Study notes, exam 1
MATH 4567 Applied Fourier Analysis
Spring 2019
University of Minnesota, Twin Cities

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November 2, 2019 Compiled on November 2, 2019 at 9:53pm [public]

Contents

1 Chapter 1, sections 1-8 (Fourier series)

1.1 section 1

definition of left and right limits. definition of piecewise continuous function.

1.2 section 2

definition of Fourier cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{T}x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$ for $0 < x < \pi$.

1.3 section 3

Examples of Fourier cosine series

1.4 section 4

definition of Fourier sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{2\pi}{T}x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(nx)$ for $0 < x < \pi$.

1.5 section 5

Examples of Fourier sine series

1.6 section 6

Fourier series For period $T = 2\pi$

$$\begin{aligned} f(x) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{T}x\right) + b_n \sin\left(n\frac{2\pi}{T}x\right) \qquad -\pi < x < \pi \\ &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \end{aligned}$$

Where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 0,1,2,\cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n = 1,2,\cdots \end{aligned}$$

If $f(x)$ is even then $b_n = 0$ and if $f(x)$ is odd, then $a_n = 0$.

1.7 section 7

Fourier series examples.

1.8 section 8 Adoption to different regions

Shows how F.S. on $-L < x < L$ can be obtained from know F.S. on $-\pi < x < \pi$. Not clear why example 2 on page 22 replaces $a = \frac{1}{\pi}$.

2 Chapter 2, sections 9-20 (Convergence of Fourier series)

2.1 section 9 (one sided derivatives)

$$
\begin{aligned}
f'_{+}(x_0) &= \lim_{\substack{x\to x_0\\ x>x_0}} \frac{f(x) - f\left(x_0^{+}\right)}{x - x_0} \\
f'_{-}(x_0) &= \lim_{\substack{x\to x_0\\ x>x_0}} \frac{f(x) - f\left(x_0^{+}\right)}{x - x_0}
\end{aligned}
$$

Smooth function is one who is continuous and its derivative is also continuous. For example $f(x) = x^2$ is smooth, but $f(x) = |x|$ is not smooth.

Piecewise smooth function is one which $f(x)$ and $f'(x)$ are piecewise continuous.

2.2 section 10 (Properties of Fourier coefficients)

Bessel's inequalities

$$
\begin{aligned}
\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 &\leq \frac{2}{\pi} \int_0^{\pi} \left[f(x)\right]^2 dx \\
\lim_{n\to\infty} a_n &= 0 \\
\sum_{n=1}^{\infty} b_n^2 &\leq \frac{2}{\pi} \int_0^{\pi} \left[f(x)\right]^2 dx \\
\lim_{n\to\infty} b_n &= 0
\end{aligned}
$$

2.3 section 11 (Two Lemmas)

Lemma 1 If $f(x)$ is P.W.C. on $0 < x < \pi$ then

$$\lim_{N\to\infty}\int_0^{\pi} f(x)\sin\left(\left(N+\frac{1}{2}\right)x\right)dx = 0$$

Lemma 2 If $g(x)$ is P.W.C. on $0 < x < \pi$ and that $g'_+(0)$ exist, then

$$\lim_{N\to\infty}\int_0^{\pi} g(x)\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}}dx = \frac{\pi}{2}g(0^+)$$

Where $\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}}$ is called the Dirichlet kernel $D_N(x)$.

$$D_N(x) = \frac{1}{2}+\sum_{n=1}^{N}\cos(nx)$$

$$D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\frac{x}{2}}$$

$$\int_0^{\pi} D_N(x)\,dx = \frac{\pi}{2}$$

2.4 Section 12 (Fourier theorem)

If $f(x)$ is P.W.C. on $-\pi < x < \pi$ and $f(x)$ is periodic on all of x with period 2π then at each x where $f'_+(x)$ and $f'_-(x)$ both exist, then $f(x)$ converges to the average of $f(x)$ at x which is $\frac{f(x^+)+f(x^-)}{2}$. Proof is long.

2.5 Section 13 (Related Fourier theorem)

Nothing new here. Seems same as last one. If $f(x)$ is PWC and $f'(x)$ is PWC, and $f(x)$ is periodic, then F.S. of $f(x)$ converges to mean of $f(x)$ at each point x.

2.6 Section 14 (Examples)

Examples on the Fourier theorem

2.7 Section 15 (Convergence on other intervals)

Nothing new here.

