ECE 3015 Final Exam

Sample 2

1. (20 points) An LTI system is described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 9\frac{d}{dt}y(t) - 10y(t) = 3\frac{d^2}{dt^2}x(t) + 7\frac{d}{dt}x(t) - 10x(t)$$

Assume the system has initial conditions $y(0^+) = 5$, $\frac{d}{dt}y(t)\Big|_{t=0^+} = -28$ and input $x(t) = u(t-1)$.

(a) Use Laplace transforms to find the natural response. $x(t) = 0$

$$s^2Y(s) - \frac{dy(t)}{dt}\Big|_{t=0^+} - sy(0^+) + 9sY(s) - 9y(0^+) - 10Y(s) = 0$$

$$Y(s) = \frac{5s + 45 - 28}{s^2 + 9s - 10} = \frac{5s + 17}{(s-1)(s+10)} = \frac{A}{s-1} + \frac{B}{s+10}$$

$$A = \frac{5s+17}{s+10}\Big|_{s=1} = \frac{22}{11} = 2 \qquad B = \frac{5s+17}{s-1}\Big|_{s=-10} = \frac{-33}{-11} = 3$$

$$y^{(n)}(t) = 2e^{t}u(t) + 3e^{-10t}u(t)$$

(b) Use Laplace transforms to find the overall output of the system.

$y(t) = y^{(n)}(t) + y^{(f)}(t)$ Note $x(0) = 0$ $\left.\frac{d}{dt}x(t)\right|_{t=0^+} = 0$

$$Y^{(f)}(s) = \frac{3s^2 + 7s - 10}{s^2 + 9s - 10} X(s) \qquad X(s) = e^{-s}\left(\frac{1}{s}\right)$$

$$= \frac{Ae^{-s}}{s} + \frac{Be^{-s}}{s-1} + \frac{Ce^{-s}}{s+10} \qquad A = \left.\frac{3s^2 + 7s - 10}{s^2 + 9s - 10}\right|_{s=0} = 1$$

$$B = \left.\frac{3s^2 + 7s - 10}{s(s+10)}\right|_{s=1} = \frac{0}{11} = 0$$

$$C = \left.\frac{3s^2 + 7s - 10}{s(s-1)}\right|_{s=-10} = \frac{220}{110} = 2$$

300−

$-10 \ (-11)$

$$y^{(f)}(t) = u(t-1) + 2e^{-10(t-1)}u(t-1)$$

$$y(t) = y^{(f)}(t) + y^{(n)}(t)$$

$$= 2e^{t}u(t) + 3e^{-10t}u(t-1) + u(t-1) + 2e^{-10(t-1)}u(t-1)$$

(c) Is this system stable? Justify your answer.

Not stable. Natural response blows up.

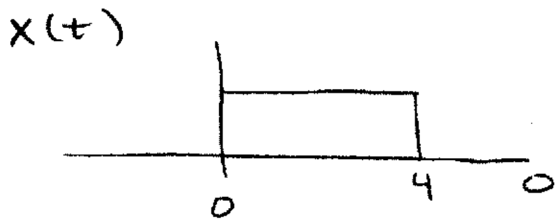
2. (20 points) Let a signal $y(t)$ have FT representation

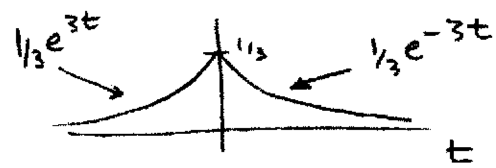
$$Y(j\omega) = e^{-j2\omega}\left(\frac{\sin(2\omega)}{\omega}\right)\left(\frac{4}{9+\omega^2}\right)$$

Find $y(t)$.

