HW 8
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

1. (10 pts) Prove the following relations.

$$\begin{aligned} (AB)^T &= B^T A^T \\ (AB)^\dagger &= B^\dagger A^\dagger \\ \mathrm{Tr}(AB) &= \mathrm{Tr}(BA) \\ \det A^T &= \det A \\ \det(AB) &= \det(A)\cdot\det(B) \end{aligned}$$

For the last one you may assume that A and B are diagonal.

Figure 1: Problem statement

1.1 part 1 $(AB)^T = B^T A^T$

Let A be an n,m matrix and B be an m,p matrix. Hence $AB = C$ is an n,p matrix. By definition of matrix product which is rows of A multiply columns of B then the ij element of C is

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Then $(AB)^T = C^T$. Hence from above, elements of C^T are given by

$$c_{ji} = \sum_{k=1}^{m} a_{jk} b_{ki} \tag{1}$$

Now let $B^T A^T = Q$. Where now B^T is order $p \times m$ and A^T is order $m \times n$, hence Q is $p \times n$.

$$\begin{aligned} q_{ij} &= \sum_{k=1}^{m} (b_{ik})^T \left(a_{kj}\right)^T \\ &= \sum_{k=1}^{m} b_{ki} a_{jk} \end{aligned}$$

But $\sum b_{ki} a_{jk}$ means to multiply column i of B by row j in A, which is the same as multiplying row j of A by column i of B. Hence we can change the order of multiplication above as

$$q_{ij} = \sum_{k=1}^{m} a_{jk} b_{ki} \tag{2}$$

Comparing (1) and (2) shows they are the same. Hence

$$C^T = Q$$

Or

$$(AB)^T = B^T A^T$$

1.2 Part 2 $(AB)^\dagger = B^\dagger A^\dagger$

By definition $A^\dagger = \left(A^T\right)^*$. Which means we take the transpose of A and then apply complex conjugate to its entries. Hence the solution follows the above, but we just have to apply complex conjugate at the end of each operation

Let A be an $n \times m$ matrix and B be $m \times p$ matrix. Hence $AB = C$ which is $n \times p$ matrix. By definition of matrix product which is row of A multiplies columns of B then the ij element of C is

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Then $(AB)_{ij}^\dagger = \left(C_{ij}^T\right)^* = c_{ji}^*$. Hence from above

$$c_{ji}^* = \sum_{k=1}^{m} \left(a_{jk} b_{ki}\right)^*$$

But complex conjugate of product is same as product of complex conjugates, hence the above is same as

$$c_{ji}^* = \sum_{k=1}^{m} a_{jk}^* b_{ki}^* \tag{1}$$

Now let $B^\dagger A^\dagger = Q$. Then

$$\begin{aligned} q_{ij} &= \sum_{k=1}^{m} \left(b_{ik}^T\right)^* \left(a_{kj}^T\right)^* \\ &= \sum_{k=1}^{m} b_{ki}^* a_{jk}^* \end{aligned}$$

But $\sum_{k=1}^{m} b_{ki}^* a_{jk}^*$ means to multiply complex conjugate of column i of B by complex conjugate of row j in A, which is the same as multiplying complex conjugate complex of row j of A by complex conjugate of column i of B. Hence the above can be written as

$$q_{ij} = \sum_{k=1}^{m} a_{jk}^* b_{ki}^* \tag{2}$$

Comparing (1) and (2) shows they are the same. Hence

$$\left(C^T\right)^* = Q$$

Or

$$(AB)^\dagger = B^\dagger A^\dagger$$

1.3 Part 3 $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$

The trace Tr of a matrix is the sum of elements on the diagonal matrix (and this applies only to square matrices). Let A be $n \times m$ And B be an $m \times n$ matrix. Hence AB is $n \times n$ matrix and BA is $m \times m$ matrix.

$$\begin{aligned}
\operatorname{Tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} b_{ji} \right) \\
&= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} b_{ji} a_{ij} \right) \\
&= \sum_{j=1}^{m} (BA)_{jj} \\
&= \operatorname{Tr}(BA)
\end{aligned}$$

1.4 Part 4 $\det\left(A^T\right) = \det A$

Proof by induction. Let base be $n = 1$. Hence $A_{1\times 1}$. It is clear that $\det(A) = \det\left(A^T\right)$ in this case. We could also have selected base case to be $n = 2$. Any base case will work in proof by induction.

