HW 7
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

1. (5 pts) Evaluate the integral

$$\int_0^{\pi} dx \int_1^2 dy\, \delta(\sin x)\delta(x^2 - y^2)$$

Figure 1: Problem statement

<u>Solution</u>

$$I = \int_0^{\pi} \left(\int_1^2 \delta(\sin(x))\, \delta\left(x^2 - y^2\right) dy \right) dx$$

Since $\delta(\sin(x))$ does not depend on y we can move it from the inner integral to the outside integral

$$I = \int_0^{\pi} \delta(\sin(x)) \left(\int_1^2 \delta\left(x^2 - y^2\right) dy \right) dx \tag{1}$$

Now we need to evaluate.

$$I_2 = \int_1^2 \delta\left(x^2 - y^2\right) dy$$

This is in the form of $\int_1^2 f(y)\, \delta(g(y))\, dy$ where now $f(y) = 1$ and $g(y) = x^2 - y^2$. Therefore the roots of $g(y)$ are $\pm x$. We see that x has to be in the range of $1 \cdots 2$, since that is where y is defined over. Hence the root $-x$ is outside this range and can not be used. So there is only one root which is $+x$. Now, using the result obtained from last HW which says

$$\int_1^2 f(y)\, \delta(g(y))\, du = \frac{f(y_0)}{\left|g'(y_0)\right|}$$

Therefore integral I_2 becomes

$$I_2 = \lim_{y \to y_0} \frac{f(y)}{\left|g'(y)\right|}$$

Where $y_0 = x$ is the root and where $g'(y) = -2y$ and where $f(y) = 1$. Hence the above becomes

$$I_2 = \frac{1}{2|x|} \left(\theta(x-1) - \theta(x-2)\right)$$

Where we added $(\theta(x-1) - \theta(x-2))$ to insure that x is $1 < x < 2$. Using this result in (1) gives (we do not need to write $|x|$ any more since $x > 0$)

$$\begin{aligned} I &= \int_0^{\pi} \frac{1}{2x} \left(\theta(x-1) - \theta(x-2)\right) \delta(\sin(x))\, dx \\ &= \int_1^2 \frac{1}{2x} \delta(\sin(x))\, dx \end{aligned}$$

Let $f(x) = \frac{1}{2x}, g(x) = \sin(x)$, then the above in the form

$$I = \int_1^2 f(x)\, \delta(g(x))\, dx = \sum_{x_0} \frac{f(x_0)}{\left|g'(x_0)\right|}$$

Where x_0 are the zero of $g(x) = \sin(x)$ inside the range $x = 1 \cdots 2$. But there are <u>no zeros</u> of $\sin(x)$ in this range. Therefore this leads to

$$I = 0$$

In other words

$$\int_0^{\pi} \left(\int_1^2 \delta\left(\sin\left(x\right)\right) \delta\left(x^2 - y^2\right) dy \right) dx = 0$$

2 Problem 2

> **2.** (5 pts) Consider the linear response formula
>
> $$x(t) = \int_{-\infty}^{\infty} G(t-t')F(t')dt'$$
>
> When the input is $F(t) = \mathrm{e}^{-\lambda t}\theta(t)$ the output is $x(t) = (1-\mathrm{e}^{-\alpha t})\mathrm{e}^{-\lambda t}$. What is $\tilde{G}(\omega)$? What is the output if $F(t) = F_0\delta(t)$?

Figure 2: Problem statement

Solution

2.1 Part (a)

Since

$$\tilde{G}(\omega) = \frac{\text{Fourier transform of output}}{\text{Fourier transform of input}} \tag{1}$$

Assuming causal system, then the output $x(t)$ is $x(t) = \left(1-e^{-\alpha t}\right)e^{-\lambda t}\theta(t)$. In other words, we added unit step $\theta(t)$ to indicate it also starts at $t=0$, since the input starts at $t=0$. Therefore the above definition becomes

$$\begin{aligned}
G(\omega) &== \frac{\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt}{\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt} \\
&= \frac{\int_{-\infty}^{\infty}\left(1-e^{-\alpha t}\right)e^{-\lambda t}\theta(t)e^{-i\omega t}dt}{\int_{-\infty}^{\infty} e^{-\lambda t}\theta(t)e^{-i\omega t}dt} \\
&= \frac{\int_{0}^{\infty}\left(1-e^{-\alpha t}\right)e^{-\lambda t}e^{-i\omega t}dt}{\int_{0}^{\infty} e^{-\lambda t}e^{-i\omega t}dt}
\end{aligned} \tag{2}$$

But

$$\begin{aligned}
\int_0^{\infty}\left(1-e^{-\alpha t}\right)e^{-\lambda t}e^{-i\omega t}dt &= \int_0^{\infty} e^{-\lambda t}e^{-i\omega t}dt - \int_0^{\infty} e^{-\alpha t}e^{-\lambda t}e^{-i\omega t}dt \\
&= \int_0^{\infty} e^{-t(\lambda+i\omega)}dt - \int_0^{\infty} e^{-t(\alpha+\lambda+i\omega)}dt \\
&= \left[\frac{e^{-t(\lambda+i\omega)}}{-(\lambda+i\omega)}\right]_0^{\infty} + \left[\frac{e^{-t(\alpha+\lambda+i\omega)}}{\alpha+\lambda+i\omega}\right]_0^{\infty} \\
&= \frac{-1}{(\lambda+i\omega)}\left[e^{-t(\lambda+i\omega)}\right]_0^{\infty} + \frac{1}{\alpha+\lambda+i\omega}\left[e^{-t(\alpha+\lambda+i\omega)}\right]_0^{\infty} \\
&= \frac{-1}{(\lambda+i\omega)}\left[e^{-t\lambda}e^{-it\omega}\right]_0^{\infty} + \frac{1}{\alpha+\lambda+i\omega}\left[e^{-t(\alpha+\lambda)}e^{-it\omega}\right]_0^{\infty}
\end{aligned}$$

With the assumptions[1] that $\lambda > 0, \alpha > 0$, then the above simplifies to

$$\begin{aligned}
\int_0^{\infty}\left(1-e^{-\alpha t}\right)e^{-\lambda t}e^{-i\omega t}dt &= \frac{-1}{(\lambda+i\omega)}[0-1] + \frac{1}{\alpha+\lambda+i\omega}[0-1] \\
&= \frac{1}{(\lambda+i\omega)} - \frac{1}{(\alpha+\lambda+i\omega)} \\
&= \frac{(\alpha+\lambda+i\omega)-(\lambda+i\omega)}{(\lambda+i\omega)(\alpha+\lambda+i\omega)} \\
&= \frac{\alpha}{(\lambda+i\omega)(\alpha+\lambda+i\omega)}
\end{aligned} \tag{3}$$

