HW 10 MATH 4567 Applied Fourier Analysis Spring 2019 University of Minnesota, Twin Cities

Nasser M. Abbasi

November 2, 2019 Compiled on November 2, 2019 at 9:52pm [public]

Contents

1	Section 69, Problem 1	2
	1.1 Part (a) .	
2	Section 72, Problem 3	5
3	Section 72, Problem 6	7
4	Section 72, Problem 9	9

1 Section 69, Problem 1

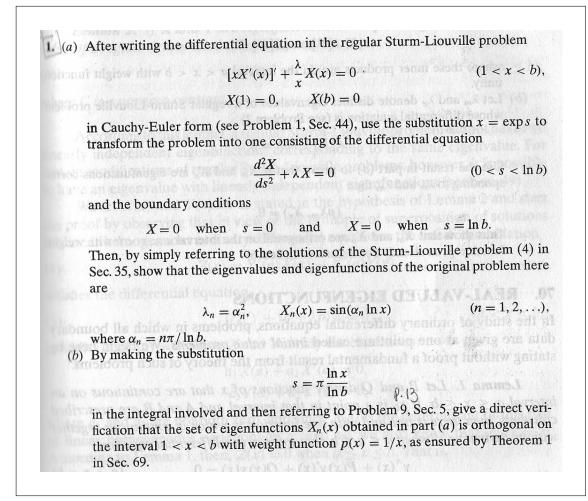


Figure 1: Problem statement

Solution

1.1 Part (a)

$$X'(x) + xX''(x) + \frac{\lambda}{x}X(x) = 0$$

x²X''(x) + xX'(x) + \lambda X(x) = 0 (1)

To transform the above to $X''(s) + \lambda X(s) = 0$, let $x = e^s$. Therefore $\frac{dx}{ds} = e^s$ or $\frac{ds}{dx} = e^{-s}$. Now

$$\frac{dX}{dx} = \frac{dX}{ds}\frac{ds}{dx}$$
$$= \frac{dX}{ds}e^{-s}$$
(2)

And

$$\frac{d^2 X}{dx^2} = \frac{d}{dx} \left(\frac{dX}{dx} \right)$$
$$= \frac{d}{dx} \left(\frac{dX}{ds} e^{-s} \right)$$

Hence, by product rule

$$\frac{d^{2}X}{dx^{2}} = \frac{d^{2}X}{ds^{2}}\frac{ds}{dx}e^{-s} + \frac{dX}{ds}\frac{d}{dx}(e^{-s})
= \frac{d^{2}X}{ds^{2}}e^{-s}e^{-s} + \frac{dX}{ds}\frac{d}{ds}(e^{-s})\frac{ds}{dx}
= \frac{d^{2}X}{ds^{2}}e^{-2s} + \frac{dX}{ds}(-e^{-s})(e^{-s})
= e^{-2s}\frac{d^{2}X}{ds^{2}} - e^{-2s}\frac{dX}{ds}$$
(3)

Substituting (2,3) back into (1) gives

$$x^{2}\left(e^{-2s}\frac{d^{2}X}{ds^{2}} - e^{-2s}\frac{dX}{ds}\right) + x\left(\frac{dX}{ds}e^{-s}\right) + \lambda X = 0$$

But $x = e^s$ and the above simplifies to

$$e^{2s} \left(e^{-2s} \frac{d^2 X}{ds^2} - e^{-2s} \frac{dX}{ds} \right) + e^s \left(\frac{dX}{ds} e^{-s} \right) + \lambda X = 0$$
$$\frac{d^2 X}{ds^2} - \frac{dX}{ds} + \frac{dX}{ds} + \lambda X = 0$$
$$\frac{d^2 X(s)}{ds^2} + \lambda X(s) = 0$$