We now assume it is true for the $n-1$ case. i.e. $\det\left(A_{(n-1)\times(n-1)}\right) = \det\left(A^T_{(n-1)\times(n-1)}\right)$ is assumed to be true. This is called the induction hypothesis step.

We need now to show it is true for the case of n, i.e. we need to show that $\det(A_{n\times n}) = \det\left(A^T_{n\times n}\right)$. Let

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Therefore

$$A^T_{n\times n} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \cdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Now we take $\det(A)$ and expand using cofactors along the first row which gives

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1} a_{1n}\det(A_{1n}) \tag{1}$$

Where A_{ij} in the above means the matrix of dimensions $(n-1, n-1)$ taken from $A_{n\times n}$ by removing the i^{th} row and the j^{th} column. Now we do the same for A^T above, but instead of expanding using first row, we expend using first column of A^T since we can pick any row or any column to expand around in order find the determinant. This gives

$$\det\left(A^T\right) = a_{11}\det\left(A^T\right)_{11} - a_{12}\det\left(A^T\right)_{21} + \cdots + (-1)^{n+1} a_{1n}\det\left(A^T\right)_{n1} \tag{2}$$

For (1) to be the same as (2) we need to show that $\det(A_{11}) = \det\left(A^T\right)_{11}$ and $\det(A_{12}) = \det\left(A^T\right)_{21}$ and all the way to $\det(A_{1n}) = \det\left(A^T\right)_{n1}$. But this is true by assumption. Since we assumed that $\det\left(A_{(n-1)\times(n-1)}\right) = \det\left(A^T_{(n-1)\times(n-1)}\right)$. In other words, by the induction hypothesis $\det\left(A_{ij}\right) = \det\left(A^T\right)_{ji}$ since both are $(n-1)\times(n-1)$ order. Hence (1) is the same as (2). This completes the proof.

1.5 Part 5 $\det(AB) = \det(A)\det(B)$

Since the matrices are diagonal they must be square. And since product AB is defined, then they must both be same dimension, say $n \times n$.

Since A, B are diagonal, then

$$\begin{aligned}
\det(A) &= a_{11}a_{22}\cdots a_{nn} = \prod_i^n a_{ii} \\
\det(B) &= b_{11}b_{22}\cdots b_{nn} = \prod_i^n b_{jj}
\end{aligned}$$

Now since A, B are diagonals, then the product is diagonal. Using definition of a row from A multiplies a column in B, we get

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}\begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & 0 & 0 & 0 \\ 0 & a_{22}b_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn}b_{nn} \end{pmatrix}$$

Then we see that

$$\begin{aligned}
\det(AB) &= (a_{11}b_{11})(a_{22}b_{22})\cdots(a_{nn}b_{nn}) \\
&= (a_{11}a_{22}\cdots a_{nn})(b_{11}b_{22}\cdots b_{nn}) \\
&= \prod_i^n a_{ii}\prod_i^n b_{jj} \\
&= \det(A)\det(B)
\end{aligned}$$

2 Problem 2

2. (7 pts) Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} \end{pmatrix}$$

Figure 2: Problem statement

We first need to find the eigenvalues λ by solving

$$\det(A - \lambda I) = 0$$

The above gives a polynomial of order 3.