[1] So that input does not blow up with time, and its follows that output also decays with time, hence $\alpha > 0$

And

$$\begin{aligned}\int_0^\infty e^{-\lambda t}e^{-i\omega t}dt &= \int_0^\infty e^{-t(\lambda+i\omega)}dt\\ &= \left[\frac{e^{-t(\lambda+i\omega)}}{-(\lambda+i\omega)}\right]_0^\infty\\ &= \frac{-1}{(\lambda+i\omega)}\left[e^{-t(\lambda+i\omega)}\right]_0^\infty\end{aligned}$$

Since we assumed that $\lambda > 0$, then the above simplifies to

$$\begin{aligned}\int_0^\infty e^{-\lambda t}e^{-i\omega t}dt &= \frac{-1}{(\lambda+i\omega)}[0-1]\\ &= \frac{1}{(\lambda+i\omega)} \qquad (4)\end{aligned}$$

Substituting (3,4) into (2) gives the transfer function

$$\tilde{G}(\omega) = \frac{\frac{\alpha}{(\lambda+i\omega)(\alpha+\lambda+i\omega)}}{\frac{1}{(\lambda+i\omega)}}$$

Therefore

$$\boxed{\tilde{G}(\omega) = \frac{\alpha}{\alpha+\lambda+i\omega}}$$

2.2 Part (b)

If the input is $F(t) = F_0\delta(t)$ then the output is

$$\begin{aligned}x(t) &= \int_{-\infty}^{\infty} G(t-t')F_0\delta(t')\,dt\\ &= F_0G(t) \qquad (4A)\end{aligned}$$

Hence we just need to find $G(t)$ which is the inverse Fourier transform of $\tilde{G}(\omega)$ we found above.

$$\begin{aligned}G(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha}{\alpha+\lambda+i\omega}e^{i\omega t}d\omega\\ &= \frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\alpha+\lambda)+i\omega}d\omega\end{aligned}$$

To integrate the the above, we will use complex contour integration. Let $\omega = z$, hence the above becomes

$$G(t) = \frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\frac{e^{izt}}{(\alpha+\lambda)+iz}dz$$

Therefore $f(z) = \frac{e^{izt}}{(\alpha+\lambda)+iz}$. The pole is at $iz = -(\alpha+\lambda)$ or $z = i(\alpha+\lambda)$. Since $\alpha+\lambda > 0$, then the pole is in upper half plane. Lets find out where we will put the half circle, if it will go on the upper half or lower half. Since numerator is $e^{izt} = e^{i(x+iy)t} = e^{iz}e^{-yt}$ and therefore, since $t > 0$, then we want to choose the upper half circle, since there y is positive, which will cause the numerator to go to zero as $R \to \infty$. This implies there is one pole inside the upper half plane, we all what we need to do is find the residue at $z_0 = i(\alpha+\lambda)$.

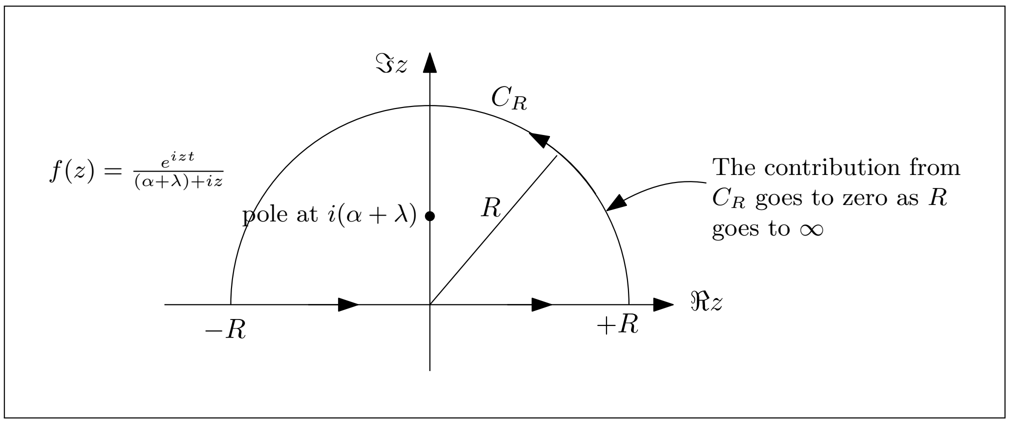

Figure 3: Contour integration used for finding inverse Fourier transform

Hence

$$\left(\frac{\alpha}{2\pi}\right)\int_{-\infty}^{\infty}\frac{e^{izt}}{(\alpha+\lambda)+iz}dz = \left(\frac{\alpha}{2\pi}\right)2\pi i\sum \text{Residue} \tag{5}$$

But, since $z_0 = i(\alpha+\lambda)$, then

$$\begin{aligned}
\text{Residue}(z_0) &= \lim_{z\to z_0}(z-z_0)f(z) \\
&= \lim_{z\to i(\alpha+\lambda)}(z-i(\alpha+\lambda))\frac{e^{izt}}{(\alpha+\lambda)+iz} \\
&= \left(\lim_{z\to i(\alpha+\lambda)} e^{izt}\right)\left(\lim_{z\to i(\alpha+\lambda)}\frac{z-i(\alpha+\lambda)}{(\alpha+\lambda)+iz}\right)
\end{aligned}$$

Applying L'Hopitals gives

$$\begin{aligned}
\text{Residue}(z_0) &= \left(\lim_{z\to i(\alpha+\lambda)} e^{izt}\right)\left(\lim_{z\to i(\alpha+\lambda)}\frac{1}{i}\right) \\
&= -i\lim_{z\to i(\alpha+\lambda)} e^{izt} \\
&= -ie^{-(\alpha+\lambda)t}
\end{aligned}$$

Now that we found the residue, then from (5)

$$\begin{aligned}
\left(\frac{\alpha}{2\pi}\right)\int_{-\infty}^{\infty}\frac{e^{izt}}{(\alpha+\lambda)+iz}dz &= \left(\frac{\alpha}{2\pi}\right)2\pi i\left(-ie^{-(\alpha+\lambda)t}\right) \\
&= \alpha e^{-(\alpha+\lambda)t}
\end{aligned}$$

We have found $G(t)$

$$G(t) = \alpha e^{-(\alpha+\lambda)t} \qquad t>0$$

From (4A), the response is

$$\begin{aligned}
x(t) &= F_0 G(t) \\
&= \alpha F_0 e^{-(\alpha+\lambda)t}\theta(t)
\end{aligned}$$

3 Problem 3

3. (5 pts) By using the integral representation

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)\, d\theta$$

find the Laplace transform of J_0.