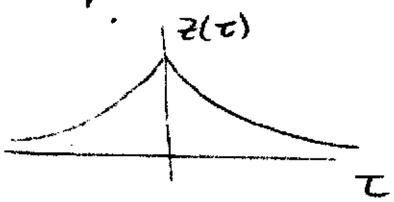
$$X(j\omega) = e^{-j2\omega}\,\frac{2\sin(2\omega)}{\omega} \qquad Z(j\omega) = \frac{2}{9+\omega^2}$$

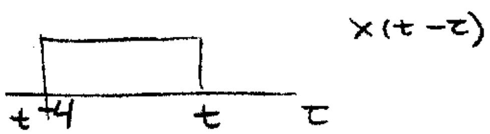
$$z(t) = \frac{1}{3}e^{-3|t|}$$





$$y(t) = x(t) * z(t)$$





1) $t<0$

$$w_t(\tau) = \begin{cases} \frac{1}{3}e^{3\tau} & t-4 \le \tau \le t \\ 0 & \text{other.} \end{cases}$$

2) $0 < t < t-4$

$$w_t(\tau) = \begin{cases} \frac{1}{3}e^{3\tau} & t-4<\tau<0 \\ \frac{1}{3}e^{-3\tau} & 0<\tau<t \\ 0 & \text{other} \end{cases}$$

3) $4<t$

$$w_t(\tau) = \begin{cases} \frac{1}{3}e^{-3\tau} & t-4<\tau<t \\ 0 & \text{other} \end{cases}$$

$t<0$

$$y(t) = \int_{t-4}^{t} \frac{1}{3}e^{3\tau}\,d\tau$$

$$= \frac{1}{9}e^{3\tau}\Big|_{t-4}^{t} = \underline{\frac{1}{9}\left[e^{3t} - e^{3(t-4)}\right]}$$

$0<t<t-4$

$$y(t) = \int_{t-4}^{0}\frac{1}{3}e^{3\tau}d\tau + \int_{0}^{t}\frac{1}{3}e^{-3\tau}d\tau$$

$$= \frac{1}{9}e^{3\tau}\Big|_{t-4}^{0} - \frac{1}{9}e^{-3\tau}\Big|_{0}^{t}$$

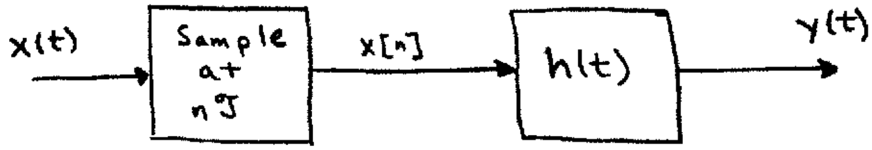
$$= \underline{\frac{1}{9}\left[1 - e^{3(t-4)} + 1 - e^{-3t}\right]}$$