$$\left| \begin{pmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \frac{5}{2} - \lambda & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} - \lambda & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{5}{2} - \lambda\right) \begin{vmatrix} \frac{7}{3} - \lambda & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{1}{18}} & \frac{13}{6} - \lambda \end{vmatrix} - \sqrt{\frac{3}{2}} \begin{vmatrix} \sqrt{\frac{3}{2}} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \frac{13}{6} - \lambda \end{vmatrix} + \sqrt{\frac{3}{4}} \begin{vmatrix} \sqrt{\frac{3}{2}} & \frac{7}{3} - \lambda \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} \end{vmatrix} = 0$$

Hence

$$\left(\frac{5}{2} - \lambda\right)\left(\left(\frac{7}{3} - \lambda\right)\left(\frac{13}{6} - \lambda\right) - \sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}\right) - \sqrt{\frac{3}{2}}\left(\sqrt{\frac{3}{2}}\left(\frac{13}{6} - \lambda\right) - \sqrt{\frac{1}{18}}\sqrt{\frac{3}{4}}\right) + \sqrt{\frac{3}{4}}\left(\sqrt{\frac{3}{2}}\sqrt{\frac{1}{18}} - \left(\frac{7}{3} - \lambda\right)\sqrt{\frac{3}{4}}\right) = 0$$

Or

$$\left(\frac{5}{2}-\lambda\right)\left(\lambda^2-\frac{9}{2}\lambda+\frac{90}{18}\right)-\sqrt{\frac{3}{2}}\left(\sqrt{6}-\frac{1}{2}\sqrt{2}\sqrt{3}\lambda\right)+\sqrt{\frac{3}{4}}\left(\sqrt{3}\left(\frac{1}{2}\lambda-1\right)\right)=0$$

$$\left(\frac{5}{2}-\lambda\right)\left(\lambda^2-\frac{9}{2}\lambda+\frac{90}{18}\right)+\left(\frac{3}{2}\lambda-3\right)+\left(\frac{3}{4}\lambda-\frac{3}{2}\right)=0$$

$$\left(\frac{5}{2}-\lambda\right)\left(\lambda^2-\frac{9}{2}\lambda+\frac{90}{18}\right)+\frac{9}{4}\lambda-\frac{9}{2}=0$$

$$-\lambda^3+7\lambda^2-14\lambda+8=0$$

$$\lambda^3-7\lambda^2+14\lambda-8=0$$

By inspection we see that $\lambda=2$ is a root. Then by long division $\frac{\lambda^3-7\lambda^2+14\lambda-8}{\lambda-2}=\lambda^2-5\lambda+4$. Therefore the above polynomial can be written as

$$\left(\lambda^2-5\lambda+4\right)(\lambda-2)=0$$

$$(\lambda-1)(\lambda-4)(\lambda-2)=0$$

Hence the eigenvalues are

$$\lambda_1=1$$
$$\lambda_2=2$$
$$\lambda_3=4$$

For each eigenvalue there is one corresponding eigenvector (unless it is degenerate). The eigenvectors are found by solving the following

$$Av_i=\lambda_i v_i$$

$$(A-\lambda_i I)\,v_i=0$$

$$\begin{pmatrix}\frac{5}{2}-\lambda_i & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{7}{3}-\lambda_i & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}-\lambda_i\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}_i=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

<u>For $\lambda_1=1$</u>

$$\begin{pmatrix}\frac{5}{2}-1 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{7}{3}-1 & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{2}} & \frac{4}{3} & \sqrt{\frac{1}{18}}\\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{7}{6}\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Let $v_1 = 1$ and the above becomes

$$\begin{pmatrix} \frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{4}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{7}{6} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We only need the first 2 equations. This results in

$$\begin{aligned} \frac{3}{2} + \sqrt{\frac{3}{2}}v_2 + \sqrt{\frac{3}{4}}v_3 &= 0 \\ \sqrt{\frac{3}{2}} + \frac{4}{3}v_2 + \sqrt{\frac{1}{18}}v_3 &= 0 \end{aligned}$$

From the first equation above

$$v_2 = \frac{-\frac{3}{2} - \sqrt{\frac{3}{4}}v_3}{\sqrt{\frac{3}{2}}} \tag{4}$$

