HW 1

EE 3015
Signals and Systems

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Contents

1 Problem 1.8, Chapter 1

Express the real part of each of the following signals in the form $Ae^{-at}\cos\left(\omega t+\phi\right)$, where A,a,ω,ϕ are real numbers with $A>0$ and $-\pi<\phi\leq\pi$: (a) $x_1(t)=-2$, (b) $x_2(t)=\sqrt{2}e^{j\frac{\pi}{4}}\cos(3t+2\pi)$, (c) $x_3(t)=e^{-t}\sin(3t+\pi)$, (d) $x_4(t)=je^{(-2+100j)t}$

Solution

1.1 part a

$$x_1(t)=-2$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$A=2,a=0,\phi=0,\omega=0,\phi=-\pi$$

1.2 part b

$$x_2(t)=\sqrt{2}e^{j\frac{\pi}{4}}\cos(3t+2\pi)$$

Since $\cos(3t+2\pi)=\cos(3t)$, the above becomes

$$\begin{aligned}x_2(t)&=\sqrt{2}e^{j\frac{\pi}{4}}\cos(3t)\\&=\sqrt{2}\left(\cos\frac{\pi}{4}+j\sin\frac{\pi}{4}\right)\cos(3t)\\&=\sqrt{2}\left(\frac{1}{2}\sqrt{2}+j\frac{1}{2}\sqrt{2}\right)\cos(3t)\\&=\left(1+j\right)\cos(3t)\end{aligned}$$

Hence the real part of $x_2(t)$ is

$$\operatorname{Re}(x_2(t))=\cos(3t)$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$A=1,a=0,\omega=3,\phi=0$$

1.3 part c

$$x_3(t)=e^{-t}\sin(3t+\pi)$$

Since $\sin(3t+\pi)=\cos\left(3t+\pi-\frac{\pi}{2}\right)=\cos\left(3t+\frac{\pi}{2}\right)$, then the above becomes

$$x_3(t)=e^{-t}\cos\left(3t+\frac{\pi}{2}\right)$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$A=1,a=1,\omega=3,\phi=\frac{\pi}{2}$$

1.4 part d

$$
\begin{aligned}
x_4(t) &= je^{(-2+100j)t} \\
&= je^{-2t}e^{j100t}
\end{aligned}
$$

But $j = e^{j\frac{\pi}{2}}$, hence the above becomes

$$
\begin{aligned}
x_4(t) &= e^{j\frac{\pi}{2}}e^{-2t}e^{j100t} \\
&= e^{-2t}e^{j\left(100t+\frac{\pi}{2}\right)} \\
&= e^{-2t}\left(\cos\left(100t+\frac{\pi}{2}\right)+j\sin\left(100t+\frac{\pi}{2}\right)\right)
\end{aligned}
$$

Therefore

$$
\operatorname{Re}(x_4(t)) = e^{-2t}\cos\left(100t+\frac{\pi}{2}\right)
$$

Comparing the above to $Ae^{-at}\cos\left(\omega t+\phi\right)$ shows that

$$
A = 1, a = 2, \omega = 100, \phi = \frac{\pi}{2}
$$

2 Problem 1.13, Chapter 1

Consider the continuous-time signal $x(t) = \delta(t+2) - \delta(t-2)$. Calculate the value of E_∞ for the signal $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

Solution

$y(t)$ is first found

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} \delta(t+2) - \delta(t-2)\, d\tau \\ &= \int_{-\infty}^{t} \delta(t+2)\, d\tau - \int_{-\infty}^{t} \delta(t-2)\, d\tau \end{aligned}$$

$\delta(t+2)$ is an impulse at $t = -2$ and $\delta(t-2)$ is an impulse at $t = 2$. Hence is $t < -2$ then the above is zero. If $-2 < t < 2$ then only the first integral contributes giving 1 and if $t > 2$ then both integral contribute 1 each, and hence cancel each others giving $y = 0$. Therefore

$$y(t) = \begin{cases} 0 & t < -2 \\ 1 & -2 < t < 2 \\ 0 & t > 2 \end{cases}$$

Now that $y(t)$ is found, its E_∞ can be calculated using the definition

$$\begin{aligned} E_\infty &= \int_{-\infty}^{\infty} |y(t)|^2\, dt \\ &= \int_{-2}^{2} 1 dt \\ &= [t]_{-2}^{2} \\ &= 2 + 2 \end{aligned}$$

Hence

$$E_\infty = 4$$

3 Problem 1.17, Chapter 1

Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by $y(t) = x(\sin(t))$. (a) Is this system causal? (b) Is this system linear?