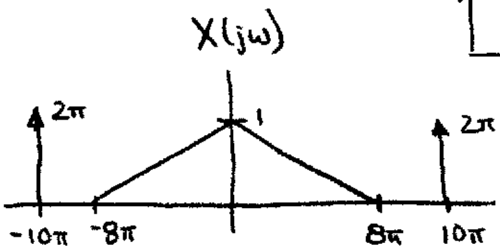
$4<t$

$$y(t) = -\frac{1}{9}e^{-3\tau}\Big|_{t-4}^{t}$$

$$= \underline{\frac{1}{9}\left[e^{-3(t-4)} - e^{-3t}\right]}$$

3. (20 points) Consider the system depicted below





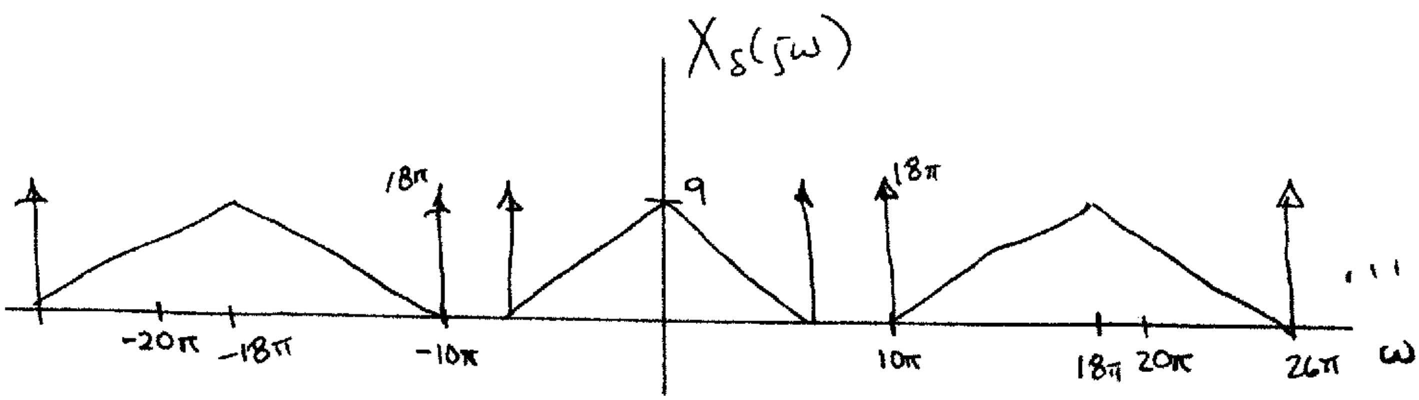
$x[n] = x(nT)$

$$h(t) = \frac{1}{\pi t}\left[\sin 11\pi t + \sin 24\pi t\right]$$

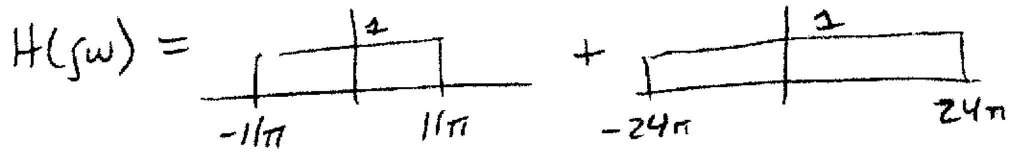
(a) Sketch the FT representation in the space given [redacted] assuming $T = \frac{1}{9}$ seconds.

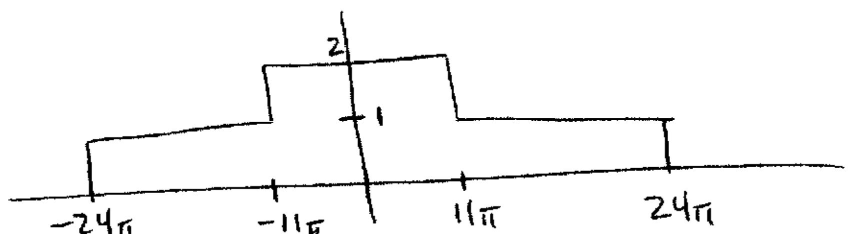
corresponding to $x[n]$ $\qquad \omega_s = 18\pi$

