

HW 4  
Physics 5041 Mathematical Methods for Physics  
Spring 2019  
University of Minnesota, Twin Cities

Nasser M. Abbasi

November 2, 2019

Compiled on November 2, 2019 at 10:26pm

[public]

# Contents

<b>1 Problem 1</b>	<b>2</b>
1.1 Part (a) . . . . .	2
1.2 Part (b) . . . . .	2
1.3 Part (c) . . . . .	3
<b>2 Problem 2</b>	<b>5</b>
<b>3 Problem 3</b>	<b>6</b>
<b>4 Problem 4</b>	<b>7</b>
<b>5 Problem 5</b>	<b>10</b>

## 1 Problem 1

Let  $C$  denote the square contour with corners at  $\pm 2, \pm 2i$  and which is taken in counter clockwise direction. Use the residue theorem to evaluate the integral  $\int_C f(x) dz$  for the following functions

(a)  $\frac{e^{-z}}{z-i\frac{\pi}{2}}$ , (b)  $\frac{\cos z}{z(z^2+8)}$  (c)  $\frac{z}{z+1}$

Solution

### 1.1 Part (a)

The function  $f(z) = \frac{e^{-z}}{z-i\frac{\pi}{2}}$  has a simple pole at  $z = i\frac{\pi}{2} \approx 1.57i$ , hence it is inside the contour.

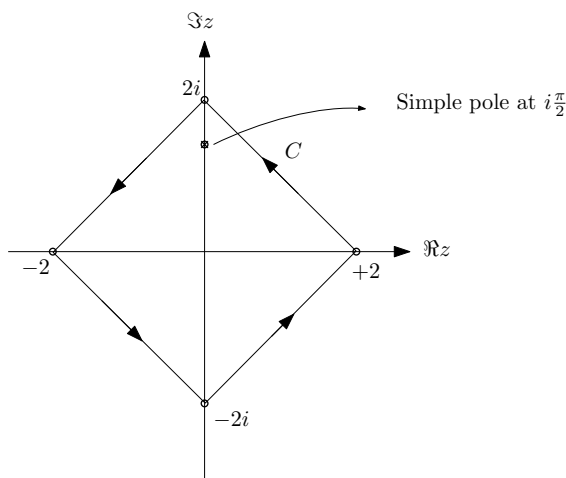


Figure 1: Location of pole relative to contour

Hence by residue theorem

$$\oint_C f(z) dz = 2\pi i \operatorname{Residue}(f(z))_{z=z_0}$$

So we just need to find the residue of  $f(z)$  at  $z = z_0 = i\frac{\pi}{2}$ . Since this is a simple plot, then the residue is given by

$$\begin{aligned} \operatorname{Residue}(f(z))_{z=z_0} &= \lim_{z \rightarrow z_0} (z - z_0) f(z) \\ &= \lim_{z \rightarrow i\frac{\pi}{2}} \left( z - i\frac{\pi}{2} \right) \frac{e^{-z}}{z - i\frac{\pi}{2}} \\ &= \lim_{z \rightarrow i\frac{\pi}{2}} e^{-z} \\ &= e^{-i\frac{\pi}{2}} \\ &= \cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right) \\ &= -i \end{aligned}$$

Therefore

$$\begin{aligned} \oint_C \frac{e^{-z}}{z - i\frac{\pi}{2}} dz &= 2\pi i (-i) \\ &= 2\pi \end{aligned}$$

### 1.2 Part (b)

The function  $f(z) = \frac{\cos z}{z(z^2+8)}$  has one simple pole at  $z = 0$  which is inside the contour, and a poles at  $z = \pm i\sqrt{8} = \pm 2i\sqrt{2} \approx \pm 2.83i$  but these are outside the contour.