Solution

3.1 Part a

A system is causal if its output at time t depends only on current t and on past t and not on future t. Picking $t = -\pi$, then $y(-\pi) = x(\sin(-\pi)) = x(0)$. This shows that $y(-\pi) = x(0)$. Hence the output depends on input at future time (since $0 > -\pi$). Therefore this system is not causal.

3.2 Part b

Let input be $x(t) = a_1 x_1(t) + a_2 x_2(t)$. If the output when the input is $x(t)$ is given by $y(t) = a_1 y_1(t) + a_2 y_2(t)$ where $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$ then the system is linear. From the definition

$$\begin{aligned} y(t) &= x(\sin(t)) \\ &= a_1 x_1(\sin(t)) + a_2 x_2(\sin(t)) \end{aligned}$$

Now, $y_1(t) = x_1(\sin t)$ and $y_2(t) = x_2(\sin t)$. Hence the above becomes

$$\begin{aligned} y(t) &= x(\sin(t)) \\ &= a_1 y_1(t) + a_2 y_2(t) \end{aligned}$$

Therefore the system is linear.

4 Problem 1.21, Chapter 1

A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals: (a) $x(t-1)$. (b) $x(2-t)$ (c) $x(2t+1)$ (d) $x\left(4-\frac{t}{2}\right)$

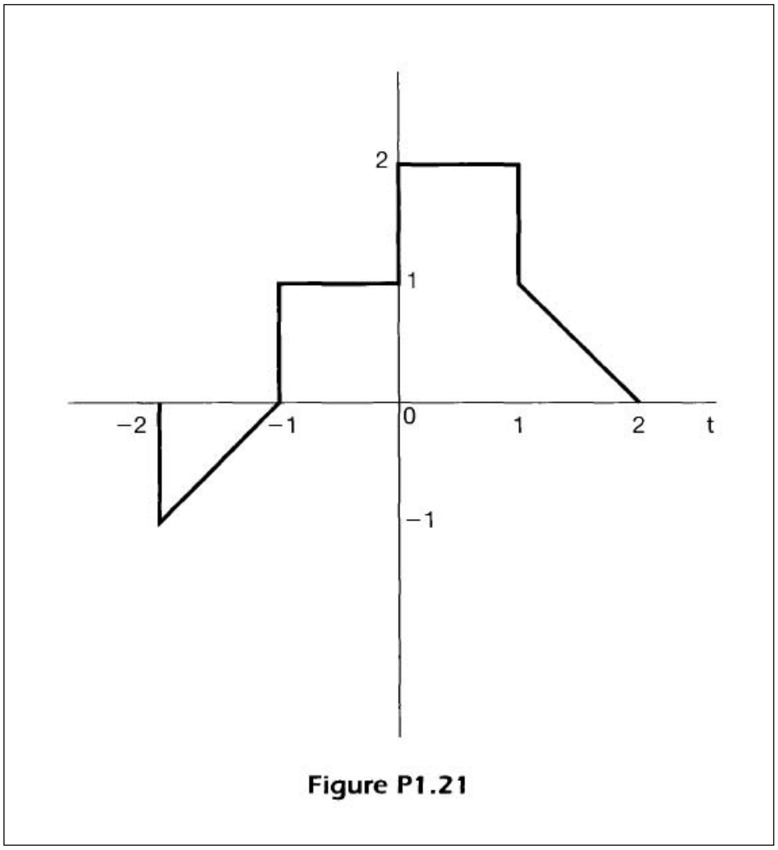

Figure 1: The function $x(t)$

Solution

Looking at the plot, it can be constructed from unit step $u(t)$ and ramp function $r(t)$ as follows

