1. (40 points)

With the aid of the expansion

i.e

So

$$\pi - x = 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad 0 < x < \pi,$$

solve the following problem.

$$u_t(x,t) = u_{xx}(x,t) + t\,(\pi - x), \; 0 < x < \pi, \; t > 0;$$

Source.

$$u(0,t) = 0, \; u(\pi,t) = 0; \; u(x,0) = 0.$$

2. (20 points) Verify that all of the conditions of the Fourier sine integral representation are satisfied by the function f defined by

$$f(x) := \begin{cases} x & \text{when } 0 \leq x \leq 1 \\ 2 - x & \text{when } 1 < x \leq 2 \\ 0 & \text{when } x < 0 \text{ or } x > 2 \end{cases}$$

and show that for $0 < x < \infty$,

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{(2 \sin \alpha - \sin 2\alpha) \sin \alpha x}{\alpha^2} \, d\alpha.$$

3. (40 points)

Find the bounded harmonic function $u(x, y)$ in the semi-infinite strip $0 < x < \infty$, $0 < y < 1$ that satisfies the conditions $u(x, 0) = 0$, $u(0, y) = 0$ and $u(x, 1) = f(x)$, where $f(x)$ is the function given in problem 2.