(1) $f(z) = z^{1/n} = r^{1/n} e^{i\theta/n}$ n sheets

R_1: $0 < \theta < 2\pi$

R_2: $2\pi < \theta < 4\pi$

$f(r, \theta + 2\pi n) = f(r, \theta)$

$\vdots \qquad \vdots$

R_n $2\pi(n-1) < \theta < 2\pi n$

(2) $w = \tan^{-1} z$ $\quad z = \tan w = \dfrac{\sin w}{\cos w} = \dfrac{1}{i} \dfrac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} =$

$= \dfrac{1}{i} \dfrac{e^{2iw} - 1}{e^{2iw} + 1} \Rightarrow iz(e^{i2w} + 1) = e^{i2w} - 1$

$e^{i2w}(1 - iz) = 1 + iz$

$e^{i2w} = \dfrac{1 + iZ}{1 - iZ} = \dfrac{i - z}{i + z}$

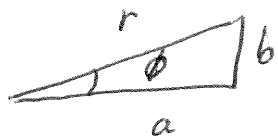
$2iw = \ln\left(\dfrac{i - z}{i + z}\right)$

$w = \dfrac{1}{2i} \ln\left(\dfrac{i - z}{i + z}\right)$

$$\boxed{w = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right)}$$

③ $w = \tan^{-1} z = \frac{i}{2}\left[\ln\left(x + i(y+1)\right) - \ln\left(-x + i(1-y)\right)\right]$

$$\ln(a+ib) = \ln r + i(\phi + 2\pi n)$$



$$r = \sqrt{a^2+b^2} \qquad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

any integer
↓

$$\ln\left(x + i(y+1)\right) = \frac{1}{2}\ln\left[x^2 + (1+y)^2\right] + i\left[\tan^{-1}\left(\frac{1+y}{x}\right) + 2\pi k\right]$$

$$\ln\left(-x + i(1-y)\right) = \ln\left(x + i(y-1)\right) + \underbrace{\ln(-1)}_{i\pi(2m+1)}$$

$$= \frac{1}{2}\ln\left[x^2 + (y-1)^2\right] + i\left[\tan^{-1}\left(\frac{y-1}{x}\right) + \pi(2m+1)\right]$$

↑
any integer
↓

$$\boxed{\tan^{-1} z = -\frac{1}{2}\tan^{-1}\left(\frac{y+1}{x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1-y}{x}\right) + \pi\left(n+\frac{1}{2}\right) + \frac{i}{4}\ln\left[\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right]}$$

The principle value of $\tan^{-1}$ is usually taken to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

(4) $$u = \ln r = \frac{1}{2}\ln r^2 = \frac{1}{2}\ln(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{x}{r^2} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4}$$

$$\frac{\partial u}{\partial y} = \frac{y}{r^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2} - \frac{2y^2}{r^4}$$

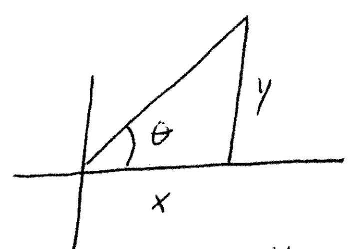
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r^2} - \frac{2(x^2+y^2)}{r^4} = 0$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \Rightarrow v = \tan^{-1}\left(\frac{y}{x}\right) + \phi(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2}$$

$$= \frac{-y}{x^2+y^2} + \phi'(x) \Rightarrow \phi = C \quad \text{constant}$$

$$\boxed{v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + C}$$



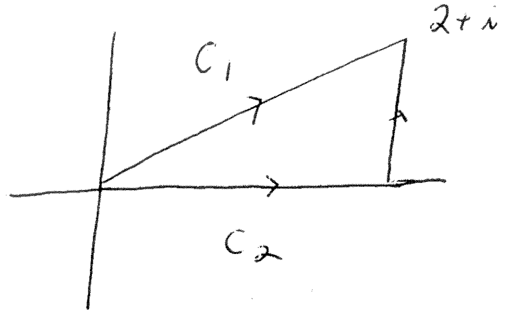
$$\tan\theta = \frac{y}{x}$$

$$\boxed{v = \theta + C}$$

Polar coordinates $$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r}$$

(5)



Along C_1 $\quad x = 2t \quad y = t \qquad 0 \le t \le 1$

$$\int_{C_1} e^z\,dz = \int_{C_1} e^{x+iy}(dx + i\,dy) = \int_0^1 e^{(2+i)t}(2+i)\,dt$$

$$= e^{(2+i)t}\Big|_0^1 = e^2 e^i - 1 = e^2(\cos(1) + i\sin(1)) - 1$$

Along C_2 $\qquad \int_{C_2} e^z\,dz = \int_0^2 e^x\,dx + i\int_0^1 e^2 e^{iy}\,dy$

$$= (e^2 - 1) + i e^2 \frac{e^i - 1}{i} = e^2 e^i - 1$$

Same answer.