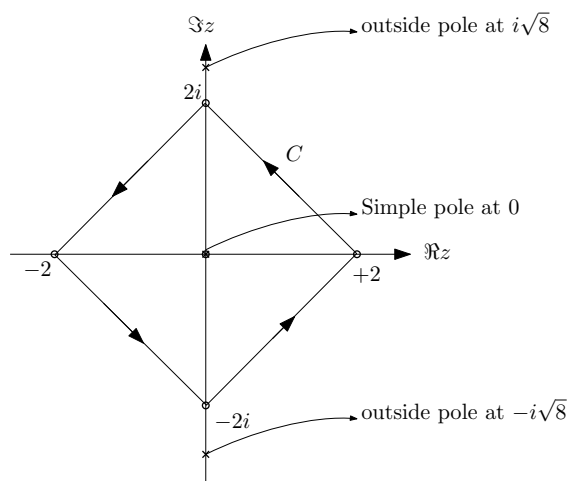


Figure 2: Location of pole relative to contour

Therefore by residue theorem, only the pole inside the contour which is at  $z = 0$  will contribute to the integral. So we just need to find residue at  $z = 0$

$$\begin{aligned}
 \text{Residue}(f(z))_{z=z_0} &= \lim_{z \rightarrow z_0} (z - z_0) f(z) \\
 &= \lim_{z \rightarrow 0} (z) \frac{\cos z}{z(z^2 + 8)} \\
 &= \lim_{z \rightarrow 0} \frac{\cos z}{(z^2 + 8)} \\
 &= \frac{1}{8}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \oint_C \frac{\cos z}{z(z^2 + 8)} dz &= 2\pi i \left( \frac{1}{8} \right) \\
 &= i \frac{\pi}{4}
 \end{aligned}$$

### 1.3 Part (c)

The function  $f(z) = \frac{z}{2(z+1)}$  has one simple pole at  $z = -1$  which is inside the contour.

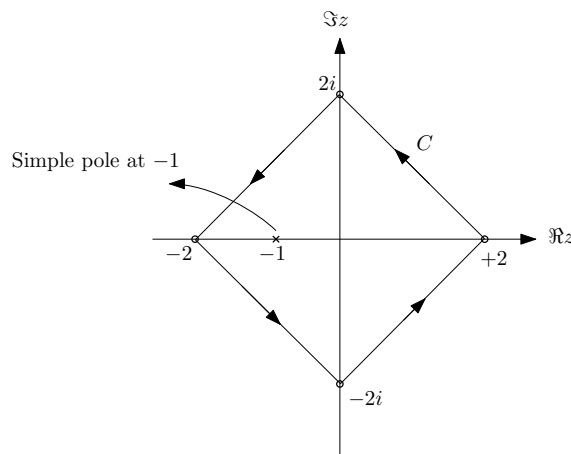


Figure 3: Location of pole relative to contour

So we just need to find residue at  $z = -1$

$$\begin{aligned}\text{Residue}(f(z))_{z=z_0} &= \lim_{z \rightarrow z_0} (z - z_0) f(z) \\ &= \lim_{z \rightarrow -1} (z + 1) \frac{z}{2(z + 1)} \\ &= \lim_{z \rightarrow -1} \frac{z}{2} \\ &= \frac{-1}{2}\end{aligned}$$

Therefore

$$\begin{aligned}\oint_C \frac{z}{2(z + 1)} dz &= 2\pi i \left( \frac{-1}{2} \right) \\ &= -i\pi\end{aligned}$$

## 2 Problem 2

---

Assume that  $f(z)$  is analytic on and interior to a closed contour  $C$  and that the point  $z_0$  lies inside  $C$ . Show that

$$\oint_C \frac{f'(z)}{z-z_0} dz = \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