Figure 4: Problem statement

Solution

Using

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)\, d\theta$$

Hence Laplace transform is

$$\begin{aligned}\hat{J}_0(s) &= \int_0^{\infty} J_0(x)\, e^{-sx} dx \\ &= \frac{1}{2\pi}\int_0^{\infty}\left(\int_0^{2\pi}\cos(x\cos\theta)\, d\theta\right) e^{-sx} dx\end{aligned}$$

Changing order of integration

$$\hat{J}_0(s) = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\infty}\cos(x\cos\theta)\, e^{-sx} dx\right) d\theta \tag{1}$$

Let $I = \int_0^{\infty} \cos(x\cos\theta)\, e^{-sx} dx$. This is solved by applying integration by parts twice

Let $u = \cos(x\cos\theta)\,, dv = e^{-sx}$, hence $du = -\cos\theta\sin(x\cos\theta)\,, v = -\frac{e^{-sx}}{s}$. Therefore

$$\begin{aligned}I &= [uv]_0^{\infty} - \int_0^{\infty} v du \\ &= -\frac{1}{s}\left[\cos(x\cos\theta)\, e^{-sx}\right]_0^{\infty} - \frac{1}{s}\cos\theta\int_0^{\infty} e^{-sx}\sin(x\cos\theta)\, dx \\ &= -\frac{1}{s}[0-1] - \frac{\cos\theta}{s}\int_0^{\infty} e^{-sx}\sin(x\cos\theta)\, dx \\ &= \frac{1}{s} - \frac{\cos\theta}{s}\int_0^{\infty} e^{-sx}\sin(x\cos\theta)\, dx\end{aligned}$$

Integration by parts again, let $\sin(x\cos\theta) = u, du = \cos\theta\cos(x\cos\theta)\,, dv = e^{-sx}, v = -\frac{e^{-sx}}{s}$, and the above becomes

$$\begin{aligned}I &= \frac{1}{s} - \frac{\cos\theta}{s}\left([uv]_0^{\infty} - \int_0^{\infty} v du\right) \\ &= \frac{1}{s} - \frac{\cos\theta}{s}\left(-\frac{1}{s}\left[\sin(x\cos\theta)\, e^{-sx}\right]_0^{\infty} + \frac{\cos\theta}{s}\int_0^{\infty}\cos(x\cos\theta)\, e^{-sx} du\right) \\ &= \frac{1}{s} - \frac{\cos\theta}{s}\left(-\frac{1}{s}[0] + \frac{\cos\theta}{s} I\right) \\ &= \frac{1}{s} - \frac{\cos\theta}{s}\left(\frac{\cos\theta}{s} I\right) \\ &= \frac{1}{s} - \frac{\cos^2\theta}{s^2} I\end{aligned}$$

Solving for I gives

$$\begin{aligned} I+\frac{\cos^2\theta}{s^2}I &= \frac{1}{s}\\ I\left(1+\frac{\cos^2\theta}{s^2}\right) &= \frac{1}{s}\\ I\left(\frac{s^2+\cos^2\theta}{s^2}\right) &= \frac{1}{s}\\ I\left(\frac{s^2+\cos^2\theta}{s}\right) &= 1\\ I &= \frac{s}{s^2+\cos^2\theta} \end{aligned}$$

Therefore

$$\int_0^\infty \cos\left(x\cos\theta\right)e^{-sx}dx = \frac{s}{s^2+\cos^2\left(\theta\right)} \tag{2}$$

Substituting (2) in (1) gives

$$\hat{J}_0\left(s\right)=\frac{1}{2\pi}\int_0^{2\pi}\frac{s}{s^2+\cos^2\left(\theta\right)}d\theta$$

Since the above is an even function, we can rewrite as

$$\hat{J}_0\left(s\right)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{s}{s^2+\cos^2\left(\theta\right)}d\theta$$

The above can be solved using contour integration or using standard method of integration using substitution which I think is simpler here.

Multiplying numerator and denominator of $\hat{J}_0\left(s\right)$ above by $\sec^2\left(\theta\right)$ gives

$$\hat{J}_0\left(s\right)=\frac{2s}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sec^2\left(\theta\right)}{s^2\sec^2\left(\theta\right)+1}d\theta$$

Let $u=\tan\left(\theta\right)$. When $\theta=0, u=0$ and when $\theta=\frac{\pi}{2}, u=\infty$. Since $du=d\theta\sec^2\left(\theta\right)$. Hence the above integral becomes, since $\sec^2\left(\theta\right)=1+\tan^2\left(\theta\right)=1+u^2$

$$\hat{J}_0\left(s\right)=\frac{2s}{\pi}\int_0^{\infty}\frac{1}{s^2\sec^2\left(\theta\right)+1}du$$

But $\sec^2\left(\theta\right)=1+\tan^2\left(\theta\right)=1+u^2$ therefore the above becomes

$$\begin{aligned} \hat{J}_0\left(s\right) &= \frac{2s}{\pi}\int_0^{\infty}\frac{1}{s^2\left(1+u^2\right)+1}du\\ &= \frac{2s}{\pi}\int_0^{\infty}\frac{1}{\left(1+s^2\right)+s^2u^2}du\\ &= \frac{2s}{\pi}\int_0^{\infty}\frac{1}{s^2}\frac{1}{\left(\frac{1+s^2}{s^2}\right)+u^2}du\\ &= \frac{2s}{\pi}\int_0^{\infty}\frac{1}{\left(\frac{1+s^2}{s^2}\right)+u^2}du \end{aligned}$$

Let $\left(\frac{1+s^2}{s^2}\right)=A$, so the integral in the form $\int\frac{1}{A+u^2}du=\frac{1}{\sqrt{A}}\arctan\left(\frac{u}{\sqrt{A}}\right)$, hence the above becomes

$$\begin{aligned}
\hat{J}_0(s) &= \frac{2s}{\pi}\left[\frac{1}{\sqrt{\frac{1+s^2}{s^2}}}\arctan\left(\frac{u}{\sqrt{\frac{1+s^2}{s^2}}}\right)\right]_0^\infty \\
&= \frac{2}{\sqrt{1+s^2}}\frac{1}{\pi}\left[\arctan\left(\frac{s}{\sqrt{1+s^2}}u\right)\right]_0^\infty \\
&= \frac{2}{\sqrt{1+s^2}}\frac{1}{\pi}\left[\frac{\pi}{2}-0\right] \\
&= \frac{1}{\sqrt{1+s^2}}
\end{aligned}$$

3.1 Appendix

This part contains attempt made using contour integration. For reference and not for grading.