2.8 Section 16 (Lemma on absolute and uniform convergence)

If $f(x)$ is continuous on $-\pi < x < \pi$ (notice it has to be continuous, not PWC) and if $f(-\pi) = f(\pi)$ and $f'(x)$ is PWC on $-\pi < x < \pi$ then

$$\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

converges. Proof is given. And

$$\sum_{n=1}^{N} \alpha_n^2 + \beta_n^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left[f'(x)\right]^2 dx \qquad N = 1, 2, 3, \cdots$$

Where

$$\begin{aligned} f'(x) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \beta_n \sin(nx) \\ \alpha_0 &= 0 \\ \alpha_n &= nb_n \\ \beta_n &= na_n \end{aligned}$$

2.9 Section 17 (Absolute and uniform convergence of Fourier series)

M test is used to check if series is U.C. (uniform convergent). If we can find $\sum_{n=1}^{\infty} M_n$ which is convergent and M_n is positive constant, and where $\left|f_n(x)\right| \leq M_n$ for each n in $a < x < b$, then series $\sum_{n=1}^{\infty} f_n(x)$ is U.C.

<u>Theorem</u> If $f(x)$ is continuous on $-\pi \leq x \leq \pi$ and $f(-\pi) = f(\pi)$ and $f'(x)$ is PWC, then $f(x)$ both absolutely and uniformly convergent,

2.10 Section 18 (Gibbs phenomenon)

Not on exam.

2.11 Section 19 (Differentiation of Fourier series)

Same conditions as section 17 theorem. If $f(x)$ is continuous on $-\pi \leq x \leq \pi$ and $f(-\pi) = f(\pi)$ and $f'(x)$ is PWC, then F.S. of $f(x)$ can be differentiated term by term.

2.12 Section 20 (Integration of Fourier series)

As long as $f(x)$ is PWC, we can integrate F.S. term by term.

3 Chapter 3 (partial differential equations of physics)

3.1 Section 21 (Linear boundary value problem)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

And definitions.

3.2 Section 22 (1D heat PDE)

Flux is $\Phi = -K\frac{du}{dn}$ where K is thermal conductivity. Flux is amount of heat passing in normal direction per unit area in one second. Derivation of heat PDE

$$u_t = ku_{xx}$$

where k is thermal diffusivity $k = \frac{K}{\sigma\delta}$ where σ is specific heat and δ is density of material.

3.3 Section 23 (Related heat equations)

Nothing much here.

3.4 Section 24 (Laplace in cylindrical and spherical)

Just need to know the equations. Will be given in exam.

3.5 Section 25 (Derivations)

Not in exam

3.6 Section 26 (Boundary conditions)

Just need to know Neumann and Dirichlet.

3.7 Section 27 (Duhamel's principle)

Do not think this will be on exam.

3.8 Section 28 (Vibrating string)

Derivation of $y_{tt} = a^2 y_{xx}$ using physics. Will not be on exam.

3.9 Section 29 (Vibrations of bars and membranes)

Generalization of section 28.

3.10 Section 30 (General solution to wave equation)

To derive solution to $y_{tt} = a^2 y_{xx}$, use $u = x + at, v = x - at$ and the PDE becomes $y_{uv} = 0$ which has solution $y = \Phi(u) + \Psi(v)$ or

$$y(x,t) = \Phi(x+at) + \Psi(x-at)$$

Where initial conditions are $y(x,0) = f(x), y_t(x,0) = g(x)$ then the solution becomes

$$y(x,t) = \frac{1}{2}\left(f(x+at) + f(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds$$

3.11 Section 31 (Types of equations and boundary conditions)

1. Hyperbolic $B^2 - 4AC > 0$
2. Elliptic $B^2 - 4AC < 0$
3. parabolic $B^2 - 4AC = 0$

4 Chapter 4 (The Fourier method)

4.1 Section 32 (linear operators)

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2$$

4.2 Section 33 (Principle of superposition)

Suppose each function u_i satisfies a linear homogeneous differential equation or boundary value problem $Lu = 0$, then $\sum_{n=1}^{\infty} u_n$ also satisfies the same equation.

4.3 Section 34 (Examples of Principle of superposition)

Some examples. Go over.

4.4 Section 35 (Eigenvalues and eigenfunctions)

Show how to solve $X'' + \lambda X = 0$ for different boundary conditions.

4.5 Section 36 (A temperature problem)

Applying Eigenvalues and eigenfunctions to heat PDE on rod.

4.6 Section 37 (Vibrating string)

Applying Eigenvalues and eigenfunctions to wave PDE On string $u_{tt} = a^2 u_{xx}$ with fixed on ends and have initial conditions.

4.7 Section 38 (Historical development)

Not on exam