$$x(t) = -u(t+2) + r(t+2) - r(t+1) + u(t+1) + u(t) - u(t-1) - r(t-1) + r(t-2)$$

Here is an implementation

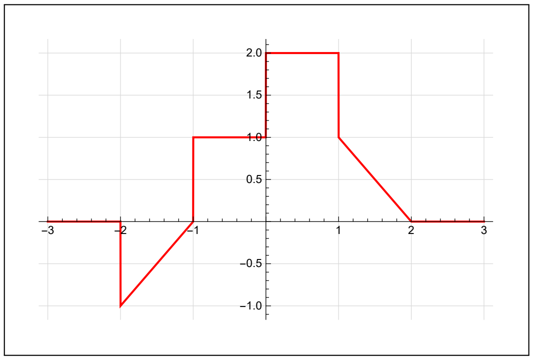

Figure 2: Construction the signal $x(t)$ from unit step and ramp functions

```
x[t_] := -UnitStep[t + 2] + Ramp[t + 2] - Ramp[t + 1] +
         UnitStep[t + 1] + UnitStep[t] - UnitStep[t - 1] - Ramp[t - 1] + Ramp[t - 2];
p = Plot[x[t], {t, -3, 3}, Exclusions → None,
         PlotStyle → Red,
         GridLines → Automatic, GridLinesStyle → LightGray];
```

Figure 3: Code for the above

4.1 Part a

$x(t-1)$ is $x(t)$ shifted to right by one unit time. Hence it becomes

$$x(t-1) = -u(t+1) + r(t+1) - r(t) + u(t) + u(t-1) - u(t-2) - r(t-2) + r(t-3)$$

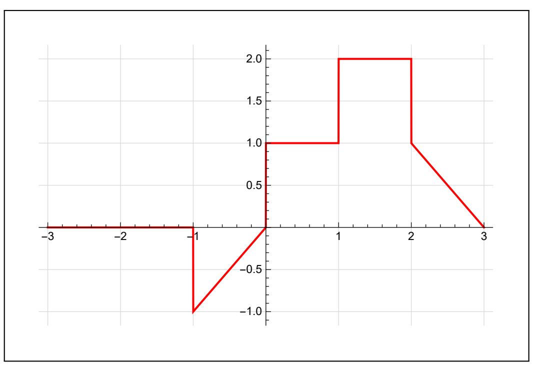

Figure 4: Part (a) plot

```
x[t_] := -UnitStep[t + 2] + Ramp[t + 2] - Ramp[t + 1] +
         UnitStep[t + 1] + UnitStep[t] - UnitStep[t - 1] - Ramp[t - 1] + Ramp[t - 2];
p = Plot[x[t - 1], {t, -3, 3}, Exclusions → None,
         PlotStyle → Red, GridLines → Automatic,
         GridLinesStyle → LightGray];
```

Figure 5: Code for the above

4.2 Part b

$$x(2-t) = x(-(t-2))$$

Hence the signal $x(t)$ is first flipped right to left (also called reflection about the $t = 0$ axis) and the resulting function is then shifted to the right by 2 units. It becomes

$$\begin{aligned} x(2-t) &= -u((2-t)+2) + r((2-t)+2) - r((2-t)+1) + u((2-t)+1) + u(2-t) - u((2-t)-1) - r((2-t)-1) + r((2-t)-2) \\ &= -u(4-t) + r(4-t) - r(3-t) + u(3-t) + u(2-t) - u(1-t) - r(1-t) + r(-t) \end{aligned}$$