Solve

$$\hat{J}_0(s) = \frac{1}{2\pi}\int_0^{2\pi}\frac{s}{s^2+\cos^2(\theta)}d\theta$$

Let $z = e^{i\theta}$, then $dz = izd\theta$, and $\cos(\theta) = \frac{z+z^{-1}}{2}$, hence the above integral becomes

$$\begin{aligned}
\hat{J}_0(s) &= \frac{1}{2\pi}\oint\frac{s}{s^2+\left(\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{iz} \\
&= \frac{1}{2\pi}\oint\frac{4s}{4s^2+\left(z+\frac{1}{z}\right)^2}\frac{dz}{iz} \\
&= -\frac{i4s}{2\pi}\oint\frac{1}{4s^2+\left(z+\frac{1}{z}\right)^2}\frac{dz}{z} \\
&= -\frac{i4s}{2\pi}\oint\frac{1}{4s^2+\left(\frac{z^2+1}{z}\right)^2}\frac{dz}{z} \\
&= -\frac{i4s}{2\pi}\oint\frac{z}{4s^2z^2+\left(z^2+1\right)^2}dz
\end{aligned}$$

Did not complete.

Alternative solution

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\,d\theta$$

Let $\cos\theta = u$, hence $\frac{du}{d\theta} = -\sin\theta$. But $\cos^2\theta+\sin^2\theta = 1$, therefore $\sin^2\theta = 1-\cos^2\theta = 1-u^2$. Hence $\sin\theta = \sqrt{1-u^2}$. When $\theta = 0, u = 1$ and when $\theta = \pi, u = -1$, therefore the above integral now can be written as

$$\begin{aligned}
J_0(x) &= \frac{1}{\pi}\int_1^{-1}\cos(xu)\left(\frac{-du}{\sqrt{1-u^2}}\right) \\
&= \frac{1}{\pi}\int_{-1}^{1}\cos(xu)\frac{du}{\sqrt{1-u^2}}
\end{aligned}$$

Since the integrand is even, then the above becomes

$$J_0(x) = \frac{2}{\pi}\int_0^1\cos(xu)\frac{du}{\sqrt{1-u^2}}$$

And the above is what will be used as starting point. I could not solve this using complex contour integration, which is probably would have been easier if I knew how to do it, but instead solved it using substitution as follows.

Changing the argument from x to α gives

$$J_0(\alpha) = \frac{2}{\pi}\int_0^1 \cos(\alpha u)\frac{du}{\sqrt{1-u^2}}$$

u is arbitrary inside the integral so we can rename it back to x and the above becomes

$$J_0(\alpha) = \frac{2}{\pi}\int_0^1 \cos(\alpha x)\frac{dx}{\sqrt{1-x^2}}$$

Which is the same as (by renaming the argument again, since it better to use t with Laplace by convention, just for notation sake)

$$J_0(\alpha t) = \frac{2}{\pi}\int_0^1 \cos(\alpha t x)\frac{dx}{\sqrt{1-x^2}}$$

Now, the Laplace transform of $J_0(\alpha t)$ is

$$\begin{aligned}
\hat{J}_0(s) &= \int_0^\infty J_0(\alpha t)\, e^{-st} dt \\
&= \int_0^\infty \left(\frac{2}{\pi}\int_0^1 \cos(\alpha t x)\frac{dx}{\sqrt{1-x^2}}\right) e^{-st} dt \\
&= \frac{2}{\pi}\int_0^\infty \left(\int_0^1 \cos(\alpha t x)\frac{dx}{\sqrt{1-x^2}}\right) e^{-st} dt
\end{aligned}$$

Changing order of integration gives

$$\hat{J}_0(s) = \frac{2}{\pi}\int_0^1 \left(\int_0^\infty \cos(\alpha t x)\, e^{-st} dt\right)\frac{1}{\sqrt{1-x^2}} dx$$

But $\int_0^\infty \cos(\alpha t x)\, e^{-st} dt$ is the Laplace transform of $\cos(\alpha t x)$ which is from tables $\frac{s}{s^2+a^2x^2}$. Hence the above simplifies to

$$\begin{aligned}
\hat{J}_0(s) &= \frac{2}{\pi}\int_0^1 \frac{s}{s^2+\alpha^2 x^2}\frac{1}{\sqrt{1-x^2}} dx \\
&= \frac{2s}{\pi}\int_0^1 \frac{1}{\left(s^2+\alpha^2x^2\right)\sqrt{1-x^2}} dx \\
&= \frac{2s}{\pi}\frac{\alpha\frac{\pi}{s}}{2\alpha\sqrt{\alpha^2+s^2}} \\
&= \frac{1}{\sqrt{\alpha^2+s^2}}
\end{aligned}$$

But we did the Laplace transform of $J_0(\alpha t)$, which is the same as $J_0(\alpha x)$ and to get Laplace transform of $J_0(x)$, we just need to set $\alpha = 1$ in the above result, which gives

$$\hat{J}_0(s) = \frac{1}{\sqrt{1+s^2}}$$

4 Problem 4

4. (5 pts) A reasonably accurate description of the atomic contribution to the dielectric function is

$$\epsilon(\omega) = 1 + \omega_P^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega}$$

There are f_j electrons per molecule with binding frequency ω_j and damping constant γ_j. The oscillator strengths f_j obey the sum rule $\sum_j f_j = Z$ which is the total number of electrons per molecule. Using the imaginary part of ϵ in the dispersion relation, show that the real part is correctly reproduced.

Figure 5: Problem statement

Solution

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega}$$

It is enough to work with one term in the sum above and verify what is being asked on that term. Then it will be valid for the sum. Hence we will use the following as the starting relation

$$\begin{aligned} \epsilon(\omega) &= 1 + \frac{\omega_p^2 f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega} \qquad j = 1, 2, 3, \cdots \\ &= 1 - \frac{\omega_p^2 f_j}{\left(\omega^2 - \omega_j^2\right) - 2i\gamma_j\omega} \end{aligned} \tag{1}$$

It is assumed that γ is much smaller than ω. In the above ω is the variable quantity and $\omega_j, \omega_p, \gamma_j$ are given parameters with known values for the problem

