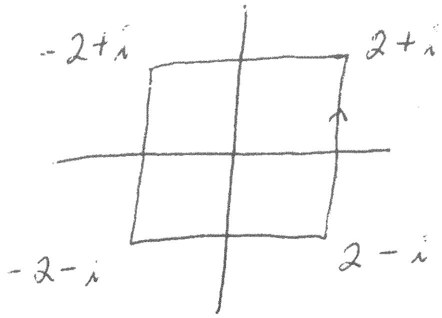
1



C

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}$$

(a) $f = e^{-z}$

$$\int_C \frac{f(z)\,dz}{z - \frac{i\pi}{2}} = 2\pi i\, e^{-\frac{i\pi}{2}} = \boxed{2\pi}$$

z_0 is within C

(b)

$$\int_C \frac{\cos z\,dz}{z\,(z + i\,2\sqrt{2})(z - i\,2\sqrt{2})} = \frac{2\pi i\,\cos(0)}{8} = \boxed{\frac{\pi}{4} i}$$

inside C

outside C

(c)

$$\int_C \frac{\frac{1}{2} z\,dz}{z + \frac{1}{2}} = 2\pi i \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \boxed{-\frac{\pi}{2} i}$$

inside C

(2) Cauchy integral formula for first derivative is

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^2}$$

Then apply the Cauchy integral formula to $f'(z)$.

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f'(z)\,dz}{z - z_0}$$

Equate them:

$$\boxed{\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}}$$

(3) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

Expansion about $z = 0$:

$$\frac{1}{z^2}\frac{1}{1-z} = \boxed{\sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots}$$

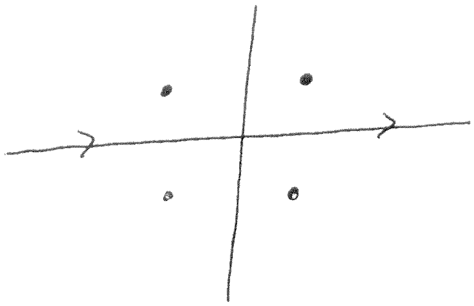
Expansion about $z = 1$: $$\frac{1}{z^2} = \frac{1}{[1+(z-1)]^2} =$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1} = 1 - 2(z-1) + 3(z-1)^2 - \cdots$$

$$\frac{1}{z^2}\frac{1}{1-z} = -\frac{1}{z-1}\sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1} =$$

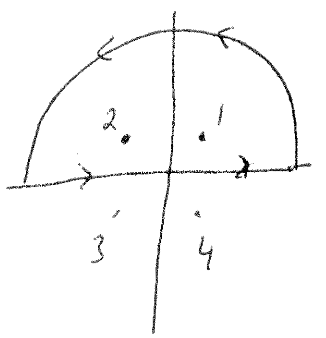
$$= \boxed{\sum_{n=1}^{\infty} (-1)^n n (z-1)^{n-2} = \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 + \cdots}$$

④ $I = \int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{1+z^4}$ Poles $z = e^{\pm i\frac{\pi}{4}}, e^{\pm i \frac{3\pi}{4}}$



Close the contour in either the upper half or lower half planes with a semi-circle of radius $R \to \infty$.

Choose upper half, use residue theorem.



$$e^{\pm i\frac{\pi}{4}} = \frac{1 \pm i}{\sqrt{2}} \qquad e^{\pm i \frac{3\pi}{4}} = \frac{-1 \pm i}{\sqrt{2}}$$

$$1 + z^4 = (z^2 + i)(z^2 - i) = (z - z_1)(z-z_2)(z-z_3)(z-z_4)$$

Relevant residues of $\frac{1}{1+z^4}$ are

$$K_1 = \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = \frac{1}{8}\,\frac{1}{i-1} \cdot 2\sqrt{2}$$

$$K_2 = \frac{1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} = \frac{1}{8}\,\frac{1}{i+1} \cdot 2\sqrt{2}$$

$$I = \frac{1}{2} \cdot 2\pi i \,(K_1 + K_2) = \frac{\pi i}{8} \underbrace{\left(\frac{1}{i+1} + \frac{1}{i-1} \right)}_{\frac{2i}{i^2-1} = -i} \cdot 2\sqrt{2}$$

$$\boxed{I = \frac{\pi}{2\sqrt{2}}}$$

(5) $$I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}$$

$$z = e^{i\theta} \qquad dz = iz\,d\theta \qquad \sin\theta = \frac{z - \frac{1}{z}}{2i} \qquad \cos\theta = \frac{z + \frac{1}{z}}{2}$$

$$I = \int_C \frac{\left(z - \frac{1}{z}\right)^2}{(2i)^2} \frac{dz}{iz} \frac{1}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} = \frac{i}{2b}\int_C \frac{\left(z^2 - 2 + \frac{1}{z^2}\right)}{\left(z^2 + 2\frac{a}{b}z + 1\right)} dz$$

↑ unit circle

There are two singularities inside the unit circle.

They are $z = 0$ and one of the roots $z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$.

For definiteness assume $b \geq 0$. Then + sign.

Residues are $-\frac{ia}{b^2}$ and $\frac{i}{4b}\sqrt{\frac{a^2}{b^2} - 1}$.

Then $I = \frac{2\pi}{b}\left[\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}\right]$ or

$$\boxed{I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{2\pi}{a + \sqrt{a^2 - b^2}}}$$