### Solution

We see that  $g(z) = \frac{f'(z)}{z-z_0}$  has a simple pole at  $z = z_0$ . Therefore

$$\oint_C g(z) dz = 2\pi i (b_1) \quad (1)$$

Where  $b_1$  is the Residue of  $g(z)$  at  $z_0$ . By definition the residue of a simple pole is found as follows

$$\begin{aligned} b_1 &= \lim_{z \rightarrow z_0} (z - z_0) g(z) \\ &= \lim_{z \rightarrow z_0} (z - z_0) \frac{f'(z)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} f'(z) \\ &= f'(z_0) \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} \oint_C g(z) dz &= (2\pi i) f'(z_0) \\ \oint_C \frac{f'(z)}{z-z_0} dz &= (2\pi i) f'(z_0) \end{aligned} \quad (2)$$

But per lecture notes, page 46 on complex analysis, it shows that

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

Substituting the above back into RHS of (2) results in

$$\oint_C \frac{f'(z)}{z-z_0} dz = 2\pi i \left( \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz \right)$$

Therefore

$$\oint_C \frac{f'(z)}{z-z_0} dz = \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

QED.

### 3 Problem 3

---

Give the Laurent series expansion both in powers of  $z$  and in powers of  $(z-1)$  for the function  $\frac{1}{z^2(1-z)}$

#### Solution

There is a pole of order 2 at  $z = 0$  and a pole of order one at  $z = 1$ . Therefore, there is a Laurent series expansion about  $z = 0$  which is valid inside a disk or radius 1 centered at  $z = 0$ . Around  $z = 1$  there is another Laurent series expansion of the function, which is valid inside a disk centered at  $z = 0$  of radius 1.

#### Laurent series expansion around $z = 0$

$$\begin{aligned}\frac{1}{z^2(1-z)} &= \frac{1}{z^2} \frac{1}{1-z} \\ &= \frac{1}{z^2} (1 + z + z^2 + z^3 + \dots) \quad |z| < 1 \\ &= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \dots\end{aligned}$$

We see from the above that the residue at  $z = 0$  is 1 which is the coefficient of  $\frac{1}{z}$  term.

#### Laurent series expansion around $z = 1$

Let  $u = z - 1$ , hence  $z = u + 1$  and the function  $\frac{1}{z^2(1-z)}$  in terms of  $u$  becomes

$$\frac{1}{(1+u)^2(-u)} = \frac{-1}{u} \frac{1}{(1+u)^2} \quad (1)$$

But  $\frac{1}{(1+u)^2} = (1+u)^{-2}$ . Applying Binomial expansion  $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$  which is valid for  $|x| < 1$  then we see that for  $n = -2$  we obtain

$$(1+u)^{-2} = 1 + (-2)u + \frac{(-2)(-2-1)}{2!}u^2 + \frac{(-2)(-2-1)(-2-2)}{3!}u^3 + \dots$$

The above is valid for  $|u| < 1$  or  $|z-1| < 1$  or  $0 < z < 2$ . Simplifying the above gives

$$\frac{1}{(1-u)^2} = 1 - 2u + 3u^2 - 4u^3 + \dots$$

Substituting the above back into (1) gives

$$\begin{aligned}\frac{-1}{u} \frac{1}{(1+u)^2} &= \frac{-1}{u} (1 - 2u + 3u^2 - 4u^3 + \dots) \\ &= \frac{-1}{u} + 2 - 3u + 4u^2 - \dots\end{aligned}$$

But since  $u = z - 1$  then the above becomes

$$\frac{1}{z^2(1-z)} = \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - 5(z-1)^3 + \dots$$

We see from the above that the residue of  $f(z)$  is  $-1$  at  $z = 1$ .

#### In summary

1. Laurent series around  $z = 0$  is  $\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \dots$  which is valid inside disk centered at  $z = 0$  of radius 1
2. Laurent series around  $z = 1$  is  $\frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - 5(z-1)^3 + \dots$  which is valid inside disk centered at  $z = 1$  of radius 1

Note that there is another Laurent series expansions that can be found, which is for the region  $1 < |z| < \infty$ , which is outside a disk of radius 1 centered at  $z = 0$ . But the problem is asking for the above two expansions only.