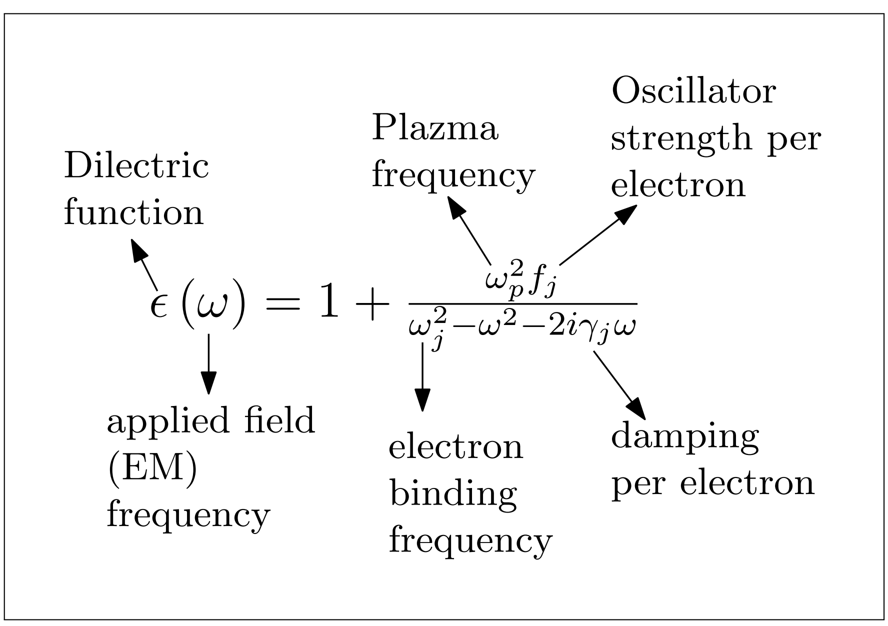

Figure 6: Physical meaning of terms involved

The real and imaginary parts are found by multiplying numerator and denominator by complex conjugate of denominator

$$\begin{aligned}
\epsilon(\omega) &= 1-\frac{\omega_p^2 f_j}{\left(\left(\omega^2-\omega_j^2\right)-2i\gamma_j\omega\right)}\frac{\left(\omega^2-\omega_j^2\right)+2i\gamma_j\omega}{\left(\left(\omega^2-\omega_j^2\right)+2i\gamma_j\omega\right)}\\
&= 1-\frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)+2i\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2-\left(2i\gamma_j\omega\right)^2}\\
&= 1-\frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)+2i\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2+4\gamma_j^2\omega^2}\\
&= 1-\frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)}{\left(\omega^2-\omega_j^2\right)^2+4\gamma_j^2\omega^2}-\frac{2i\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2+4\gamma_j^2\omega^2}\\
&= \left(1-\frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)}{\left(\omega^2-\omega_j^2\right)^2+4\gamma_j^2\omega^2}\right)-i\left(\frac{2\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2+4\gamma^2\omega^2}\right)
\end{aligned}$$

Hence we see that

$$\operatorname{Re}(\epsilon(\omega)) = 1-\frac{\omega_p^2 f_j\left(\omega^2-\omega_j^2\right)}{\left(\omega^2-\omega_j^2\right)^2+4\gamma_j^2\omega^2}\qquad j=1,2,3,\cdots \tag{1}$$

$$\operatorname{Im}(\epsilon(\omega)) = -\frac{2\gamma_j\omega\omega_p^2 f_j}{\left(\omega^2-\omega_j^2\right)^2+4\gamma^2\omega^2}\qquad j=1,2,3,\cdots \tag{2}$$

Now, the dispersion relations for the above are, as derived in class notes

$$\operatorname{Re}(\epsilon(\omega)) = 1+\frac{1}{\pi}(P.V.)\int_{-\infty}^{\infty}\frac{\operatorname{Im}(\epsilon(\omega'))}{\omega'-\omega}d\omega' \tag{3}$$

$$\operatorname{Im}(\epsilon(\omega)) = -\frac{1}{\pi}(P.V.)\int_{-\infty}^{\infty}\frac{\operatorname{Re}(\epsilon(\omega'))}{\omega'-\omega}d\omega' \tag{4}$$

The question is asking to use (2) in (3) in order to obtain and verify (1).

Substituting (2) into (3) gives

$$\begin{aligned}
\operatorname{Re}(\epsilon(\omega)) &= 1-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega'-\omega}\overbrace{\left(\frac{2\omega_p^2\gamma_j\omega' f_j}{\left((\omega')^2-\omega_j^2\right)^2+4\gamma^2(\omega')^2}\right)}^{\operatorname{Im}(\epsilon(\omega'))}d\omega'\\
&= 1-\frac{2\gamma\omega_p^2 f_j}{\pi}\int_{-\infty}^{\infty}\frac{1}{(\omega'-\omega)}\left(\frac{\omega'}{\left((\omega')^2-\omega_j^2\right)^2+4\gamma^2(\omega')^2}\right)d\omega' \qquad (5)
\end{aligned}$$

To find the poles in (5), it is easier to start from the original function

$$\frac{\omega_p^2 f_j}{\omega^2-2i\gamma_j\omega-\omega_j^2}$$

The roots of the denominator are $r_{1,2}=\frac{-b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}=\frac{2i\gamma_j}{2}\pm\frac{1}{2}\sqrt{\left(-2i\gamma_j\right)^2+4\omega_j^2}=i\gamma_j\pm\frac{1}{2}\sqrt{-4\gamma_j^2+4\omega_j^2}=i\gamma_j\pm\sqrt{\omega_j^2-\gamma_j^2}$. Hence after multiplying by the complex conjugate as we did above, we obtain the new term which is $\omega^2+2i\gamma_j\omega-\omega_j^2$. This one has roots $r_{3,4}=\frac{-2i\gamma_j}{2}\pm\frac{1}{2}\sqrt{\left(2i\gamma_j\right)^2+4\omega_j^2}=-i\gamma_j\pm\sqrt{\omega_j^2-\gamma_j^2}$. Therefore, we see that the poles for the term $\frac{\omega'}{\left((\omega')^2-\omega_j^2\right)^2+4\gamma^2(\omega')^2}$ are

$$\begin{aligned}
r_1 &= i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} \\
r_2 &= i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} \\
r_3 &= -i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} \\
r_4 &= -i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}
\end{aligned}$$

We now need to handle the term $\frac{1}{(\omega'-\omega)}$ in (5) in order to find all the poles. To do this, we use

$$\begin{aligned}
\frac{1}{\omega' - \omega - i\Delta} &= \frac{1}{\omega' - \omega} + i\pi\delta\left(\omega' - \omega\right) \\
\frac{1}{\omega' - \omega + i\Delta} &= \frac{1}{\omega' - \omega} - i\pi\delta\left(\omega' - \omega\right)
\end{aligned}$$

Where Δ is very small quantity. Adding the above two equations gives

$$\begin{aligned}
\frac{1}{\omega' - \omega - i\Delta} + \frac{1}{\omega' - \omega + i\Delta} &= \frac{2}{\omega' - \omega} \\
\frac{1}{\omega' - \omega} &= \frac{1}{2}\left(\frac{1}{\omega' - (\omega + i\Delta)} + \frac{1}{\omega' - (\omega - i\Delta)}\right)
\end{aligned}$$

