HW 5
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

Evaluate the following integral for $t > 0$ and for $t < 0$ when $\omega_0 > 0$ and $\epsilon \to 0^+$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega$$

Solution

Case $t > 0$

We select the upper half for contour C since when $t > 0$ the integral on upper half will vanish as will be shown below.

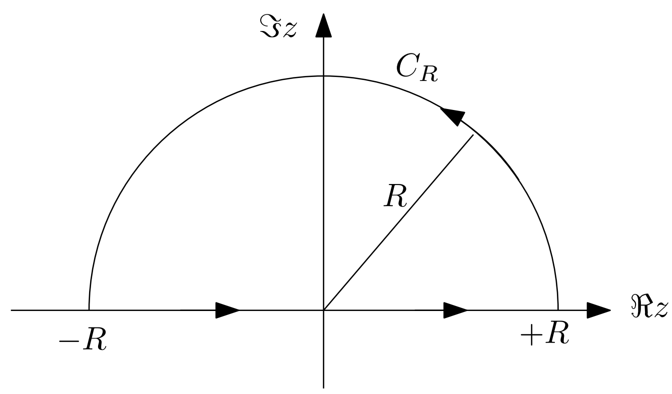

Figure 1: Contour used for $t > 0$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega &= \oint_C \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \\
&= \lim_{R\to\infty} (P.V.) \int_{-R}^{R} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz + \int_{CR} \frac{e^{izt}}{(z - i\epsilon)^2 - \omega_0^2} dz \\
&= 2\pi i \sum \text{Residue}
\end{aligned}$$

Therefore, if we can show that $\lim_{R\to\infty} \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} dz = 0$, then the above implies that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = 2\pi i \sum \text{Residue} \tag{1}$$

Now we need to find the residues inside the contour shown. There is a pole when $(\omega - i\epsilon)^2 = \omega_0^2$ or $\omega - i\epsilon = \pm\omega_0$ or $\omega = i\epsilon \pm \omega_0$. Hence there are two simple poles, they are

$$\begin{aligned}
z_1 &= i\epsilon + \omega_0 \\
z_2 &= i\epsilon - \omega_0
\end{aligned}$$

They are both in upper half, inside the contour (since $\omega_0 > 0$ and ϵ is positive).

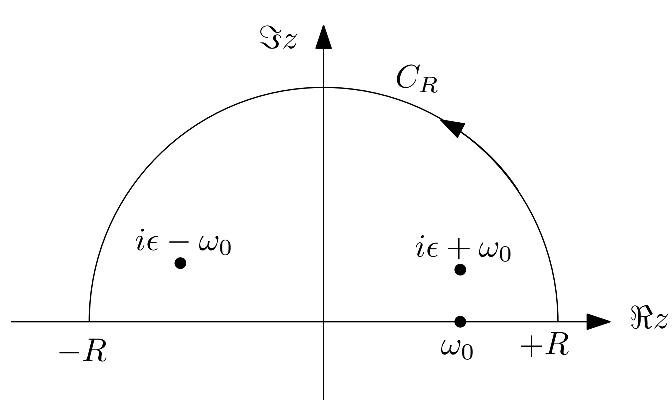

Figure 2: Locations of poles

Now we find the residues

$$\begin{aligned}
\text{Residue}(z_1) &= \lim_{z\to z_1}(z-z_1)\frac{e^{izt}}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_1}\frac{e^{izt}}{(z-z_2)}\\
&= \frac{e^{it(i\epsilon+\omega_0)}}{(i\epsilon+\omega_0)-(i\epsilon-\omega_0)}\\
&= \frac{e^{-t\epsilon}e^{it\omega_0}}{2\omega_0} \qquad (2)
\end{aligned}$$

And

$$\begin{aligned}
\text{Residue}(z_2) &= \lim_{z\to z_2}(z-z_2)\frac{e^{izt}}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_2}\frac{e^{izt}}{(z-z_1)}\\
&= \frac{e^{it(i\epsilon-\omega_0)}}{(i\epsilon-\omega_0)-(i\epsilon+\omega_0)}\\
&= \frac{e^{-t\epsilon}e^{-it\omega_0}}{-2\omega_0} \qquad (3)
\end{aligned}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega-i\epsilon)^2-\omega_0^2}d\omega &= 2\pi i\left(\frac{e^{-t\epsilon}e^{it\omega_0}}{2\omega_0}+\frac{e^{-t\epsilon}e^{-it\omega_0}}{-2\omega_0}\right)\\
&= \frac{2\pi i}{2\omega_0}e^{-t\epsilon}\left(e^{it\omega_0}-e^{-it\omega_0}\right)\\
&= \frac{2\pi}{\omega_0}e^{-t\epsilon}\left(\frac{e^{it\omega_0}-e^{-it\omega_0}}{-2i}\right)\\
&= -\frac{2\pi}{\omega_0}e^{-t\epsilon}\left(\frac{e^{it\omega_0}-e^{-it\omega_0}}{2i}\right)\\
&= -\frac{2\pi}{\omega_0}e^{-t\epsilon}\sin(t\omega_0)
\end{aligned}$$

Now, to finish the solution, we must show that $\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz=0$. But

$$\begin{aligned}
\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz &\le \left|\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz\right|_{\max}\\
&\le \int_{CR}\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max}dz\\
&= \left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max}\int_{CR}dz\\
&= \left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max}\int_0^{\pi}Rd\theta\\
&= R\pi\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max} \qquad (4)
\end{aligned}$$