## 4 Problem 4

Evaluate the integral  $\int_0^{\infty} \frac{dx}{1+x^4}$

solution

Since the integrand is even, then

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx$$

Now we consider the following contour

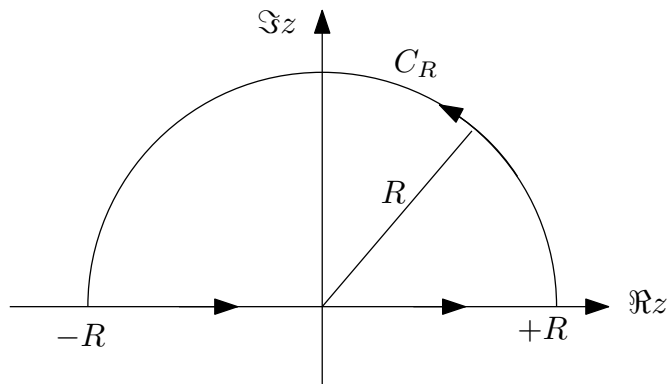


Figure 4: contour used for problem 4

Therefore

$$\oint_C f(z) dz = \left( \lim_{R \rightarrow \infty} \int_{-R}^0 f(x) dx + \lim_{\bar{R} \rightarrow \infty} \int_0^{\bar{R}} f(x) dx \right) + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned} \oint_C f(z) dz &= \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz \\ &= 2\pi i \sum \text{Residue} \end{aligned}$$

Where  $\sum \text{Residue}$  is sum of residues of  $\frac{1}{z^4+1}$  for poles that are inside the contour  $C$ . Therefore the above becomes

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx &= 2\pi i \sum \text{Residue} - \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz \\ \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx &= 2\pi i \sum \text{Residue} - \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{z^4+1} dz \end{aligned} \quad (1)$$

Now we will show that  $\lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{z^4+1} dz = 0$ . Since

$$\begin{aligned} \left| \int_{C_R} \frac{1}{z^4+1} dz \right| &\leq ML \\ &= |f(z)|_{\max} (\pi R) \end{aligned} \quad (2)$$

But

$$f(z) = \frac{1}{(z^2 - i)(z^2 + i)}$$

Hence, and since  $z = R e^{i\theta}$  then

$$|f(z)|_{\max} \leq \frac{1}{|z^2 - i|_{\min} |z^2 + i|_{\min}}$$

Using the inverse triangle inequality then  $|z^2 - i| \geq |z|^2 - 1$  and  $|z^2 + i| \geq |z|^2 - 1$ , and because



$|z| = R$  then the above becomes

$$\begin{aligned} |f(z)|_{\max} &\leq \frac{1}{(R^2 + 1)(R^2 - 1)} \\ &= \frac{1}{R^4 - 1} \end{aligned}$$

Therefore (2) becomes

$$\left| \int_{C_R} \frac{1}{z^4 + 1} dz \right| \leq \frac{\pi R}{R^4 - 1}$$

Then it is clear that as  $R \rightarrow \infty$  the above goes to zero since  $\lim_{R \rightarrow \infty} \frac{\pi R}{R^4 - 1} = \lim_{R \rightarrow \infty} \frac{\frac{\pi}{R^3}}{1 - \frac{1}{R^4}} = \frac{0}{1} = 0$ . Equation (1) now simplifies to

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx = 2\pi i \sum \text{Residue} \quad (2A)$$