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

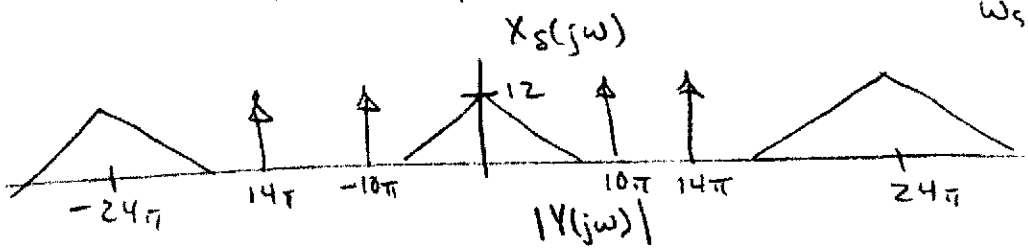


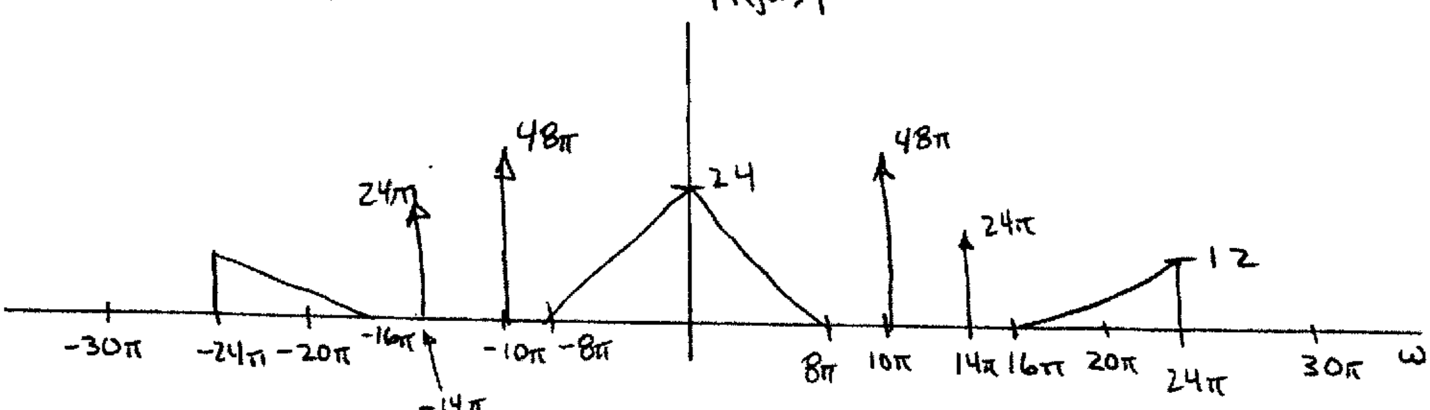
(b) Sketch $|Y(j\omega)|$ in the space provided assuming $T = \frac{1}{12}$. Hint: $y(t) = x_\delta(t) * h(t)$.





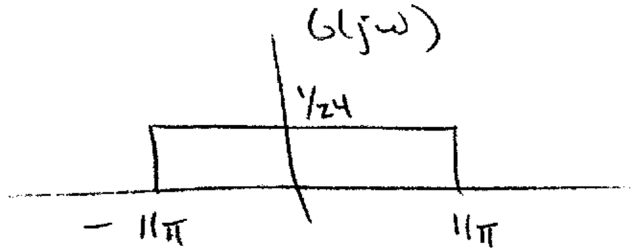
$\omega_s = 24\pi$



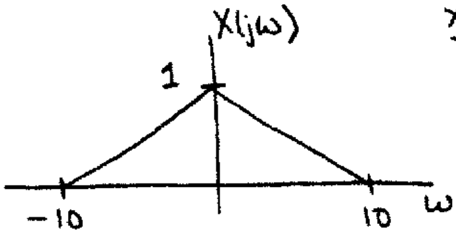


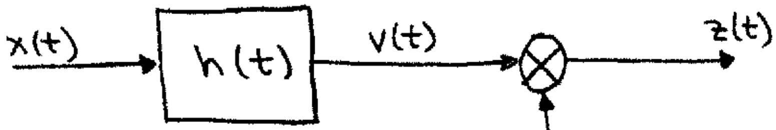
(c) Determine the frequency response $G(j\omega)$ of the inverse system for this system assuming $T = \frac{1}{12}$.

Recover $X(j\omega)$ from $Y(j\omega)$



4. (20 points) Consider the system shown below.

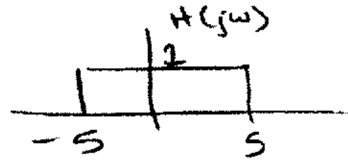


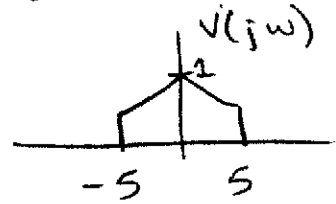


$w(t) = \cos(5t) + 2\cos(15t)$

$h(t) = \dfrac{\sin 5t}{\pi t}$

(a) Find the largest sampling interval T such that $v(t)$ can be uniquely reconstructed from its samples $v[n] = v(nT)$.