But

$$
\begin{aligned}
\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max} &\leq \frac{\left|e^{izt}\right|_{\max}}{\left|(z-z_1)(z-z_2)\right|_{\min}} \\
&\leq \frac{\left|e^{izt}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&= \frac{\left|e^{it(x+iy)}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&= \frac{\left|e^{itx-ty}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&= \frac{\left|e^{itx}\right|_{\max}\left|e^{-ty}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}} \\
&\leq \frac{\left|e^{-ty}\right|_{\max}}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}}
\end{aligned}
$$

Now, since $y > 0$ (we are in the upper half) and also since $t > 0$, then $\left|e^{-ty}\right|_{\max} = 1$, which occurs when $y = 0$. Hence the above becomes

$$
\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max} \leq \frac{1}{\left|(z-z_1)\right|_{\min}\left|(z-z_2)\right|_{\min}}
$$

By inverse triangle inequality $\left|(z-z_1)\right|_{\min} \geq |z|^2 + |z_1|^2 = R^2 + \left|\epsilon^2+\omega_0^2\right|^2$ and $\left|(z-z_2)\right|_{\min} \geq |z|^2 + |z_2|^2 = R^2 + \left|\epsilon^2+\omega_0^2\right|^2$. The above becomes

$$
\left|\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}\right|_{\max} \leq \frac{1}{2R^2 + 2\left|\epsilon^2+\omega_0^2\right|^2}
$$

Substituting the above in (4) gives

$$
\begin{aligned}
\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz &\leq \lim_{R\to\infty} R\pi\left(\frac{1}{2R^2+2\left|\epsilon^2+\omega_0^2\right|^2}\right) \\
&= \pi\lim_{R\to\infty}\frac{R}{2R^2+2\left|\epsilon^2+\omega_0^2\right|^2}
\end{aligned}
$$

But $2\left|\epsilon^2+\omega_0^2\right|^2$ is a finite value, say β so the above is

$$
\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz \leq \pi\lim_{R\to\infty}\frac{R}{2R^2+\beta}
$$

And it is clear now that the above limit goes to zero. In other words, $\lim_{R\to\infty}\frac{R}{2R^2+\beta} = \lim_{R\to\infty}\frac{\frac{1}{R}}{2+\frac{\beta}{R^2}} = \frac{0}{2} = 0$.

Hence <u>The final solution is</u>

$$
\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega-i\epsilon)^2-\omega_0^2}d\omega = -\frac{2\pi}{\omega_0}e^{-t\epsilon}\sin(t\omega_0)
$$

<u>Case $t < 0$</u>

Here, we must use the lower half for the contour in order for the half circle contour integral to vanish.

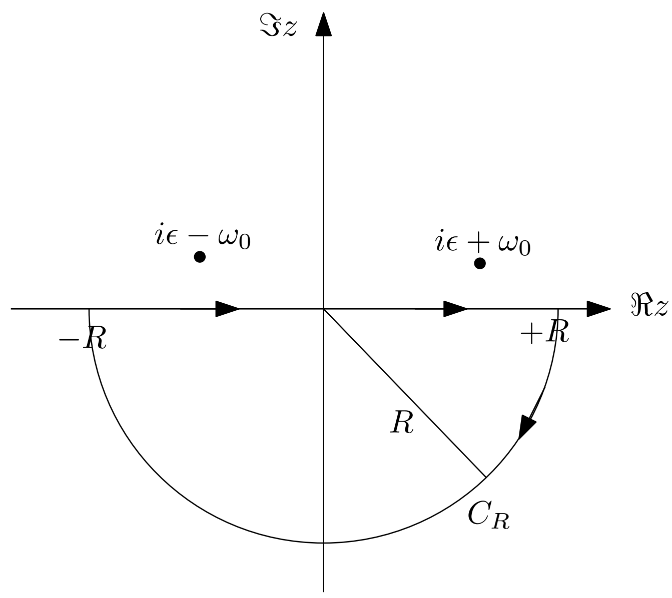

Figure 3: Contour for $t < 0$

In this case the sum of residues is zero (since both poles are in the upper half), then we see right away that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2} d\omega = 0 \qquad t < 0$$

But we must show that $\lim_{R\to\infty} \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2} dz = 0$ here as well for the above result to be valid. Similar to what was done earlier:

$$\begin{aligned}
\int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2} dz &\leq \left| \int_{CR} \frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2} dz \right|_{\max} \\
&\leq \int_{CR} \left| \frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2} \right|_{\max} dz \qquad (1) \\
&= R\pi \left| \frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2} \right|_{\max} \qquad (4)
\end{aligned}$$

But

$$\begin{aligned}
\left| \frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2} \right|_{\max} &\leq \frac{\left|e^{izt}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&\leq \frac{\left|e^{it(x+iy)}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&= \frac{\left|e^{itx-ty}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&= \frac{\left|e^{itx}\right|_{\max} \left|e^{-ty}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}} \\
&\leq \frac{\left|e^{-ty}\right|_{\max}}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}}
\end{aligned}$$

Since $y < 0$ (we are now in lower half) and also since $t < 0$, then $\left|e^{-ty}\right|_{\max} = 1$, which occurs when $y = 0$. Hence

$$\left| \frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2} \right|_{\max} \leq \frac{1}{|(z-z_1)|_{\min} |(z-z_2)|_{\min}}$$

But by inverse triangle inequality $|(z-z_1)|_{\min} \geq |z|^2 + |z_1|^2 = R^2 + \left|\epsilon^2 + \omega_0^2\right|^2$ and $|(z-z_2)|_{\min} \geq$