The flipped signal is

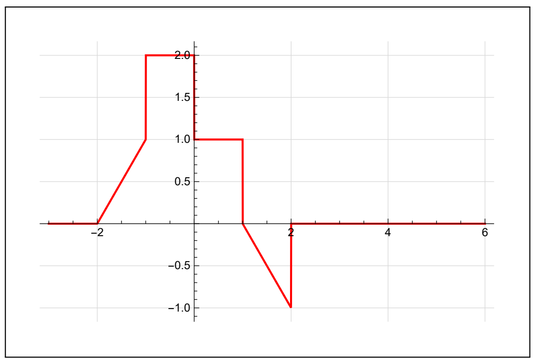

Figure 6: Part (b) signal after reflection

Now the above is shifted to the right by 2 units giving

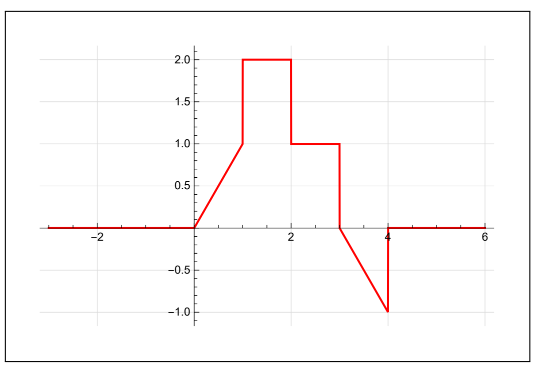

Figure 7: Part (b) final plot

It also possible to first do the shifting, followed by the reflection. Same output will result.

4.3 Part c

$x(2t+1) = x\left(2\left(t+\frac{1}{2}\right)\right)$. The signal is first shifted to the left by $\frac{1}{2}$ due to the $+\frac{1}{2}$ term, and then the resulting signal is squashed (contraction) by factor of 2. Since the original signal is from

−1 to 3, then after first shifting it to the left by 0.5 it becomes from −1.5 to 2.5. Hence the original ramp that went from −2 to −1 now goes from −1.5 to −1 and the line that originally went from −1 to 0 now goes from −1 to $-\frac{1}{2}$ (half the length) and so on. This is the result

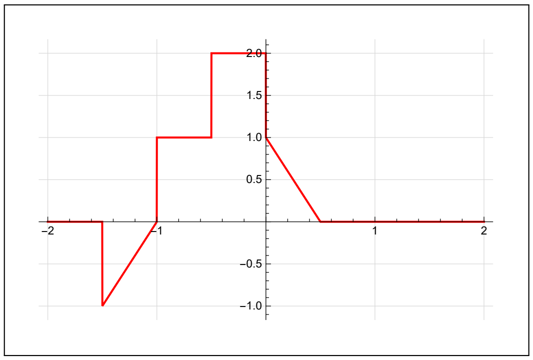

Figure 8: Part (c) plot

```
ClearAll[x, t];
x[t_] := -UnitStep[t + 2] + Ramp[t + 2] - Ramp[t + 1] +
         UnitStep[t + 1] + UnitStep[t] - UnitStep[t - 1] - Ramp[t - 1] + Ramp[t - 2];
p = Plot[x[2 t + 1], {t, -2, 2}, PlotRange → All, Exclusions → None,
         PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray];
```

Figure 9: Code for the above

4.4 Part d

$x\left(4-\frac{t}{2}\right)=x\left(-\left(\frac{t}{2}-4\right)\right)=x\left(-\frac{1}{2}\left(t-8\right)\right)$. Hence, the signal is first shifted to the right by 8 due to the −8 term, and then the resulting signal is flipped across the $t=0$ axis, and then the resulting signal is stretched (expanded) by factor of 2 due to the multiplication by $\frac{1}{2}$ term. This is the result showing each step

Figure 10: Part (d) plot

5 Problem 1.22, Chapter 1

A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals (a) $x[n-4]$ (b) $x[3-n]$ (c) $x[3n]$ (d) $x[3n+1]$

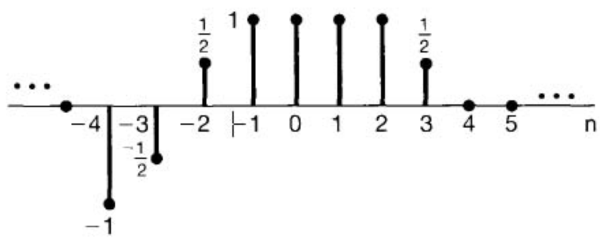

Figure P1.22

Figure 11: The function $x[n]$

<u>Solution</u>

5.1 Part a

$x[n-4]$ is $x[n]$ shifted to the right by 4 positions. Hence it becomes

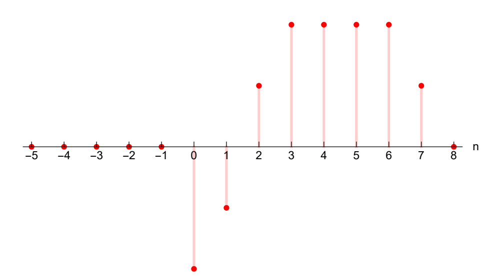

Figure 12: Part (a) plot

```
x[n_] := Piecewise[{{-1, n == -4}, {-1/2, n == -3}, {1/2, n == -2},
         {1, n == -1}, {1, n == 0}, {1, n == 1}, {1, n == 2}, {1/2, n == 3}}]
p = DiscretePlot[x[n - 4], {n, -5, 8}, PlotStyle → {Thick, Red}, AxesLabel → {"n", "x[n]"},
         Axes → {True, False}, Ticks → {Range[-5, 8], Automatic}];
```