Where in the above final steps we let $\Delta^n \to 0$ for $n > 1$ since Δ is very small. The above is what we will use in (6). Hence (5) becomes

$$\begin{aligned}
\operatorname{Re}\left(\epsilon\left(\omega\right)\right) &= 1 - \frac{\gamma\omega_p^2 f_j}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\omega' - (\omega + i\Delta)} + \frac{1}{\omega' - (\omega - i\Delta)}\right)\left(\frac{\omega'}{(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}\right)d\omega' \\
&= 1 - \frac{\gamma\omega_p^2 f_j}{2\pi}\int_{-\infty}^{\infty}\frac{\omega' - (\omega - i\Delta) + \omega' - (\omega + i\Delta)}{(\omega' - (\omega + i\Delta))(\omega' - (\omega - i\Delta))}\left(\frac{\omega'}{(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}\right)d\omega' \\
&= 1 - \frac{\gamma\omega_p^2 f_j}{2\pi}\int_{-\infty}^{\infty}\frac{(2\omega' - 2\omega)}{(\omega' - (\omega + i\Delta))(\omega' - (\omega - i\Delta))}\left(\frac{\omega'}{(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}\right)d\omega' \\
&= 1 - \frac{\gamma\omega_p^2 f_j}{\pi}\int_{-\infty}^{\infty}\frac{(\omega')^2 - \omega\omega'}{(\omega' - r_5)(\omega' - r_6)(\omega' - r_1)(\omega' - r_2)(\omega' - r_3)(\omega' - r_4)}d\omega'
\end{aligned} \tag{5A}$$

There are 6 poles in total

$$\begin{aligned}
r_1 &= i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} \\
r_2 &= i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} \\
r_3 &= -i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} \\
r_4 &= -i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} \\
r_5 &= \omega + i\Delta \\
r_6 &= \omega - i\Delta
\end{aligned}$$

Three of the above poles are in lower half plane, and three are in the upper half plane. Here is a diagram which shows the location of the poles. Recalling that Δ is small quantity.

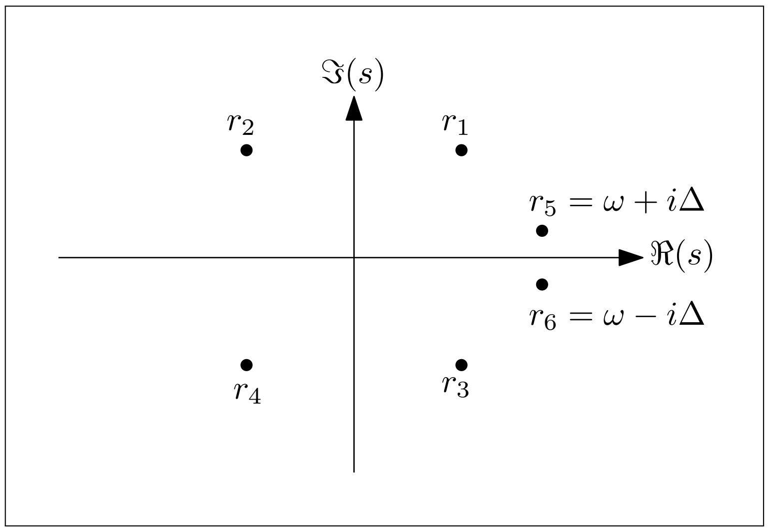

Figure 7: Location of the 6 poles

We will use the following contour

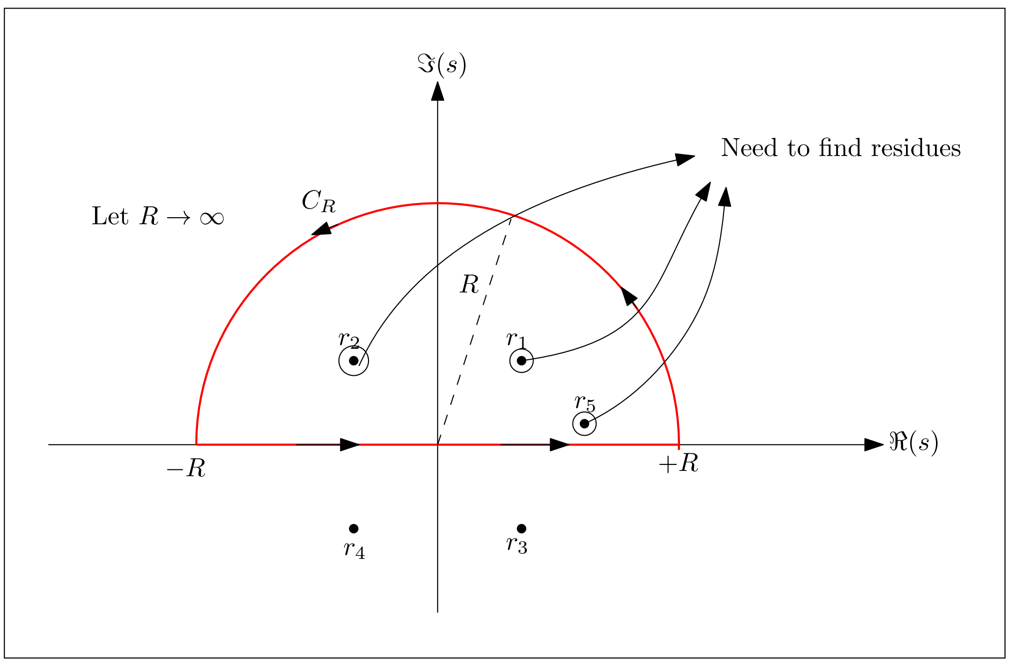

Figure 8: Countour used for the integral

The integrand which is a function of ω' is analytic except for the 3 poles in the upper half. Let the integrand be $g(\omega')$, then using residue theorem gives

$$\begin{aligned}\lim_{R\to\infty} \oint g(z)\,dz &= \lim_{R\to\infty} (P.V.) \int_{-R}^{R} g(\omega')\,d\omega' + \lim_{R\to\infty} \int_{C_R} g(z)\,dz \\ &= 2\pi i \sum \text{Residue}\end{aligned}$$

Hence

$$\lim_{R\to\infty} (P.V.) \int_{-R}^{R} g(\omega')\,d\omega' = 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} g(z)\,dz$$

Since the denominator in (5A) has higher powers of ω' than in the numerator (6^{th} order vs. 2^{nd} order), then this shows that $\lim_{R\to\infty} \int_{C_R} g(z)\,dz \to 0$, and the above reduces to

$$(P.V.) \int_{-\infty}^{\infty} g(\omega')\,d\omega' = 2\pi i \sum \text{Residue} \tag{8}$$

Therefore we just need to find the three residues at r_1, r_2, r_5 in order to find the integral above.