$|z|^2 + |z_2|^2 = R^2 + \left|\epsilon^2 + \omega_0^2\right|^2$. Hence the above becomes

$$\left|\frac{e^{izt}}{(z-i\epsilon)^2 - \omega_0^2}\right|_{\max} \leq \frac{1}{2R^2 + 2\left|\epsilon^2 + \omega_0^2\right|^2}$$

The rest follows what was done in first part. Therefore

$$\begin{aligned}\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz &\leq \lim_{R\to\infty} R\pi\left(\frac{1}{2R^2+2\left|\epsilon^2+\omega_0^2\right|^2}\right)\\ &= \pi\lim_{R\to\infty}\frac{R}{2R^2+2\left|\epsilon^2+\omega_0^2\right|^2}\end{aligned}$$

But $2\left|\epsilon^2 + \omega_0^2\right|^2$ is finite number, say β so the above is

$$\lim_{R\to\infty}\int_{CR}\frac{e^{izt}}{(z-i\epsilon)^2-\omega_0^2}dz \leq \pi\lim_{R\to\infty}\frac{R}{2R^2+\beta}$$

And it is clear now that the above limit goes to zero.

The final solution is

$$\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega-i\epsilon)^2-\omega_0^2}d\omega = 0 \qquad t<0$$

2 Problem 2

Evaluate the following integrals $\int_0^\infty \frac{\ln x}{1+x^2}dx$ and $\int_0^\infty \frac{\ln^2 x}{1+x^2}dx$. In order to find the second one you need to consider the integral $\int_0^\infty \frac{\ln^3 x}{1+x^2}dx$

Solution

2.1 Part (a)

There are two ways to find $\int_0^\infty \frac{\ln x}{1+x^2}dx$. One uses a substitution method and requires no complex contour integration and the second method uses $\int_0^\infty \frac{\ln^2 x}{1+x^2}dx$ with complex integration to find $\int_0^\infty \frac{\ln x}{1+x^2}dx$.

Method one

Let $x = \frac{1}{y}$. Hence $dx = -\frac{1}{y^2}dy$. When $x = 0 \rightarrow y = \infty$ and when $x = \infty \rightarrow y = 0$. Hence the integral $\int_0^\infty \frac{\ln x}{1+x^2}dx$ becomes

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{1+x^2}dx &= \int_\infty^0 \frac{\ln\left(\frac{1}{y}\right)}{1+\frac{1}{y^2}}\left(-\frac{1}{y^2}dy\right) \\
&= -\int_\infty^0 \frac{\ln\left(\frac{1}{y}\right)}{\frac{y^2+1}{y^2}}\left(\frac{1}{y^2}dy\right) \\
&= -\int_\infty^0 \frac{\ln\left(\frac{1}{y}\right)}{y^2+1}dy \\
&= \int_\infty^0 \frac{\ln\left(y\right)}{y^2+1}dy \\
&= -\int_0^\infty \frac{\ln\left(y\right)}{y^2+1}dy
\end{aligned}$$

Since on the RHS y is arbitrary integration variable, we can rename it back to x. Hence the above becomes

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{1+x^2}dx &= -\int_0^\infty \frac{\ln(x)}{x^2+1}dy \\
2\int_0^\infty \frac{\ln x}{1+x^2}dx &= 0
\end{aligned}$$

Therefore

$$\boxed{\int_0^\infty \frac{\ln x}{1+x^2}dx = 0}$$

Method two

In this method will use complex integration on $\int_0^\infty \frac{\ln^2 z}{1+z^2}dz$ to show that $\int_0^\infty \frac{\ln z}{1+z^2}dz = 0$. The following contour will be used.

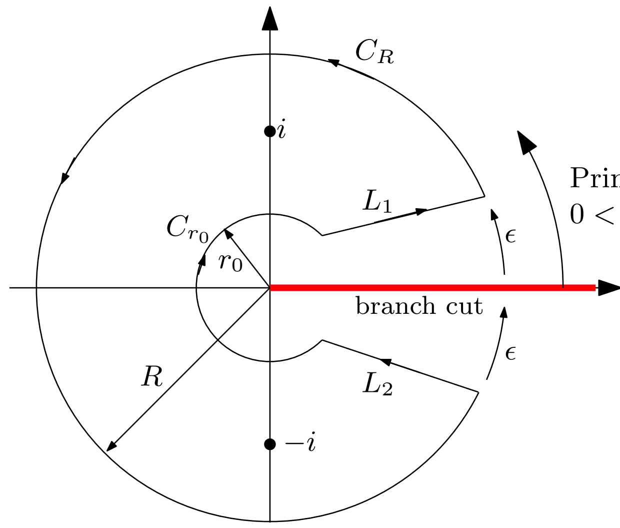

Figure 4: Contour for problem 2, showing location of poles at $\pm i$

$$
\begin{aligned}
\oint \frac{\ln^2 z}{1+z^2} dz &= \oint f(z)\, dz \\
&= \int_{L_2} f(z)\, dz + \int_{C_{r_0}} f(z)\, dz + \int_{L_1} f(z)\, dz + \int_{C_R} f(z)\, dz \\
&= 2\pi i \sum \text{Residue}
\end{aligned}
$$

Hence

$$
\int_{L_2} f(z)\, dz + \int_{C_{r_0}} f(z)\, dz + \int_{L_1} f(z)\, dz + \int_{C_R} f(z)\, dz = 2\pi i \sum \text{Residue} \tag{1}
$$