Figure 13: Code used for the above

5.2 Part b

$x[3-n] = x[-(n-3)]$. Hence $x[n]$ is first reflected to obtain $x[-n]$ and then the result is shifted to right by 3. This is the result showing each step

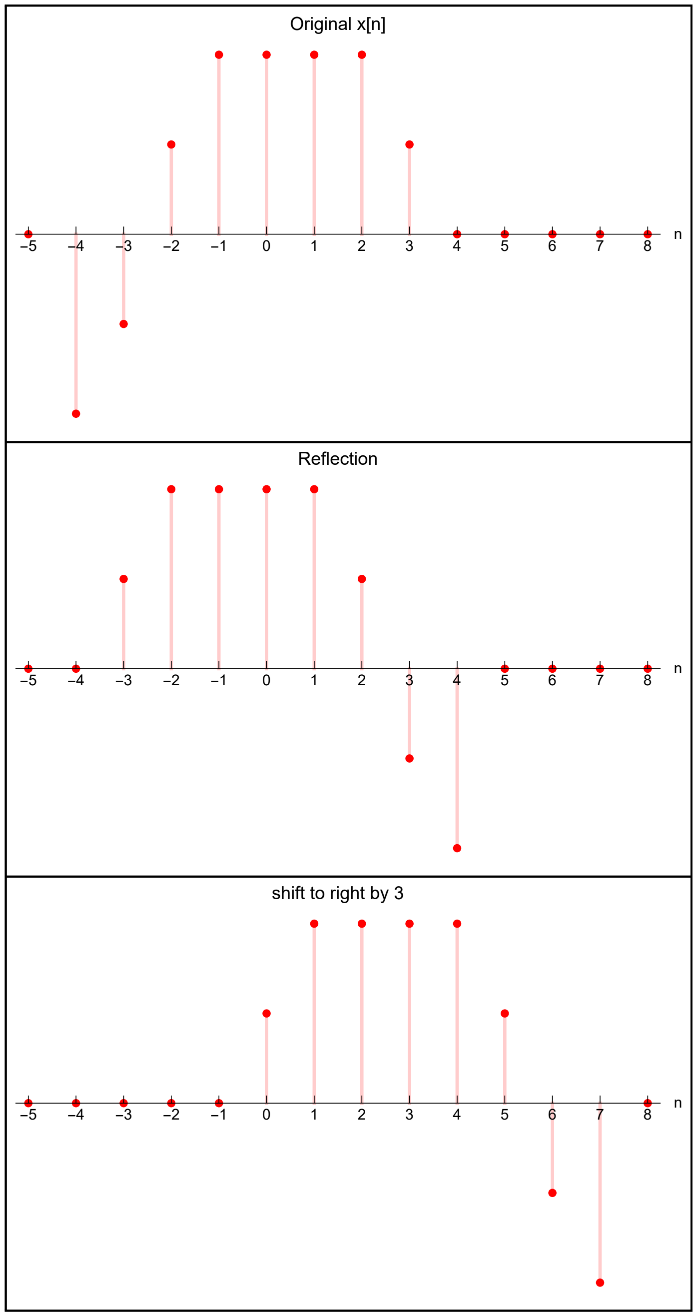

Figure 14: Part (b) plot

5.3 Part c

$x[3n]$. Sample at $n = 0$ remains the same. Sample at $n = -1$ gets the value of the sample that was at -3 which is $-\frac{1}{2}$. Sample at $n = -2$ gets the value of the sample that was at $n = -6$ which is zero. Hence for all n less than -1 new values are all zero. Same for the right side. The sample at $n = 1$ gets the value of the sample that was at 3 which is $\frac{1}{2}$ and sample at $n = 2$ gets the value of the sample that was at 6 which is 0 and all $n > 1$ are therefore zero. Notice that this operation causes samples to be lost from the original signal. This is the final result