We just now need to find the residues of  $\frac{1}{z^4 + 1}$  located in upper half plane. The zeros of the denominator  $z^4 + 1 = 0$  are at  $z = -1^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}}$ , then the first zero is at  $e^{i\frac{\pi}{4}}$ , and the second zero at  $e^{i(\frac{\pi}{4} + \frac{\pi}{2})} = e^{i(\frac{3}{4}\pi)}$  and the third zero at  $e^{i(\frac{3}{4}\pi + \frac{\pi}{2})} = e^{i(\frac{5}{4}\pi)}$  and the fourth zero at  $e^{i(\frac{5}{4}\pi + \frac{\pi}{2})} = e^{i\frac{7}{4}\pi}$ . Hence poles are at

$$\begin{aligned} z_1 &= e^{i\frac{\pi}{4}} \\ z_2 &= e^{i\frac{3}{4}\pi} \\ z_3 &= e^{i\frac{5}{4}\pi} \\ z_4 &= e^{i\frac{7}{4}\pi} \end{aligned}$$

Out of these only the first two are in upper half plane. Hence since these are simple poles, we can use the following to find the residues

$$\begin{aligned} \text{Residue}(z_1) &= \lim_{z \rightarrow z_1} (z - z_1) f(z) \\ &= \lim_{z \rightarrow z_1} (z - z_1) \frac{1}{z^4 - 1} \end{aligned}$$

Applying L'Hopitals rule, the above becomes

$$\begin{aligned} \text{Residue}(z_1) &= \lim_{z \rightarrow z_1} \frac{\frac{d}{dz}(z - z_1)}{\frac{d}{dz}(z^4 - 1)} \\ &= \lim_{z \rightarrow e^{i\frac{\pi}{4}}} \frac{1}{4z^3} \\ &= \frac{1}{4 \left( e^{i\frac{\pi}{4}} \right)^3} \\ &= \frac{1}{4e^{i\frac{3\pi}{4}}} \end{aligned}$$

Similarly for the other residue

$$\begin{aligned} \text{Residue}(z_2) &= \lim_{z \rightarrow z_2} (z - z_2) f(z) \\ &= \lim_{z \rightarrow z_2} (z - z_2) \frac{1}{z^4 - 1} \end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}
 \text{Residue}(z_2) &= \lim_{z \rightarrow e^{i\frac{3}{4}\pi}} \frac{1}{4z^3} \\
 &= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3} \\
 &= \frac{1}{4e^{i\frac{9\pi}{4}}} \\
 &= \frac{1}{4e^{i\frac{\pi}{4}}}
 \end{aligned}$$

Now that we found all the residues, then (2A) becomes

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx &= 2\pi i \left( \frac{1}{4e^{i\frac{3\pi}{4}}} + \frac{1}{4e^{i\frac{\pi}{4}}} \right) \\
 &= 2\pi i \left( \frac{\sqrt{2}}{4i} \right) \\
 &= \frac{1}{2} \sqrt{2} \pi
 \end{aligned}$$

But  $\int_0^{\infty} \frac{1}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$ , therefore

$$\begin{aligned}
 \int_0^{\infty} \frac{1}{x^4+1} dx &= \frac{1}{2} \left( \frac{1}{2} \sqrt{2} \pi \right) \\
 &= \frac{1}{4} \sqrt{2} \pi \\
 &= \frac{2}{4\sqrt{2}} \pi \\
 &= \frac{1}{2\sqrt{2}} \pi
 \end{aligned}$$

## 5 Problem 5

---

Evaluate the following integral  $\int_0^{2\pi} \frac{\sin^2 \theta}{a+b \cos \theta} d\theta$  when  $a > |b|$

solution

This is converted to complex integration by using  $z = re^{i\theta} = e^{i\theta}$  since  $r = 1$ . Therefore  $dz = ie^{i\theta} d\theta$  or

$$dz = izd\theta$$

In addition,

$$\begin{aligned} \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ &= \frac{z + z^{-1}}{2} \end{aligned}$$

And

$$\begin{aligned} \sin^2 \theta &= \frac{1}{2} - \frac{1}{2} \cos 2\theta \\ &= \frac{1}{2} - \frac{1}{2} \left( \frac{e^{i2\theta} + e^{-i2\theta}}{2} \right) \\ &= \frac{1}{2} - \frac{1}{2} \left( \frac{z^2 + z^{-2}}{2} \right) \\ &= \frac{1}{2} - \frac{1}{4} (z^2 + z^{-2}) \end{aligned}$$

