^2\left(\frac{\pi}{4}\right)}}{\sqrt{4+1}} \\
&= \frac{\sqrt{\frac{3}{2}-\sqrt{2}+\frac{1}{2}}}{\sqrt{5}} \\
&= \frac{\sqrt{2-\sqrt{2}}}{\sqrt{5}} \\
&= 0.342\,28
\end{aligned}$$

And

$$\begin{aligned}
\arg H(\Omega) &= \arg\frac{1-e^{-j\Omega}}{2+e^{-2j\Omega}} \\
&= \arg\frac{(1-\cos\Omega)+j\sin\Omega}{(2+\cos(2\Omega))-j\sin(2\Omega)} \\
&= \arctan\left(\frac{\sin\Omega}{1-\cos\Omega}\right) - \arctan\left(\frac{-\sin(2\Omega)}{2+\cos(2\Omega)}\right)
\end{aligned}$$

When $\Omega = \frac{\pi}{4}$ the above becomes

$$\begin{aligned}
\arg H\left(\frac{\pi}{4}\right) &= \arctan\left(\frac{\sin\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}\right) - \arctan\left(\frac{-\sin\left(\frac{\pi}{2}\right)}{2+\cos\left(\frac{\pi}{2}\right)}\right) \\
&= \arctan\left(\frac{\sin\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}\right) - \arctan\left(\frac{-1}{2}\right) \\
&= \arctan\left(\frac{\sin\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}\right) + \arctan\left(\frac{1}{2}\right) \\
&= \arctan(2.4142) + 0.46365 \\
&= 1.1781 + 0.46365 \\
&= 1.6418 \text{ rad}
\end{aligned}$$

Hence (1) becomes

$$y[n] = 0.3423\cos\left(\frac{\pi}{4}n + \frac{\pi}{3} + 1.6418\right)$$