There are two poles in $\frac{\ln^2 z}{(z-i)(z+i)}$. Residue at $z_1 = i$ is

$$
\begin{aligned}
\text{Residue}(i) &= \lim_{z\to i} (z-i) \frac{\ln^2 z}{(z-i)(z+i)} \\
&= \lim_{z\to i} \frac{\ln^2 z}{(z+i)} \\
&= \frac{\ln^2 i}{2i} \\
&= \frac{\left(\ln(1) + i\frac{\pi}{2}\right)^2}{2i} \\
&= \frac{\left(i\frac{\pi}{2}\right)^2}{2i} \\
&= \frac{-\frac{\pi^2}{4}}{2i} \\
&= \frac{-\pi^2}{8i}
\end{aligned} \tag{2}
$$

And

$$
\begin{aligned}
\text{Residue}(-i) &= \lim_{z\to -i} (z+i) \frac{\ln^2 z}{(z-i)(z+i)} \\
&= \lim_{z\to -i} \frac{\ln^2 z}{(z-i)} \\
&= \frac{\ln^2(-i)}{-2i}
\end{aligned}
$$

But $\ln(-i) = \ln(1) + i\frac{3}{2}\pi$. Notice that the phase is $\frac{3}{2}\pi$ and not $-\frac{\pi}{2}$ since we are using principle branch defined as $0 < \theta < 2\pi$. Therefore the above becomes

$$
\begin{aligned}
\operatorname{Residue}(-i) &= \frac{\left(\ln(1) + i\frac{3}{2}\pi\right)^2}{-2i} \\
&= \frac{-\frac{9}{4}\pi^2}{-2i} \\
&= \frac{9\pi^2}{8i} \qquad (3)
\end{aligned}
$$

Adding (2+3) and substituting in (1) gives

$$
\begin{aligned}
\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz &= 2\pi i\left(\frac{-\pi^2}{8i} + \frac{9\pi^2}{8i}\right) \\
\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz &= 2\pi^3
\end{aligned}
$$

We will show at the end that $\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz = 0$ and that $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$. Given this, the above simplifies to only two integrals to evaluate

$$
\int_{L_2} f(z)\,dz + \int_{L_1} f(z)\,dz = 2\pi^3 \qquad (3A)
$$

We will now work on finding $\int_{L_1} f(z)\,dz$. Let $z = re^{i\epsilon}$, hence $dz = dre^{i\epsilon}$ and the integral becomes

$$
\begin{aligned}
\int_{L_1} \frac{\ln^2 z}{1+z^2}dz &= \int_0^\infty \frac{\ln^2\left(re^{i\epsilon}\right)}{1+\left(re^{i\epsilon}\right)^2}dre^{i\epsilon} \\
&= e^{i\epsilon}\int_0^\infty \frac{(\ln r + i\epsilon)^2}{1+r^2e^{2i\epsilon}}dr \\
&= e^{i\epsilon}\int_0^\infty \frac{\ln^2 r + i^2\epsilon^2 + 2i\epsilon\ln r}{1+r^2e^{2i\epsilon}}dr
\end{aligned}
$$

Now taking the limit as $\epsilon \to 0$ the above becomes

$$
\int_{L_1} \frac{\ln^2 z}{1+z^2}dz = \int_0^\infty \frac{\ln^2 r}{1+r^2}dr \qquad (4)
$$

We will now work on finding $\int_{L_2} f(z)\,dz$. Let $z = re^{i(2\pi-\epsilon)}$, hence $dz = dre^{i(2\pi-\epsilon)}$ and the integral becomes

$$
\begin{aligned}
\int_{L_2} \frac{\ln^2 z}{1+z^2}dz &= \int_\infty^0 \frac{\ln^2\left(re^{i(2\pi-\epsilon)}\right)}{1+\left(re^{i(2\pi-\epsilon)}\right)^2}dre^{i(2\pi-\epsilon)} \\
&= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{(\ln(r) + i(2\pi-\epsilon))^2}{1+r^2e^{2i(2\pi-\epsilon)}}dr \\
&= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{\ln^2(r) - (2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln r}{1+r^2e^{2i(2\pi-\epsilon)}}dr \\
&= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{\ln^2(r) - \left(4\pi^2 + \epsilon^2 - 4\pi\epsilon\right) + 2i(2\pi-\epsilon)\ln r}{1+r^2e^{2i(2\pi-\epsilon)}}dr
\end{aligned}
$$

Taking the limit as $\epsilon \to 0$ the above becomes

$$
\int_{L_2} \frac{\ln^2 z}{1+z^2}dz = e^{i2\pi}\int_\infty^0 \frac{\ln^2(r) - 4\pi^2 + 4\pi i\ln r}{1+r^2e^{i4\pi}}dr
$$

But $e^{i2\pi} = 1$ and $e^{i4\pi} = 1$ then the above becomes

$$\begin{aligned}
\int_{L_2} \frac{\ln^2 z}{1+z^2} dz &= \int_{\infty}^{0} \frac{\ln^2(r) - 4\pi^2 + 4\pi i \ln r}{1+r^2} dr \\
&= \int_{\infty}^{0} \frac{\ln^2 r}{1+r^2} dr - \int_{\infty}^{0} \frac{4\pi^2}{1+r^2} dr + 4\pi i \int_{\infty}^{0} \frac{\ln r}{1+r^2} dr \\
&= -\int_{0}^{\infty} \frac{\ln^2 r}{1+r^2} dr + \int_{0}^{\infty} \frac{4\pi^2}{1+r^2} dr - 4\pi i \int_{0}^{\infty} \frac{\ln r}{1+r^2} dr
\end{aligned} \tag{5}$$

