1 (a) $(AB)^T_{ij} = \sum_l A_{jl} B_{li} = \sum_l (B^T)_{il} (A^T)_{lj} = (B^T A^T)_{ij}$

$\uparrow$ ij component of $(AB)^T$ $\Rightarrow$ $\boxed{(AB)^T = B^T A^T}$

(b) $(AB)^\dagger_{ij} = \sum_l A^*_{jl} B^*_{li} = \sum_l (B^\dagger)_{il} (A^\dagger)_{lj} = (B^\dagger A^\dagger)_{ij}$

$\Rightarrow \boxed{(AB)^\dagger = B^\dagger A^\dagger}$

(c) $Tr(AB) = \sum_i (AB)_{ii} = \sum_{i,j} A_{ij} B_{ji} = \sum_{i,j} B_{ji} A_{ij}$

$= \sum_j (BA)_{jj} = Tr(BA)$ $\boxed{Tr(AB) = Tr(BA)}$

(d) $\det A^T = \det A$ is clearly true when A is 1×1.

Proceed by induction. Assume it is true for $n \times n$.

Then for $(n+1) \times (n+1)$ $\det A^T = \sum_{j=1}^{n+1} A^T_{ij} (-1)^{i+j} \det A^T(i,j)$

$= \sum_{j=1}^{n+1} A_{ji} (-1)^{i+j} \det A(j,i) = \det A$

(e) $A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}$ $B = \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix}$ $AB = \begin{pmatrix} a_1 b_1 & & & 0 \\ & a_2 b_2 & & \\ & & \ddots & \\ 0 & & & a_n b_n \end{pmatrix}$

$\det AB = \prod_{i=1}^n a_i b_i = \left(\prod_{i=1}^n a_i\right)\left(\prod_{i=1}^n b_i\right) = \det A \cdot \det B$

(2) $T = \begin{pmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} \end{pmatrix}$ Note: symmetric matrix

$$\det(T-\lambda I) = \left(\frac{5}{2}-\lambda\right)\left(\frac{7}{3}-\lambda\right)\left(\frac{13}{6}-\lambda\right) + 2\cdot\sqrt{\frac{3}{2}\cdot\frac{1}{18}\cdot\frac{3}{4}}$$

$$-\frac{3}{4}\left(\frac{7}{3}-\lambda\right) - \frac{1}{18}\left(\frac{5}{2}-\lambda\right) - \frac{3}{2}\left(\frac{13}{6}-\lambda\right) = -\lambda^3+7\lambda^2-14\lambda+8$$

$$= -(\lambda-1)(\lambda-2)(\lambda-4)$$

Eigenvalues: $\boxed{\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 4}$

Find the eigenvectors by solving the algebraic equations $(T-\lambda_i I)x_i = 0$. The results are

$$x_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right)$$

$$x_2 = \left(0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$$

$$x_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)$$

These are orthogonal and chosen to be normalized to one.

③ $Ux = \lambda x \qquad U^\dagger U = U U^\dagger = I$

Take Hermitian conjugate $\Rightarrow x^\dagger U^\dagger = \lambda^* x^\dagger$

Multiply $(x^\dagger U^\dagger)(Ux) = (\lambda^* x^\dagger)(\lambda x)$

$$x^\dagger \underbrace{(U^\dagger U)}_{I} x = \lambda^* (x^\dagger x) \lambda \Rightarrow \lambda^* \lambda = 1$$

$\Rightarrow \boxed{|\lambda| = 1}$

$x_1^\dagger x_2 = x_1^\dagger (U^\dagger U) x_2 = (x_1^\dagger U^\dagger)(U x_2) = (\lambda_1^* x_1^\dagger)(\lambda_2 x_2)$

$= \lambda_1^* \lambda_2 \, x_1^\dagger x_2 \Rightarrow$ Either $\lambda_1^* \lambda_2 = 1$ or $x_1^\dagger x_2 = 0$

Since $|\lambda| = 1$ write $\lambda_1 = e^{i\theta_1}$ and $\lambda_2 = e^{i\theta_2}$.

Then $\lambda_1^* \lambda_2 = e^{i(\theta_2 - \theta_1)}$. If $\lambda_1^* \lambda_2 = 1$ then

$\theta_2 - \theta_1 = 2\pi n \Rightarrow \lambda_1 = \lambda_2$.

$\uparrow$ integer

$\boxed{\text{Thus if } \lambda_1 \neq \lambda_2 \text{ then } x_1^\dagger x_2 = 0.}$

(4) Expand around the first row.

$$\det\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & i & 1 \end{pmatrix} = \underbrace{(-1)^{1+2}(-i)}_{i}\det\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix}$$

$$= \underbrace{i(-1)^{1+1}i}_{-1}\det\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix} = -(-1)^{1+1}3\det\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = -6$$

As a check expand around the second row.

$$\underbrace{(-1)^{2+1}i}_{-i}\det\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} = -i(-i)\det\begin{pmatrix} 3 & 0 & 0 \\ 6 & 1 & i \\ 0 & i & 1 \end{pmatrix}$$

$$= -[3+3] = -6$$

$$\boxed{\text{determinant} = -6}$$