Using all of the above back in the original integral gives

$$\begin{aligned} I &= \int_0^{2\pi} \frac{\sin^2 \theta}{a + b \cos \theta} d\theta \\ &= \oint_C \frac{\frac{1}{2} - \frac{1}{4} (z^2 + z^{-2}) dz}{a + b \left( \frac{z+z^{-1}}{2} \right) iz} \end{aligned}$$

Where the contour  $C$  is around the unit circle in counter clockwise direction. Therefore

$$\begin{aligned}
I &= \frac{1}{i} \oint_C \frac{\frac{1}{2} - \frac{1}{4} \left( z^2 + \frac{1}{z^2} \right) dz}{a + \frac{b}{2} \left( z + \frac{1}{z} \right)} \\
&= \frac{1}{i} \oint_C \frac{\frac{1}{2} - \frac{1}{4} \left( \frac{z^4+1}{z^2} \right) dz}{a + \frac{b}{2} \left( \frac{z^2+1}{z} \right)} \\
&= \frac{1}{i} \oint_C \frac{\frac{z^2}{2} - \frac{1}{4}(z^4+1)}{az + \frac{b}{2}(z^2+1)} \frac{dz}{z} \\
&= \frac{1}{i} \oint_C \frac{\frac{z^2}{2} - \frac{1}{4}(z^4+1)}{az + \frac{b}{2}(z^2+1)} \frac{dz}{z^2} \\
&= \frac{1}{i} \oint_C \frac{\frac{2z^2}{4} - \frac{1}{4}(z^4+1)}{\frac{2az}{2} + \frac{b}{2}(z^2+1)} \frac{dz}{z^2} \\
&= \frac{1}{i} \oint_C \frac{2z^2 - z^4 - 1}{4az + 2bz^2 + 2b} \frac{dz}{z^2} \\
&= \frac{1}{i} \oint_C \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{2bz^2 + 4az + 2b} dz \\
&= \frac{1}{i} \oint_C \frac{1}{z^2} \frac{\frac{1}{b}z^2 - \frac{1}{2b}z^4 - \frac{1}{2b}}{z^2 + \frac{2a}{b}z + 1} dz \\
&= \frac{1}{2bi} \oint_C \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left( z^2 + \frac{2a}{b}z + 1 \right)} dz
\end{aligned}$$

Now we can use the residue theorem. There is a pole at  $z = 0$  of order 2 and two poles which are the roots of  $z^2 + \frac{2a}{b}z + 1 = 0$ . Hence

$$I = 2\pi i \sum \text{Residue}$$

First we find the roots of  $z^2 + \frac{2a}{b}z + 1 = 0$  to see the location of the poles and if there are inside the unit circle or not. These are

$$\begin{aligned}
-\frac{b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac} &= -\frac{\frac{2a}{b}}{2} \pm \frac{1}{2} \sqrt{\left(\frac{2a}{b}\right)^2 - 4} \\
&= -\frac{a}{b} \pm \frac{1}{2} \sqrt{4\frac{a^2}{b^2} - 4} \\
&= -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}
\end{aligned}$$

Since  $a > |b|$  then  $\frac{a^2}{b^2} > 1$  and the value under the square root is real. Hence both roots are real. Roots are

$$\begin{aligned}
z_1 &= -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} \\
z_2 &= -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}
\end{aligned}$$

Now we need to decide the location of these poles. Let  $\frac{a}{b} = x$ . Where  $x > 1$  since  $a > |b|$ . Then the roots can be written as

$$\begin{aligned}
z_1 &= -x + \sqrt{x^2 - 1} \\
z_2 &= -x - \sqrt{x^2 - 1}
\end{aligned}$$