Substituting in the second equation gives

$$\begin{aligned} \sqrt{\frac{3}{2}} + \frac{4}{3}\left(\frac{-\frac{3}{2} - \sqrt{\frac{3}{4}}v_3}{\sqrt{\frac{3}{2}}}\right) + \sqrt{\frac{1}{18}}v_3 &= 0 \\ -\frac{1}{2}\sqrt{2}v_3 - \frac{1}{6}\sqrt{2}\sqrt{3} &= 0 \\ v_3 &= -\frac{\frac{1}{6}\sqrt{2}\sqrt{3}}{\frac{1}{2}\sqrt{2}} \\ &= -\frac{2\sqrt{3}}{6} \\ &= -\frac{\sqrt{3}}{3} \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

Hence from (4)

$$\begin{aligned} v_2 &= \frac{-\frac{3}{2} - \sqrt{\frac{3}{4}}\left(-\frac{1}{\sqrt{3}}\right)}{\sqrt{\frac{3}{2}}} \\ &= -\frac{\sqrt{2}}{\sqrt{3}} \end{aligned}$$

Therefore the eigenvector associated with $\lambda_1 = 1$ is $\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$ or by scaling it all by $-\sqrt{3}$ it becomes

$$\vec{v}_1 = \begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}$$

We now do the same for the second eigenvalue.

<u>For $\lambda_2 = 2$</u>

$$\begin{pmatrix} \frac{5}{2} - 2 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} - 2 & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} - 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{1}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_1 = 1$ and the above becomes

$$\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{1}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We only need the first 2 equations. This results in

$$\frac{1}{2} + \sqrt{\frac{3}{2}} v_2 + \sqrt{\frac{3}{4}} v_3 = 0$$
$$\sqrt{\frac{3}{2}} + \frac{1}{3} v_2 + \sqrt{\frac{1}{18}} v_3 = 0$$

From the first equation above

$$v_2 = \frac{-\frac{1}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}} \tag{4A}$$

Substituting in the second equation gives

$$
\begin{aligned}
\sqrt{\frac{3}{2}}+\frac{1}{3}\left(\frac{-\frac{1}{2}-\sqrt{\frac{3}{4}}v_3}{\sqrt{\frac{3}{2}}}\right)+\sqrt{\frac{1}{18}}v_3 &= 0\\
\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{18}}v_3-\frac{1}{18}\sqrt{2}\sqrt{3}+\sqrt{\frac{1}{18}}v_3 &= 0\\
0 &= \sqrt{\frac{3}{2}}+\frac{1}{18}\sqrt{2}\sqrt{3}
\end{aligned}
$$

This is not possible. So out choice of setting $v_1 = 1$ does not work. Let us try to set $v_2 = 1$ and repeat the process

$$
\begin{pmatrix} \frac{1}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{1}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} v_1 \\ 1 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

Again, we only need the first two equations. This results in

$$
\begin{aligned}
\frac{1}{2}v_1+\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{4}}v_3 &= 0\\
\sqrt{\frac{3}{2}}v_1+\frac{1}{3}+\sqrt{\frac{1}{18}}v_3 &= 0
\end{aligned}
$$

From the first equation above

$$
v_1 = \frac{-\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{4}}v_3}{\frac{1}{2}} \tag{4A}
$$

Substituting in the second equation gives

$$\begin{aligned}
\sqrt{\frac{3}{4}}\left(\frac{-\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{4}}v_3}{\frac{1}{2}}\right)+\frac{1}{3}+\sqrt{\frac{1}{18}}v_3 &= 0\\
-\frac{3}{2}v_3-\frac{3}{2}\sqrt{2}+\frac{1}{3}+\sqrt{\frac{1}{18}}v_3 &= 0\\
\frac{1}{6}\sqrt{2}v_3-\frac{3}{2}v_3-\frac{3}{2}\sqrt{2}+\frac{1}{3} &= 0\\
v_3\left(\frac{1}{6}\sqrt{2}-\frac{3}{2}\right) &= \frac{3}{2}\sqrt{2}-\frac{1}{3}\\
v_3 &= \frac{\frac{3}{2}\sqrt{2}-\frac{1}{3}}{\frac{1}{6}\sqrt{2}-\frac{3}{2}}\\
&= -\sqrt{2}
\end{aligned}$$