$$\begin{aligned}
\text{Residue}\,(r_1) &= \lim_{\omega'\to r_1} (\omega' - r_1)\,\frac{(\omega')^2 - \omega\omega'}{(\omega' - (\omega + i\Delta))\,(\omega' - (\omega - i\Delta))\,(\omega' - r_1)\,(\omega' - r_2)\,(\omega' - r_3)\,(\omega' - r_4)} \\
&= \lim_{\omega'\to r_1} \frac{(\omega')^2 - \omega\omega'}{(\omega' - (\omega + i\Delta))\,(\omega' - (\omega - i\Delta))\,(\omega' - r_2)\,(\omega' - r_3)\,(\omega' - r_4)} \\
&= \frac{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - r_2\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - r_3\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - r_4\right)} \\
&= \frac{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - \left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - \left(-i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - \left(-i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)} \\
&= \frac{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} + i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} + i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}
\end{aligned}$$

Hence

$$\text{Residue}\,(r_1) = \frac{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j\right)\left(2i\gamma_j + 2\sqrt{\omega_j^2 - \gamma_j^2}\right)} \tag{9}$$

And

$$\begin{aligned}
\text{Residue}\,(r_2) &= \lim_{\omega'\to r_2} (\omega' - r_2)\,\frac{(\omega')^2 - \omega\omega'}{(\omega' - (\omega + i\Delta))\,(\omega' - (\omega - i\Delta))\,(\omega' - r_1)\,(\omega' - r_2)\,(\omega' - r_3)\,(\omega' - r_4)} \\
&= \lim_{\omega'\to r_2} \frac{(\omega')^2 - \omega\omega'}{(\omega' - (\omega + i\Delta))\,(\omega' - (\omega - i\Delta))\,(\omega' - r_1)\,(\omega' - r_3)\,(\omega' - r_4)} \\
&= \frac{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - r_1\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - r_3\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - r_4\right)} \\
&= \frac{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - \left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - \left(-i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - \left(-i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)} \\
&= \frac{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} + i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} + i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}
\end{aligned}$$

Hence

$$\text{Residue}\,(r_2) = \frac{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(-2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j - 2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j\right)} \tag{10}$$

And finally

$$\begin{aligned}
\text{Residue}\,(r_5) &= \lim_{\omega'\to r_5} (\omega' - r_5)\,\frac{(\omega')^2 - \omega\omega'}{(\omega' - r_5)\,(\omega' - r_6)\,(\omega' - r_1)\,(\omega' - r_2)\,(\omega' - r_3)\,(\omega' - r_4)} \\
&= \lim_{\omega'\to r_5} \frac{(\omega')^2 - \omega\omega'}{(\omega' - r_6)\,(\omega' - r_1)\,(\omega' - r_2)\,(\omega' - r_3)\,(\omega' - r_4)} \\
&= \frac{(\omega + i\Delta)^2 - \omega\,(\omega + i\Delta)}{\left(\omega + i\Delta - \left(\omega_j - i\Delta\right)\right)(\omega + i\Delta - r_1)\,(\omega + i\Delta - r_2)\,(\omega + i\Delta - r_3)\,(\omega + i\Delta - r_4)} \\
&= \frac{\omega^2 - \Delta^2 + 2i\omega\Delta - \omega^2 - i\Delta\omega}{(2i\Delta)\left(\omega + i\Delta - \left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(\omega + i\Delta - \left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(\omega + i\Delta - \left(-i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\right)\left(\omega + i\Delta - \left(-i\gamma_j - \right.\right.} \\
&= \frac{i\omega\Delta - \Delta^2}{(2i\Delta)\left(\omega + i\Delta - i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta - i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right.}
\end{aligned} \tag{11}$$

We found all residues for I. Hence

$$\int_{-\infty}^{\infty} \frac{(\omega')^2 - \omega\omega'}{(\omega' - r_5)\,(\omega' - r_6)\,(\omega' - r_1)\,(\omega' - r_2)\,(\omega' - r_3)\,(\omega' - r_4)}\,d\omega' = 2\pi i \sum \text{Residue}$$

Where $\sum \text{Residue}$ is given by adding (9,10,11) giving

$$\sum \text{Residue} = \frac{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j\right)\left(2i\gamma_j + 2\sqrt{\omega_j^2 - \gamma_j^2}\right)}$$
$$+ \frac{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)^2 - \omega\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)}{\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega + i\Delta)\right)\left(i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2} - (\omega - i\Delta)\right)\left(-2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j - 2\sqrt{\omega_j^2 - \gamma_j^2}\right)\left(2i\gamma_j\right)}$$
$$+ \frac{i\omega\Delta - \Delta^2}{(2i\Delta)\left(\omega + i\Delta - i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta - i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j - \sqrt{\omega_j^2 - \gamma_j^2}\right)\left(\omega + i\Delta + i\gamma_j + \sqrt{\omega_j^2 - \gamma_j^2}\right)}$$

Therefore (5A) becomes

$$\text{Re}\left(\epsilon(\omega)\right) = 1 - \frac{\gamma\omega_p^2 f_j}{\pi}(2\pi i)\sum \text{Residue}$$

To make some progress, I had to simplify the $\sum \text{Residue}$ by assuming γ is very small compared to ω_j and hence terms such as $\sqrt{\omega_j^2 - \gamma_j^2} \to \omega_j$. Using this gives

$$\sum \text{Residue} = \frac{\left(i\gamma_j + \omega_j\right)^2 - \omega\left(i\gamma_j + \omega_j\right)}{\left(i\gamma_j + \omega_j - (\omega + i\Delta)\right)\left(i\gamma_j + \omega_j - (\omega - i\Delta)\right)\left(2\omega_j\right)\left(2i\gamma_j\right)\left(2i\gamma_j + 2\omega_j\right)}$$
$$+ \frac{\left(i\gamma_j - \omega_j\right)^2 - \omega\left(i\gamma_j - \omega_j\right)}{\left(i\gamma_j - \omega_j - (\omega + i\Delta)\right)\left(i\gamma_j - \omega_j - (\omega - i\Delta)\right)\left(-2\omega_j\right)\left(2i\gamma_j - 2\omega_j\right)\left(2i\gamma_j\right)}$$
$$+ \frac{i\omega\Delta - \Delta^2}{(2i\Delta)\left(\omega + i\Delta - i\gamma_j - \omega_j\right)\left(\omega + i\Delta - i\gamma_j + \omega_j\right)\left(\omega + i\Delta + i\gamma_j - \omega_j\right)\left(\omega + i\Delta + i\gamma_j + \omega_j\right)}$$

Or

$$\sum \text{Residue} = \frac{-\gamma_j^2 + \omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j - 2\omega\omega_j - \gamma_j^2 + 2i\gamma_j\omega_j + \omega_j^2\right)\left(8i\gamma_j\omega_j^2 - 8\gamma_j^2\omega_j\right)}$$
$$+ \frac{-\gamma_j^2 + \omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j + 2\omega\omega_j - \gamma_j^2 - 2i\gamma_j\omega_j + \omega_j^2\right)\left(8\gamma_j^2\omega_j + 8i\gamma_j\omega_j^2\right)}$$
$$+ \frac{i\omega\Delta - \Delta^2}{(2i\Delta)\left(\omega + i\Delta - i\gamma_j - \omega_j\right)\left(\omega + i\Delta - i\gamma_j + \omega_j\right)\left(\omega + i\Delta + i\gamma_j - \omega_j\right)\left(\omega + i\Delta + i\gamma_j + \omega_j\right)}$$