Using (4,5) in (3A) gives

$$\begin{aligned}
-\int_{0}^{\infty} \frac{\ln^2 r}{1+r^2} dr + \int_{0}^{\infty} \frac{4\pi^2}{1+r^2} dr - 4\pi i \int_{0}^{\infty} \frac{\ln r}{1+r^2} dr + \int_{0}^{\infty} \frac{\ln^2 r}{1+r^2} dr &= 2\pi^3 \\
4\pi^2 \int_{0}^{\infty} \frac{1}{1+r^2} dr - 4\pi i \int_{0}^{\infty} \frac{\ln r}{1+r^2} dr &= 2\pi^3
\end{aligned}$$

But $\int_0^\infty \frac{1}{1+r^2} dr = \arctan(r)_0^\infty = \arctan(\infty) - \arctan(0) = \frac{\pi}{2}$, hence the above becomes

$$\begin{aligned}
4\pi^2 \left(\frac{\pi}{2}\right) - 4\pi i \int_{0}^{\infty} \frac{\ln r}{1+r^2} dr &= 2\pi^3 \\
-4\pi i \int_{0}^{\infty} \frac{\ln r}{1+r^2} dr &= 0
\end{aligned}$$

Which implies

$$\boxed{\int_0^\infty \frac{\ln r}{1+r^2} dr = 0}$$

Which is the same result obtained using method one above.

Appendix Here we will show that $\lim_{r_0 \to 0} \int_{C_{r_0}} f(z)\, dz = 0$ and $\lim_{R \to \infty} \int_{C_R} f(z)\, dz = 0$.

For $\underline{\lim_{r_0 \to 0} \int_{C_{r_0}} f(z)\, dz}$, let $z = r_0 e^{i\theta}$. Hence $dz = r_0 i e^{i\theta} d\theta$ and the integral becomes

$$\lim_{r_0 \to 0} \int_{2\pi - \epsilon}^{\epsilon} \frac{\ln^2\left(r_0 e^{i\theta}\right)}{1 + r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta = \lim_{r_0 \to 0} i \int_{2\pi - \epsilon}^{\epsilon} \frac{\ln^2\left(r_0 e^{i\theta}\right)}{1 + r_0^2 e^{i\theta}} r_0 d\theta$$

As $\epsilon \to 0$ the above becomes

$$\begin{aligned}
\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{2i\theta}}r_0ie^{i\theta}d\theta &= \lim_{r_0\to 0} i\int_{2\pi}^{0}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}r_0d\theta \\
&\leq \lim_{r_0\to 0}\left| i\int_{2\pi}^{0}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}r_0d\theta\right|_{\max} \\
&\leq \lim_{r_0\to 0}\int_{2\pi}^{0}\left|\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}\right|_{\max} r_0d\theta \\
&\leq \lim_{r_0\to 0}\left|\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}\right|_{\max}\int_{2\pi}^{0} r_0d\theta \\
&= 2\pi r_0\lim_{r_0\to 0}\left|\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{i\theta}}\right|_{\max} \\
&\leq 2\pi r_0\lim_{r_0\to 0}\frac{\left|\ln^2\left(r_0e^{i\theta}\right)\right|_{\max}}{\left|1+r_0^2e^{i\theta}\right|_{\min}} \\
&= 2\pi r_0\lim_{r_0\to 0}\frac{\left|\ln r_0+i\theta\right|_{\max}^2}{\left|1+r_0^2e^{i\theta}\right|_{\min}} \\
&= 2\pi r_0\lim_{r_0\to 0}\frac{\left|\ln^2 r_0+(i\theta)^2+2i\theta\ln r_0\right|_{\max}}{1-r_0^2} \\
&= 2\pi\lim_{r_0\to 0}\frac{r_0\ln^2 r_0-4\pi^2r_0+4\pi r_0\ln r_0}{1-r_0^2} \\
&= 2\pi\lim_{r_0\to 0}\left(\frac{r_0\ln^2 r_0}{1-r_0^2}-4\pi^2\frac{r_0}{1-r_0^2}+4\pi\frac{r_0\ln r_0}{1-r_0^2}\right)
\end{aligned}$$

But $\lim_{r_0\to 0}\frac{r_0\ln^2 r_0}{1-r_0^2}=0$ and $\lim_{r_0\to 0}\frac{r_0}{1-r_0^2}=0$ and $\lim_{r_0\to 0}\frac{r_0\ln r_0}{1-r_0^2}=0$ Hence all terms on the RHS above become zero in the limit. Therefore

$$\begin{aligned}
\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon}\frac{\ln^2\left(r_0e^{i\theta}\right)}{1+r_0^2e^{2i\theta}}r_0ie^{i\theta}d\theta &= \lim_{r_0\to 0}\int_{C_{r_0}}\frac{\ln^2 z}{1+z^2}dz \\
&= 0
\end{aligned}$$

Now we will do the same $\underline{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}$, let $z=Re^{i\theta}$. Hence $dz=Rie^{i\theta}d\theta$ and the integral becomes

$$\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta = \lim_{R\to\infty} i\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta$$