Hence from (4A) $v_1 = \frac{-\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{4}}(-\sqrt{2})}{\frac{1}{2}} = \frac{-\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{2}}}{\frac{1}{2}} = 0$. Therefore the eigenvector associated with $\lambda_2 = 2$ is $\begin{pmatrix} 0 \\ 1 \\ -\sqrt{2} \end{pmatrix}$ or by scaling it all by $-\frac{1}{\sqrt{2}}$ it becomes

$$\vec{v}_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

We now do the same for the final eigenvalue

<u>For $\lambda_3 = 4$</u>

$$\begin{pmatrix} \frac{5}{2}-4 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3}-4 & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6}-4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & -\frac{5}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & -\frac{11}{6} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_1 = 1$ and the above becomes

$$\begin{pmatrix} -\frac{3}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & -\frac{5}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & -\frac{11}{6} \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We only need the first 2 equations. This results in

$$-\frac{3}{2} + \sqrt{\frac{3}{2}} v_2 + \sqrt{\frac{3}{4}} v_3 = 0$$
$$\sqrt{\frac{3}{2}} - \frac{5}{3} v_2 + \sqrt{\frac{1}{18}} v_3 = 0$$

From the first equation above

$$v_2 = \frac{\frac{3}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}} \tag{4B}$$

Substituting in the second equation gives

$$\begin{aligned} \sqrt{\frac{3}{2}} - \frac{5}{3} \left(\frac{\frac{3}{2} - \sqrt{\frac{3}{4}} v_3}{\sqrt{\frac{3}{2}}} \right) + \sqrt{\frac{1}{18}} v_3 &= 0 \\ \frac{5}{6} \sqrt{2} v_3 - \frac{1}{3} \sqrt{2} \sqrt{3} + \sqrt{\frac{1}{18}} v_3 &= 0 \\ \sqrt{2} v_3 - \frac{1}{3} \sqrt{2} \sqrt{3} &= 0 \\ v_3 &= \frac{\frac{1}{3} \sqrt{2} \sqrt{3}}{\sqrt{2}} \\ &= \frac{1}{3} \sqrt{3} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

Hence from (4B) $v_2 = \frac{\frac{3}{2} - \sqrt{\frac{3}{4}} \left(\frac{1}{\sqrt{3}} \right)}{\sqrt{\frac{3}{2}}} = \frac{1}{3} \sqrt{2} \sqrt{3} = \frac{\sqrt{2}}{\sqrt{3}}$. Therefore the eigenvector associated with

$\lambda_3 = 4$ is $\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ or by scaling it all by $\sqrt{3}$ it becomes

$$\vec{v}_3 = \begin{pmatrix} \sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}$$

<u>Therefore the final solution is</u>

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 4$$

And

$$\vec{v}_1 = \begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} \sqrt{3} \\ \sqrt{2} \\ 1 \end{pmatrix}$$

3 Problem 3

> **3.** (5 pts) Let U be a unitary matrix and let x_1 and x_2 be two eigenvectors of U with eigenvalues λ_1 and λ_2, respectively. Show that $|\lambda_1| = |\lambda_2| = 1$. Also show that if $\lambda_1 \neq \lambda_2$ then $x_1^\dagger x_2 = 0$.