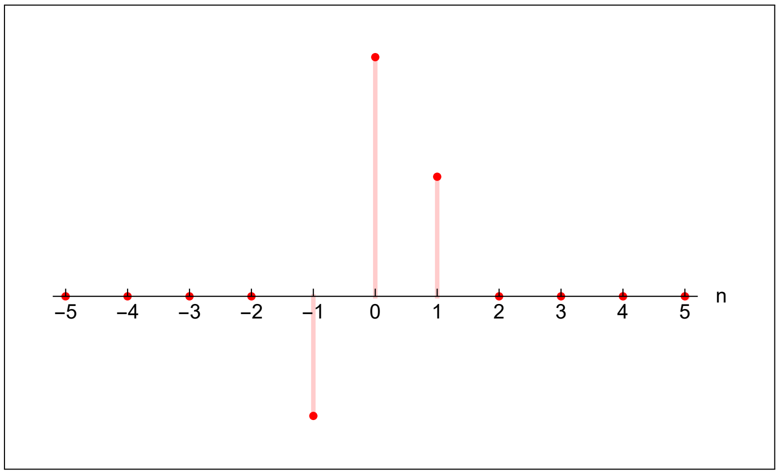

Figure 15: Part (c) plot

5.4 Part d

$x[3n+1]$. Sample at $n = 0$ gets the value that was at $n = 1$ which is 1. Sample at $n = -1$ gets the value of the sample that was at $-3+1 = -2$ which is $\frac{1}{2}$. Sample at $n = -2$ gets the value of the sample that was at $n = -6+1 = -5$ which is zero. Hence for all $n < -1$ all values are zero. Same for the right side. Sample at $n = 1$ gets the value of the sample that was at $3+1 = 4$ which is 0 and therefore for all $n > 1$ samples are zero. Notice that this operation causes samples to be lost from the original signal. The result is

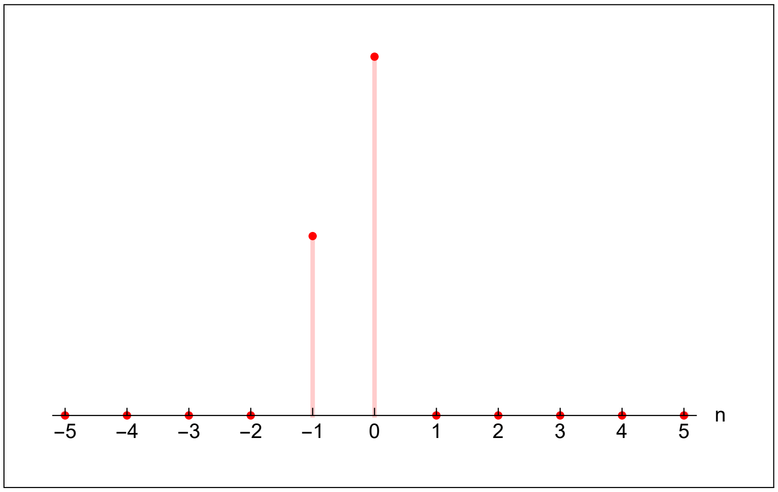

Figure 16: Part (d) plot

6 Problem 1.26, Chapter 1

Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period (a) $\sin\left(\frac{6}{7}\pi n+1\right)$ (b) $x[n]=\cos\left(\frac{n}{8}-\pi\right)$ (c) $x[n]=\cos\left(\frac{\pi}{8}n^2\right)$ (d) $x[n]=\cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$

Solution

The signal $x[n]$ is periodic, if integer N can be found that $x[n]=x[n+N]$ for all n. Fundamental period is the smallest such integer N.