Now  $\sqrt{x^2 - 1}$  is always smaller than  $x$  but  $(\sqrt{x^2 - 1} - x)$  can not be larger than 1 in magnitude. Hence  $z_1$  will always be inside the unit disk. On the other hand,  $(\sqrt{x^2 - 1} + x)$  will always be larger than 1 in magnitude (the sign is not important, we just wanted to know which pole is smaller or larger than 1 only). Therefore we conclude that  $z_1$  is inside the unit disk and  $z_2$  is outside

Therefore, we need to find residue at  $z = 0$  and  $z = z_1$  and not at  $z = z_2$ . The function  $f(z)$  is from above is

$$\begin{aligned} f(z) &= \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b}z + 1\right)} \\ &= \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{(z - z_1)(z - z_2)} \end{aligned}$$

Residue of  $f(z)$  at  $z = 0$

Since this pole is of order  $n = 2$ , then

$$\begin{aligned} \text{Residue} &= \lim_{z \rightarrow 0} \frac{d^{n-1}}{dz^{n-1}} \left( \frac{(z - z_0)^n f(z)}{(n-1)!} \right) \\ &= \lim_{z \rightarrow 0} \frac{d}{dz} \left( (z - z_0)^2 \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b}z + 1\right)} \right) \\ &= \lim_{z \rightarrow 0} \frac{d}{dz} \left( z^2 \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b}z + 1\right)} \right) \\ &= \lim_{z \rightarrow 0} \frac{d}{dz} \left( \frac{2z^2 - z^4 - 1}{z^2 + \frac{2a}{b}z + 1} \right) \\ &= \lim_{z \rightarrow 0} \frac{(4z - 4z^3) \left(z^2 + \frac{2a}{b}z + 1\right) - (2z^2 - z^4 - 1) \left(2z + \frac{2a}{b}\right)}{\left(z^2 + \frac{2a}{b}z + 1\right)^2} \\ &= -(-1) \left(\frac{2a}{b}\right) \\ &= \frac{2a}{b} \end{aligned}$$

Residue at  $z_1 = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}$

Since this pole is of order 1, then the residue is

$$\begin{aligned} \text{Residue} &= \lim_{z \rightarrow z_1} \left( (z - z_1) f(z) \right) \\ &= \lim_{z \rightarrow z_1} \left( (z - z_1) \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{(z - z_1)(z - z_2)} \right) \\ &= \lim_{z \rightarrow z_1} \left( \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{z - z_2} \right) \\ &= \frac{1}{z_1^2} \frac{2z_1^2 - z_1^4 - 1}{z_1 - z_2} \\ &= \frac{-1 z_1^4 - 2z_1^2 + 1}{z_1^2 (z_1 - z_2)} \\ &= \frac{-1 (z_1^2 - 1)^2}{z_1^2 (z_1 - z_2)} \end{aligned}$$

Let  $\frac{a}{b} = x$ , hence  $\sqrt{\frac{a^2}{b^2} - 1} = \sqrt{x^2 - 1}$ . Therefore we can write  $z_1 = -x + \sqrt{x^2 - 1}$  and  $z_2 =$

$-x - \sqrt{x^2 - 1}$  and now the above becomes

$$\begin{aligned} \text{Residue} &= \frac{-1 \left( \left( (-x + \sqrt{x^2 - 1})^2 - 1 \right)^2 \right)}{\left( (-x + \sqrt{x^2 - 1})^2 \right) \left( (-x + \sqrt{x^2 - 1}) - (-x - \sqrt{x^2 - 1}) \right)} \\ &= \frac{-1 \left( \left( (-x + \sqrt{x^2 - 1})^2 - 1 \right)^2 \right)}{2 \left( (-x + \sqrt{x^2 - 1})^2 \right) \sqrt{x^2 - 1}} \end{aligned}$$