Expanding the denominator in 3rd term above, lots of terms cancel since they contain higher powers of Δ. Removing all terms that contain Δ^2 or higher gives

$$\sum \text{Residue} = \frac{-\gamma_j^2 + \omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j - 2\omega\omega_j - \gamma_j^2 + 2i\gamma_j\omega_j + \omega_j^2\right)\left(8i\gamma_j\omega_j^2 - 8\gamma_j^2\omega_j\right)}$$
$$+ \frac{-\gamma_j^2 + \omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j + 2\omega\omega_j - \gamma_j^2 - 2i\gamma_j\omega_j + \omega_j^2\right)\left(8\gamma_j^2\omega_j + 8i\gamma_j\omega_j^2\right)}$$
$$+ \frac{i\omega\Delta}{2i\Delta\omega^4 + 4i\Delta\omega^2\gamma_j^2 - 4i\Delta\omega^2\omega_j^2 + 2i\Delta\gamma_j^4 + 4i\Delta\gamma_j^2\omega_j^2 + 2i\Delta\omega_j^4}$$

Removing terms that contain only γ_j^2 since γ_j is small gives

$$\sum \text{Residue} = \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j - 2\omega\omega_j + 2i\gamma_j\omega_j + \omega_j^2\right)\left(8i\gamma_j\omega_j^2 - 8\gamma_j^2\omega_j\right)}$$
$$+ \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j + 2\omega\omega_j - 2i\gamma_j\omega_j + \omega_j^2\right)\left(8\gamma_j^2\omega_j + 8i\gamma_j\omega_j^2\right)}$$
$$+ \frac{\omega\Delta}{2\Delta\omega^4 + 4\Delta\omega^2\gamma_j^2 - 4\Delta\omega^2\omega_j^2 + 2\Delta\gamma_j^4 + 4\Delta\gamma_j^2\omega_j^2 + 2\Delta\omega_j^4}$$

Canceling all terms with $\Delta\gamma_j^2, \Delta\gamma_j^4$ in them, since both are small gives

$$\sum \text{Residue} = \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j - 2\omega\omega_j + 2i\gamma_j\omega_j + \omega_j^2\right)\left(8i\gamma_j\omega_j^2 - 8\gamma_j^2\omega_j\right)} + \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j + 2\omega\omega_j - 2i\gamma_j\omega_j + \omega_j^2\right)\left(8\gamma_j^2\omega_j + 8i\gamma_j\omega_j^2\right)} + \frac{\omega\Delta}{2\Delta\omega^4 + 4\Delta\omega^2\gamma_j^2 - 4\Delta\omega^2\omega_j^2 + 2\Delta\omega_j^4}$$

Canceling Δ in last term gives

$$\sum \text{Residue} = \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j - 2\omega\omega_j + 2i\gamma_j\omega_j + \omega_j^2\right)\left(8i\gamma_j\omega_j^2 - 8\gamma_j^2\omega_j\right)} + \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{\left(\omega^2 - 2i\omega\gamma_j + 2\omega\omega_j - 2i\gamma_j\omega_j + \omega_j^2\right)\left(8\gamma_j^2\omega_j + 8i\gamma_j\omega_j^2\right)} + \frac{\omega}{2\omega^4 + 4\omega^2\gamma_j^2 - 4\omega^2\omega_j^2 + 2\omega_j^4}$$

Expanding

$$\sum \text{Residue} = \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{-8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 + 16i\omega\gamma_j^3\omega_j + 32\omega\gamma_j^2\omega_j^2 - 16i\omega\gamma_j\omega_j^3 - 16i\gamma_j^3\omega_j^2 - 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} + \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 - 16i\omega\gamma_j^3\omega_j + 32\omega\gamma_j^2\omega_j^2 + 16i\omega\gamma_j\omega_j^3 - 16i\gamma_j^3\omega_j^2 + 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} + \frac{\omega}{2\omega^4 + 4\omega^2\gamma_j^2 - 4\omega^2\omega_j^2 + 2\omega_j^4}$$

Removing terms with γ_j^3 and higher, since γ is small gives

$$\sum \text{Residue} = \frac{\omega_j^2 + 2i\gamma_j\omega_j - \omega i\gamma_j - \omega\omega_j}{-8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 + 32\omega\gamma_j^2\omega_j^2 - 16i\omega\gamma_j\omega_j^3 - 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} + \frac{\omega_j^2 - 2i\gamma_j\omega_j - i\omega\gamma_j + \omega\omega_j}{8\omega^2\gamma_j^2\omega_j + 8i\omega^2\gamma_j\omega_j^2 + 32\omega\gamma_j^2\omega_j^2 + 16i\omega\gamma_j\omega_j^3 + 24\gamma_j^2\omega_j^3 + 8i\gamma_j\omega_j^4} + \frac{\omega}{2\omega^4 + 4\omega^2\gamma_j^2 - 4\omega^2\omega_j^2 + 2\omega_j^4}$$

Or

$$\sum \text{Residue} = \overbrace{-\frac{\left(2i\omega^2\gamma_j^2 + i\omega^2\omega_j^2 + 2\omega\gamma_j\omega_j^2 - 6i\gamma_j^2\omega_j^2 - i\omega_j^4\right)}{4\gamma_j\left(\omega^2 - \omega_j^2\right)\left(-\omega^2\gamma_j^2 - \omega^2\omega_j^2 + 4i\omega\gamma_j\omega_j^2 + 9\gamma_j^2\omega_j^2 + \omega_j^4\right)}}^{\text{This term needs to be simplified. Error somewhere}} + \frac{\omega}{2\left(\omega^2 - \omega_j^2\right)^2 + 4\gamma_j^2\omega_j^2}$$

Hence the result becomes

$$\operatorname{Re}(\epsilon(\omega)) = 1 - \frac{\gamma\omega_p^2 f_j}{\pi}(2\pi i)\sum \text{Residue}$$

The above should come out to be as shown in (1) which is

$$\operatorname{Re}(\epsilon(\omega)) = 1 - \omega_p^2 f_j \frac{\left(\omega^2 - \omega_j^2\right)}{\left(\omega^2 - \omega_j^2\right)^2 + 4\gamma_j^2\omega^2}$$

I was not able to fully simplify the first term in $\sum$ Residue above, I seem to have made an error somewhere and not able to find it now, but the second terms looks OK. All complex i terms should cancel out since the result must be real.