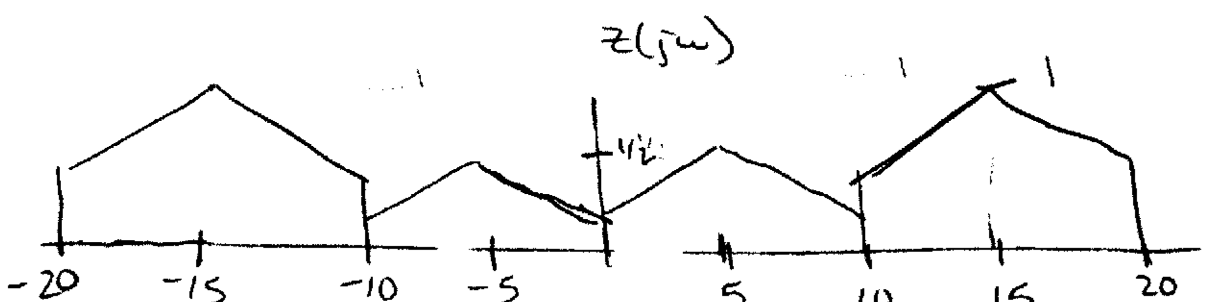
$$\omega_s = \frac{2\pi}{T} = 10 \quad \Rightarrow \quad T = \frac{2\pi}{10}$$

(b) Find the FT representation for $w(t)$.

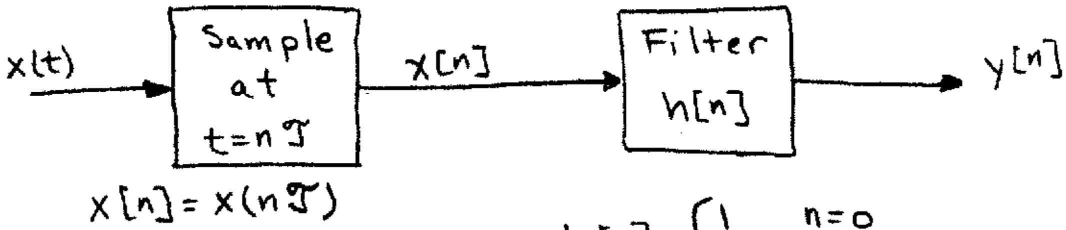
$$W(j\omega) = \pi\,\delta(\omega-5) + \pi\,\delta(\omega+5) + 2\pi\,\delta(\omega-15) + 2\pi\,\delta(\omega+15)$$

(c) Sketch the FT representation for $z(t)$ in the space provided.

$$Z(j\omega) = \frac{1}{2\pi} V(j\omega) * W(j\omega)$$



5. (20 points) The system depicted below has input $x(t)$ and output $y[n]$. The filter is LTI and has impulse response $h[n] = \delta[n] + \alpha\delta[n-1]$ with $|\alpha| < 1$.



$$h[n] = \begin{cases} 1 & n=0 \\ \alpha & n=1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Is this system stable? Prove your answer.

$$y[n] = h[n] * x[n] = x[n] + \alpha x[n-1]$$

$$\begin{aligned} |y[n]| &\leq |x[n]| + |\alpha|\,|x[n-1]| \\ &= |x(nT)| + |\alpha|\,|x((n-1)T)| \\ &\leq M_x(1+|\alpha|) \\ &< \infty \end{aligned}$$

stable

(b) Is this system time-invariant? Prove your answer.

No, sampling looses values in between sample times. Shifting the input does not correspond to shifting the output except when the input shift is an integral multiple of T

(c) Is this system linear? Prove your answer.

$$x(t) = a x_1(t) + b x_2(t) \implies$$

$$y[n] = a x_1[n] + b x_2[n] + \alpha(a x_1[n-1] + b x_1[n-1])$$

$$= a[x_1[n] + \alpha x_1[n-1]] + b[x_2[n] + \alpha x_2[n-1]])$$

$$= a y_1[n] + b y_2[n]$$

yes, linear