As $\epsilon \to 0$ the above becomes

$$
\begin{aligned}
\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta &= \lim_{R\to\infty} i\int_0^{2\pi}\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta \\
&\leq \lim_{R\to\infty}\left|i\int_0^{2\pi}\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta\right|_{\max} \\
&\leq \lim_{R\to\infty}\int_0^{2\pi}\left|\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}R\right|_{\max} d\theta \\
&\leq \lim_{R\to\infty}\left|\frac{\ln^2\left(\mathrm{Re}^{i\theta}\right)}{1+R^2e^{2i\theta}}\right|\int_0^{2\pi} Rd\theta \\
&= 2\pi\lim_{R\to\infty} R\frac{\left|\ln^2\left(\mathrm{Re}^{i\theta}\right)\right|_{\max}}{\left|1+R^2e^{2i\theta}\right|_{\min}} \\
&= 2\pi\lim_{R\to\infty} R\frac{\left|\ln R+i\theta\right|^2_{\max}}{1-R^2} \\
&= 2\pi\lim_{R\to\infty} R\frac{\left|\ln^2 R-\theta^2+2i\theta\ln R\right|_{\max}}{1-R^2} \\
&\leq 2\pi\lim_{R\to\infty}\frac{R\ln^2 R-4\pi^2R+4\pi R\ln R}{1-R^2} \\
&= 2\pi\lim_{R\to\infty}\left(\frac{R\ln^2 R}{1-R^2}-4\pi^2\frac{R}{1-R^2}+4\pi\frac{R\ln R}{1-R^2}\right)
\end{aligned}
$$

But $\lim_{R\to\infty}\frac{R\ln^2 R}{1-R^2}=0$ and $\lim_{R\to\infty}\frac{R}{1-R^2}=0$ and $\lim_{R\to\infty}\frac{R\ln R}{1-R^2}=0$ Hence all terms on the RHS above become zero in the limit. Therefore

$$
\begin{aligned}
\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^2\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta = \lim_{R\to\infty}\int_{C_R}\frac{\ln^2 z}{1+z^2}dz = 0 \\
= 0
\end{aligned}
$$

2.2 Part (b)

We will now find $\int_0^\infty\frac{\ln^3 z}{1+z^2}dz$ in order to determine $\int_0^\infty\frac{\ln^2 z}{1+z^2}dz$. We will use the same contour integration as part (a) above.

$$
\int_{L_2} f(z)\,dz+\int_{C_{r_0}} f(z)\,dz+\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz = 2\pi i\sum \text{Residue} \tag{1}
$$

There are two poles in $\frac{\ln^3 z}{(z-i)(z+i)}$. Residue at $z_1 = i$ is

$$\begin{aligned}
\text{Residue}(i) &= \lim_{z\to i}(z-i)\frac{\ln^3 z}{(z-i)(z+i)} \\
&= \lim_{z\to i}\frac{\ln^3 z}{(z+i)} \\
&= \frac{\ln^3 i}{2i} \\
&= \frac{\left(\ln(1)+i\frac{\pi}{2}\right)^3}{2i} \\
&= \frac{\left(i\frac{\pi}{2}\right)^3}{2i} \\
&= \frac{-i\frac{\pi^3}{8}}{2i} \\
&= \frac{-\pi^3}{16} \qquad (2)
\end{aligned}$$

And

$$\begin{aligned}
\text{Residue}(-i) &= \lim_{z\to -i}(z+i)\frac{\ln^3 z}{(z-i)(z+i)} \\
&= \lim_{z\to -i}\frac{\ln^3 z}{(z-i)} \\
&= \frac{\ln^3(-i)}{-2i}
\end{aligned}$$

But $\ln(-i) = \ln(1) + i\frac{3}{2}\pi$. Notice that the phase is $\frac{3}{2}\pi$ and not $-\frac{\pi}{2}$ since we are using principle branch defined as $0 < \theta < 2\pi$. Therefore the above becomes

$$\begin{aligned}
\text{Residue}(-i) &= \frac{\left(\ln(1)+i\frac{3}{2}\pi\right)^3}{-2i} \\
&= \frac{\left(i\frac{3}{2}\pi\right)^3}{-2i} \\
&= \frac{-i\frac{27}{8}\pi^3}{-2i} \\
&= \frac{27\pi^3}{16} \qquad (3)
\end{aligned}$$

Adding (2+3) and substituting in (1) gives

$$\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i\left(\frac{-\pi^3}{16} + \frac{27\pi^3}{16}\right)$$

$$\int_{L_2} f(z)\,dz + \int_{C_{r_0}} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz = \frac{13}{4}\pi^4 i$$

We will show below that $\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz = 0$ and that $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$, which simplifies the above to

$$\int_{L_2} f(z)\,dz + \int_{L_1} f(z)\,dz = \frac{13}{4}\pi^4 i \qquad (3A)$$

We will now work on finding $\int_{L_1} f(z)\,dz$. Let $z = re^{i\epsilon}$, hence $dz = dre^{i\epsilon}$ and the integral becomes

$$\begin{aligned}\int_{L_1}\frac{\ln^3 z}{1+z^2}dz &= \int_0^\infty \frac{\ln^3\left(re^{i\epsilon}\right)}{1+\left(re^{i\epsilon}\right)^2}dre^{i\epsilon}\\ &= e^{i\epsilon}\int_0^\infty \frac{(\ln r + i\epsilon)^3}{1+r^2e^{2i\epsilon}}dr\\ &= e^{i\epsilon}\int_0^\infty \frac{\left(\ln^2 r + i^2\epsilon^2 + 2i\epsilon\ln r\right)(\ln r + i\epsilon)}{1+r^2e^{2i\epsilon}}dr\\ &= e^{i\epsilon}\int_0^\infty \frac{\left(\ln^3 r + i^2\epsilon^2\ln r + 2i\epsilon\ln^2 r\right)+\left(i\epsilon\ln^2 r + i^3\epsilon^3 + 2i^2\epsilon^2\ln r\right)}{1+r^2e^{2i\epsilon}}dr\end{aligned}$$