6.1 Part a

In this part, $x[n]=\sin\left(\frac{6}{7}\pi n+1\right)$. Hence the signal is periodic if

$$
\begin{aligned}
x[n] &= x[n+N] \\
\sin\left(\frac{6}{7}\pi n+1\right) &= \sin\left(\frac{6}{7}\pi(n+N)+1\right) \\
&= \sin\left(\left(\frac{6}{7}\pi n+1\right)+\frac{6}{7}\pi N\right)
\end{aligned}
$$

The above will be true if

$$\frac{6}{7}\pi N = 2\pi m$$

For some integer m and N. This is because $\sin$ has 2π period. This implies that

$$\frac{3}{7}=\frac{m}{N}$$

Therefore $N=7, m=3$. Since it was possible to find n,N integers, then it is periodic. Since m,N are relatively prime then $N=7$ is the fundamental period.

6.2 Part b

In this part, $x[n]=\cos\left(\frac{n}{8}-\pi\right)$. Hence the signal is periodic if

$$
\begin{aligned}
x[n] &= x[n+N] \\
\cos\left(\frac{n}{8}-\pi\right) &= \cos\left(\frac{n+N}{8}-\pi\right) \\
&= \cos\left(\left(\frac{n}{8}-\pi\right)+\frac{N}{8}\right)
\end{aligned}
$$

The above will be true if

$$
\begin{aligned}
\frac{N}{8} &= 2\pi m \\
\frac{1}{16\pi} &= \frac{m}{N}
\end{aligned}
$$

For some integer N, m. It is not possible to find integers m, N to satisfy the above since π is an irrational number. Hence not periodic.

6.3 Part c

$x[n] = \cos\left(\frac{\pi}{8}n^2\right)$. Hence the signal is periodic if

$$\begin{aligned} x[n] &= x[n+N] \\ \cos\left(\frac{\pi}{8}n^2\right) &= \cos\left(\frac{\pi}{8}(n+N)^2\right) \\ &= \cos\left(\frac{\pi}{8}\left(n^2+N^2+2nN\right)\right) \\ &= \cos\left(\frac{\pi}{8}n^2+\frac{\pi}{8}\left(N^2+2nN\right)\right) \end{aligned}$$

The above will be true if

$$\frac{\pi}{8}\left(N^2+2nN\right) = 2\pi m$$

Need to find smallest integer N to satisfy this for all n. Choosing $N = 8$ the above becomes

$$\begin{aligned} \frac{\pi}{8}(64+16n) &= m(2\pi) \\ 8\pi + 2n\pi &= m(2\pi) \end{aligned}$$

Hence for all n, $N = 8$ satisfies the equation (since m is arbitrary integer). Therefore it is periodic and fundamental period $N = 8$.

6.4 Part d

Using $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ then

$$\begin{aligned} \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right) &= \frac{1}{2}\left(\cos\left(\frac{\pi}{2}n+\frac{\pi}{4}n\right)+\cos\left(\frac{\pi}{2}n-\frac{\pi}{4}n\right)\right) \\ &= \frac{1}{2}\left(\cos\left(\frac{3\pi}{4}n\right)+\cos\left(\frac{\pi}{4}n\right)\right) \end{aligned}$$

Considering each signal separately. $x[n] = \cos\left(\frac{3\pi}{4}n\right)$. This is periodic if

$$\begin{aligned} x[n] &= x[n+N] \\ \cos\left(\frac{3\pi}{4}n\right) &= \cos\left(\frac{3\pi}{4}(n+N)\right) \\ &= \cos\left(\left(\frac{3\pi}{4}n\right)+\frac{3\pi}{4}N\right) \end{aligned}$$

The above will be true if

$$\frac{3\pi}{4}N = 2\pi m$$
$$\frac{3}{8} = \frac{m}{N}$$

It was possible to find integers N, m to satisfy this, where period $N = 8$. Considering the second signal $x[n] = \cos\left(\frac{\pi}{4}n\right)$. This is periodic if

$$x[n] = x[n+N]$$
$$\cos\left(\frac{\pi}{4}n\right) = \cos\left(\frac{\pi}{4}(n+N)\right)$$
$$= \cos\left(\left(\frac{\pi}{4}n\right) + \frac{\pi}{4}N\right)$$

The above will be true if

$$\frac{\pi}{4}N = 2\pi m$$
$$\frac{1}{8} = \frac{m}{N}$$

It was possible to find integers N, m to satisfy this, where period $N = 8$. Therefore both signals periodic with same period, the sum is therefore periodic and the fundamental period is $N = 8$.