But  $\left( (-x + \sqrt{x^2 - 1})^2 \right) = x^2 + (x^2 - 1) - 2x\sqrt{x^2 - 1} = 2x^2 - 2x\sqrt{x^2 - 1} - 1$  and the above becomes

$$\begin{aligned} \text{Residue} &= \frac{-1 \left( 2x^2 - 2x\sqrt{x^2 - 1} - 1 - 1 \right)^2}{2 \left( 2x^2 - 2x\sqrt{x^2 - 1} - 1 \right) \sqrt{x^2 - 1}} \\ &= \frac{-1 \left( 2x^2 - 2x\sqrt{x^2 - 1} - 2 \right)^2}{2 \left( 2x^2 - 2x\sqrt{x^2 - 1} - 1 \right) \sqrt{x^2 - 1}} \\ &= \frac{-1 \cdot 4 \left( x^2 - x\sqrt{x^2 - 1} - 1 \right)^2}{2 \left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right) \sqrt{x^2 - 1}} \\ &= -2 \frac{\left( (x^2 - 1) - x\sqrt{x^2 - 1} \right)^2}{\left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right) \sqrt{x^2 - 1}} \end{aligned}$$

Expanding gives

$$\begin{aligned} \text{Residue} &= -2 \frac{\left( x^2 - 1 \right)^2 + \left( x\sqrt{x^2 - 1} \right)^2 - 2 \left( x^2 - 1 \right) x\sqrt{x^2 - 1}}{\left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right) \sqrt{x^2 - 1}} \\ &= -2 \frac{\left( x^2 - 1 \right)^2 + x^2 \left( x^2 - 1 \right) - 2 \left( x^2 - 1 \right) x\sqrt{x^2 - 1}}{\left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right) \sqrt{x^2 - 1}} \\ &= -2 \frac{\left( x^2 - 1 \right) \left( x^2 - 1 + x^2 - 2x\sqrt{x^2 - 1} \right)}{\left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right) \sqrt{x^2 - 1}} \\ &= -2 \frac{\left( x^2 - 1 \right) \left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right)}{\left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right) \sqrt{x^2 - 1}} \end{aligned}$$

Dividing numerator and denominator by  $(x^2 - 1)$

$$\begin{aligned} \text{Residue} &= -2 \frac{\sqrt{x^2 - 1} \left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right)}{\left( 2x^2 - 1 - 2x\sqrt{x^2 - 1} \right)} \\ &= -2\sqrt{x^2 - 1} \end{aligned}$$

Since  $x = \frac{a}{b}$  then the above becomes

$$\text{Residue} = -2\sqrt{\frac{a^2}{b^2} - 1}$$

We found all residues. The sum is

$$\sum \text{Residue} = \frac{2a}{b} - 2\sqrt{\frac{a^2}{b^2} - 1}$$

From the above we see now that

$$\begin{aligned} I &= \frac{1}{2bi} \oint_C \frac{1}{z^2} \frac{2z^2 - z^4 - 1}{\left(z^2 + \frac{2a}{b}z + 1\right)} dz \\ &= \frac{1}{2bi} (2\pi i \sum \text{Residue}) \\ &= \frac{1}{2bi} \left( 2\pi i \left( \frac{2a}{b} - 2\sqrt{\frac{a^2}{b^2} - 1} \right) \right) \\ &= \frac{\pi}{b} \left( \frac{2a}{b} - 2\sqrt{\frac{a^2}{b^2} - 1} \right) \\ &= \frac{\pi}{b} \left( \frac{2a}{b} - \frac{2}{b} \sqrt{a^2 - b^2} \right) \\ &= \frac{2\pi}{b^2} \left( a - \sqrt{a^2 - b^2} \right) \end{aligned}$$

Hence the final result is

$$\int_0^{2\pi} \frac{\sin^2 \theta}{a + b \cos \theta} d\theta = \frac{2\pi}{b^2} \left( a - \sqrt{a^2 - b^2} \right)$$