Now taking the limit as $\epsilon\to 0$ the above becomes

$$\int_{L_1}\frac{\ln^3 z}{1+z^2}dz = \int_0^\infty \frac{\ln^3 r}{1+r^2}dr \tag{4}$$

We will now work on finding $\int_{L_2} f(z)\,dz$. Let $z = re^{i(2\pi-\epsilon)}$, hence $dz = dre^{i(2\pi-\epsilon)}$ and the integral becomes

$$\begin{aligned}\int_{L_2}\frac{\ln^3 z}{1+z^2}dz &= \int_\infty^0 \frac{\ln^3\left(re^{i(2\pi-\epsilon)}\right)}{1+\left(re^{i(2\pi-\epsilon)}\right)^2}dre^{i(2\pi-\epsilon)}\\ &= e^{i(2\pi-\epsilon)}\int_\infty^0 \frac{(\ln(r)+i(2\pi-\epsilon))^3}{1+r^2e^{2i(2\pi-\epsilon)}}dr\end{aligned}$$

But $\lim_{\epsilon\to 0} e^{i(2\pi-\epsilon)} = e^{2\pi i} = 1$ and the above becomes

$$\begin{aligned}\int_{L_2}\frac{\ln^3 z}{1+z^2}dz &= \int_\infty^0 \frac{\left(\ln^2 r - (2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln r\right)(\ln(r)+i(2\pi-\epsilon))}{1+r^2e^{2i(2\pi-\epsilon)}}dr\\ &= \int_\infty^0 \frac{\ln^3 r - \ln r(2\pi-\epsilon)^2 + 2i(2\pi-\epsilon)\ln^2 r + i(2\pi-\epsilon)\ln^2 r - i(2\pi-\epsilon)^3 + 2i^2(2\pi-\epsilon)^2\ln r}{1+r^2e^{2i(2\pi-\epsilon)}}dr\end{aligned}$$

Taking the limit as $\epsilon\to 0$ the above becomes

$$\begin{aligned}\int_{L_2}\frac{\ln^3 z}{1+z^2}dz &= \int_\infty^0 \frac{\ln^3 r - 4\pi^2\ln r + 4\pi i\ln^2 r + 2\pi i\ln^2 r - i(2\pi-\epsilon)^2(2\pi-\epsilon) + 2i^2\left(4\pi^2+\epsilon^2-4\pi\epsilon\right)\ln r}{1+r^2e^{4\pi i}}dr\\ &= \int_\infty^0 \frac{\ln^3 r - 4\pi^2\ln r + 4\pi i\ln^2 r + 2\pi i\ln^2 r - i\left(4\pi^2+\epsilon^2-4\pi\epsilon\right)(2\pi-\epsilon) - 8\pi^2\ln r}{1+r^2}dr\\ &= \int_\infty^0 \frac{\ln^3 r - 4\pi^2\ln r + 6\pi i\ln^2 r - i\left(8\pi^3+2\pi\epsilon^2-8\pi^2\epsilon\right) - \left(4\pi^2\epsilon+\epsilon^3-4\pi\epsilon^2\right) - 8\pi^2\ln r}{1+r^2}dr\end{aligned}$$

Taking the limit as $\epsilon\to 0$ the above becomes

$$\begin{aligned}\int_{L_2}\frac{\ln^3 z}{1+z^2}dz &= \int_\infty^0 \frac{\ln^3(r) - 4\pi^2\ln r + 6\pi i\ln^2 r - 8i\pi^3 - 8\pi^2\ln r}{1+r^2}dr\\ &= \int_\infty^0 \frac{\ln^3(r) - 12\pi^2\ln r + 6\pi i\ln^2 r - 8i\pi^3}{1+r^2}dr\end{aligned}$$

Hence the above becomes

$$\begin{aligned}\int_{L_2}\frac{\ln^3 z}{1+z^2}dz &= \int_\infty^0 \frac{\ln^3(r)}{1+r^2}dr - 12\pi^2\int_\infty^0\frac{\ln r}{1+r^2}dr + 6\pi i\int_\infty^0\frac{\ln^2 r}{1+r^2}dr - 8i\pi^3\int_\infty^0\frac{1}{1+r^2}dr\\ &= -\int_0^\infty \frac{\ln^3 r}{1+r^2}dr + 12\pi^2\int_0^\infty\frac{\ln r}{1+r^2}dr - 6\pi i\int_0^\infty\frac{\ln^2 r}{1+r^2}dr + 8i\pi^3\int_0^\infty\frac{1}{1+r^2}dr\end{aligned}$$

But $\int_0^\infty \frac{\ln r}{1+r^2}dr = 0$ from part (a) and $\int_0^\infty \frac{1}{1+r^2}dr = \frac{\pi}{2}$, hence the above becomes

$$\int_{L_2}\frac{\ln^3 z}{1+z^2}dz = -\int_0^\infty \frac{\ln^3 r}{1+r^2}dr - 6\pi i\int_0^\infty \frac{\ln^2 r}{1+r^2}dr + 4i\pi^4 \tag{5}$$