Figure 3: Problem statement

A unitary matrix U means $U^{-1} = U^\dagger$. Let λ, x be the eigenvalue and the associated eigenvector. We also assume that the eigenvalue is not zero. Hence

$$Ux = \lambda x \tag{1}$$

Applying † operation (i.e. Transpose followed by complex conjugate) on the above gives

$$\begin{aligned}(Ux)^\dagger &= (\lambda x)^\dagger \\ x^\dagger U^\dagger &= x^\dagger \lambda^* \end{aligned} \tag{2}$$

Multiplying (2) by (1) gives

$$x^\dagger U^\dagger U x = x^\dagger \lambda^* \lambda x$$

But U is unitary, hence $U^\dagger = U^{-1}$ and the above becomes after replacing $\lambda^*\lambda$ by $|\lambda|^2$

$$\begin{aligned} x^\dagger U^{-1} U x &= |\lambda|^2 \left(x^\dagger x\right) \\ x^\dagger x &= |\lambda|^2 \left(x^\dagger x\right) \end{aligned}$$

Hence $|\lambda|^2 = 1$ or $|\lambda| = 1$ since this is a length, and so can not be negative. But since λ is an arbitrary eigenvalue, then any complex eigenvalue has absolute value of 1. Therefore

$$|\lambda_1| = |\lambda_2| = 1$$

Now we consider the specific case when $\lambda_1 \neq \lambda_2$ but we still require that $|\lambda_1| = 1$ and $|\lambda_2| = 1$ which was shown in first part above. We also assume for generality that the eigenvalues are not zero.

Given that

$$\begin{aligned} Ux_1 &= \lambda_1 x_1 \qquad (1) \\ Ux_2 &= \lambda_2 x_2 \qquad (2) \end{aligned}$$

From (1) we obtain

$$\begin{aligned}(Ux_1)^\dagger &= (\lambda_1 x_1)^\dagger \\ x_1^\dagger U^\dagger &= x_1^\dagger \lambda_1^* \end{aligned} \tag{3}$$

Multiplying (3) by (2) gives

$$\begin{aligned} x_1^\dagger U^\dagger U x_2 &= x_1^\dagger \lambda_1^* \lambda_2 x_2 \\ x_1^\dagger U^{-1} U x_2 &= \left(\lambda_1^* \lambda_2\right)\left(x_1^\dagger x_2\right) \\ x_1^\dagger x_2 &= \left(\lambda_1^* \lambda_2\right)\left(x_1^\dagger x_2\right) \end{aligned}$$

Since $|\lambda_1| = |\lambda_2| = 1$ but $\lambda_1 \neq \lambda_2$, therefore $\left(\lambda_1^* \lambda_2\right) \neq 1$. From the above this implies that $x_1^\dagger x_2 = 0$.

4 Problem 4

4. (3 pts) Calculate the determinant of the sparse matrix (sparse means that most of the entries are zero)

$$\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & i & 1 \end{pmatrix}$$

Figure 4: Problem statement

$$A = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & i & 1 \end{pmatrix}$$

We want to expand using a row or column which has most zeros in it since this leads to lots of cancellations and more efficient. Expanding using first row, then

$$\begin{aligned}
\det(A) &= 0 + i \det \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} + 0 + 0 + 0 \\
&= i \left(i \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix} \right) \\
&= i \left(i \left(3 \det \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) \right) \\
&= i \left(i \left(3 \left(1 - i^2 \right) \right) \right) \\
&= 3i^2 \left(1 - i^2 \right) \\
&= -3 (1 + 1) \\
&= -6
\end{aligned}$$

To verify this, we will now do expansion along the second row. To get the sign of a_{21} we use $(-1)^{2+1} = -1^3 = -1$. Hence

$$
\begin{aligned}
\det(A) &= -i \det \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} \\
&= -i \left(-i \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix} \right) \\
&= -i \left(-i \left(3 \det \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) \right) \\
&= -i \left(-i \left(3 \left(1 - i^2 \right) \right) \right) \\
&= 3i^2 \left(1 - i^2 \right) \\
&= -3 (1 + 1) \\
&= -6
\end{aligned}
$$

Which is the same as the expansion using the first row. Verified OK.