Using (4,5) in (3A) gives

$$\int_{L_2} f(z)\,dz + \int_{L_1} f(z)\,dz = \frac{13}{4}\pi^4 i$$

$$\left(-\int_0^\infty \frac{\ln^3 r}{1+r^2}dr - 6\pi i\int_0^\infty \frac{\ln^2 r}{1+r^2}dr + 4i\pi^4\right) + \left(\int_0^\infty \frac{\ln^3 r}{1+r^2}dr\right) = \frac{13}{4}\pi^4 i$$

$$-6\pi i\int_0^\infty \frac{\ln^2 r}{1+r^2}dr + 4i\pi^4 = \frac{13}{4}\pi^4 i$$

$$\int_0^\infty \frac{\ln^2 r}{1+r^2}dr = \frac{\frac{13}{4}\pi^4 i - 4i\pi^4}{-6\pi i}$$

$$\int_0^\infty \frac{\ln^2 r}{1+r^2}dr = \frac{13\pi^4 i - 16i\pi^4}{-24\pi i}$$

$$= \frac{-3\pi^4 i}{-24\pi i}$$

$$= \frac{\pi^3}{8}$$

Which implies

$$\boxed{\int_0^\infty \frac{\ln^2 x}{1+x^2}dx = \frac{\pi^3}{8}}$$

Appendix Here we will show that $\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz = 0$ and $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$.

For $\underline{\lim_{r_0\to 0}\int_{C_{r_0}} f(z)\,dz}$, let $z = r_0 e^{i\theta}$. Hence $dz = r_0 i e^{i\theta}d\theta$ and the integral becomes

$$\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta = \lim_{r_0\to 0} i\int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}} r_0 d\theta$$

As $\epsilon \to 0$ the above becomes

$$\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{2i\theta}} r_0 i e^{i\theta} d\theta = \lim_{r_0\to 0} i\int_{2\pi}^{0} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}} r_0 d\theta$$

$$\le \lim_{r_0\to 0}\left| i\int_{2\pi}^{0} \frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}} r_0 d\theta\right|_{\max}$$

$$\le \lim_{r_0\to 0}\int_{2\pi}^{0} \left|\frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}}\right|_{\max} r_0 d\theta$$

$$\le \lim_{r_0\to 0}\left|\frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}}\right|_{\max}\int_{2\pi}^{0} r_0 d\theta$$

$$= 2\pi r_0 \lim_{r_0\to 0}\left|\frac{\ln^3\left(r_0 e^{i\theta}\right)}{1+r_0^2 e^{i\theta}}\right|_{\max}$$

$$\le 2\pi r_0 \lim_{r_0\to 0}\frac{\left|\ln^3\left(r_0 e^{i\theta}\right)\right|_{\max}}{\left|1+r_0^2 e^{i\theta}\right|_{\min}}$$

$$\le 2\pi r_0 \lim_{r_0\to 0}\frac{\left|\ln r_0 + i\theta\right|_{\max}^2 \left|\ln r_0 + i\theta\right|_{\max}}{\left|1+r_0^2 e^{i\theta}\right|_{\min}}$$

But from part (a) we showed that $2\pi r_0 \lim_{r_0\to 0}\frac{\left|\ln r_0+i\theta\right|_{\max}^2}{\left|1+r_0^2 e^{i\theta}\right|_{\min}} = 0$, hence it follows that the

RHS above goes to zero. Therefore

$$\lim_{r_0\to 0}\int_{2\pi-\epsilon}^{\epsilon}\frac{\ln^3\left(r_0e^{i\theta}\right)}{1+r_0^2e^{2i\theta}}r_0ie^{i\theta}d\theta = \lim_{r_0\to 0}\int_{C_{r_0}}\frac{\ln^3 z}{1+z^2}dz$$
$$= 0$$

Now we will do the same $\lim_{R\to\infty}\int_{C_R} f(z)\,dz$, let $z = Re^{i\theta}$. Hence $dz = Rie^{i\theta}d\theta$ and the integral becomes

$$\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta = \lim_{R\to\infty} i\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(\mathrm{R}e^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta$$

As $\epsilon\to 0$ the above becomes

$$\begin{aligned}
\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta &= \lim_{R\to\infty} i\int_{0}^{2\pi}\frac{\ln^3\left(\mathrm{R}e^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta\\
&\le \lim_{R\to\infty}\left|i\int_{0}^{2\pi}\frac{\ln^3\left(\mathrm{R}e^{i\theta}\right)}{1+R^2e^{2i\theta}}Rd\theta\right|_{\max}\\
&\le \lim_{R\to\infty}\int_{0}^{2\pi}\left|\frac{\ln^3\left(\mathrm{R}e^{i\theta}\right)}{1+R^2e^{2i\theta}}R\right|_{\max} d\theta\\
&\le \lim_{R\to\infty}\left|\frac{\ln^3\left(\mathrm{R}e^{i\theta}\right)}{1+R^2e^{2i\theta}}\right|\int_{0}^{2\pi} Rd\theta\\
&= 2\pi\lim_{R\to\infty} R\frac{\left|\ln^3\left(\mathrm{R}e^{i\theta}\right)\right|_{\max}}{\left|1+R^2e^{2i\theta}\right|_{\min}}\\
&\le 2\pi\lim_{R\to\infty} R\frac{\left|\ln R+i\theta\right|^2_{\max}\left|\ln\left(\mathrm{R}e^{i\theta}\right)\right|_{\max}}{1-R^2}
\end{aligned}$$

But from part (a) we showed that $2\pi\lim_{R\to\infty} R\frac{|\ln R+i\theta|^2_{\max}}{1-R^2} = 0$, hence it follows that the RHS above goes to zero. Therefore

$$\lim_{R\to\infty}\int_{\epsilon}^{2\pi-\epsilon}\frac{\ln^3\left(Re^{i\theta}\right)}{1+R^2e^{2i\theta}}Rie^{i\theta}d\theta = \lim_{R\to\infty}\int_{C_R}\frac{\ln^3 z}{1+z^2}dz = 0$$
$$= 0$$