When X(1) = 0, which means when x = 1, and since $x = e^s$, then when s = 0. Hence X(1) = 0 becomes X(0) = 0. And when x = b, then $s = \ln(b)$. Hence the second condition becomes $X(\ln(b)) = 0$. Therefore the new B.C. are

$$X(0) = 0$$
$$X(\ln(b)) = 0$$

By referring to problem (4) in section 35 we see that the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{c}\right)^2$$

Where here $c = \ln(b)$. Hence

$$\lambda_n = \left(\frac{n\pi}{\ln(b)}\right)^2 \qquad n = 1, 2, 3, \cdots$$
$$= \alpha_n^2$$

Where $\alpha_n = \frac{n\pi}{\ln(b)}$. And the eigenfunctions are, per section 35

$$X_n(s) = \sin\left(\alpha_n s\right)$$

In terms of x, the eigenfunctions become

$$X_n(s) = \sin\left(\alpha_n \ln x\right)$$

1.2 Part (b)

$$\langle X_n(x), X_m(x) \rangle = \int_1^b \sin(\alpha_n \ln x) \sin(\alpha_m \ln x) p(x) dx$$

But from $(xX'(x))' + \frac{\lambda}{x}X(x) = 0$ and comparing this to $(rX')' + (\lambda p + q)X = 0$, we see that r(x) = x and q = 0 and $p = \frac{1}{x}$. Hence the above integral becomes

$$\langle X_n(x), X_m(x) \rangle = \int_1^b \frac{1}{x} \sin(\alpha_n \ln x) \sin(\alpha_m \ln x) dx$$

Let $s = \frac{\ln x}{\ln b}\pi$. Then $\frac{ds}{dx} = \frac{1}{x}\frac{\pi}{\ln b}$ or $dx = \frac{x}{\pi}\ln(b) ds$. When x = 1 then s = 0 and when x = b then $s = \pi$. Hence the above integral becomes

$$\langle X_n(x), X_m(x) \rangle = \int_{s=0}^{s=\pi} \frac{1}{x} \sin\left(\alpha_n \frac{s\ln b}{\pi}\right) \sin\left(\alpha_m \frac{s\ln b}{\pi}\right) \left(\frac{x}{\pi}\ln(b)\,ds\right)$$
$$= \frac{1}{\pi} \ln(b) \int_0^{\pi} \sin\left(\alpha_n \frac{s\ln b}{\pi}\right) \sin\left(\alpha_m \frac{s\ln b}{\pi}\right) ds$$

But $\alpha_n = \frac{n\pi}{\ln(b)}$ and $\alpha_m = \frac{m\pi}{\ln(b)}$, therefore the above becomes

$$\langle X_n(x), X_m(x) \rangle = \frac{1}{\pi} \ln(b) \int_0^\pi \sin\left(\frac{n\pi}{\ln(b)} \frac{s \ln b}{\pi}\right) \sin\left(\frac{m\pi}{\ln(b)} \frac{s \ln b}{\pi}\right) ds$$

$$= \frac{1}{\pi} \ln(b) \int_0^\pi \sin(ns) \sin(ms) ds$$
(1)

Referring to Problem 9., section 5 which says that

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = 0 \end{cases}$$

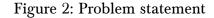
Applying this to (1) shows that

$$\langle X_n(x), X_m(x) \rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = 0 \end{cases}$$

Hence $X_n(x)$ and $X_m(x)$ are orthogonal, since this is the definition of orthogonality.

(3.)
$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(c) = 0.$$

Answer: $\lambda_n = \alpha_n^2, \quad \phi_n(x) = \sqrt{\frac{2}{c}} \cos \alpha_n x \quad (n = 1, 2, ...); \quad \alpha_n = \frac{(2n-1)\pi}{2c}.$



Solution

Solve for eigenvalues and normalized eigenfunctions.

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X(c) = 0$$

Writing the boundary conditions in SL standard form

$$a_1 X(0) + a_2 X'(0) = 0$$

 $b_1 X(c) + b_2 X'(c) = 0$

Shows that $a_1 = 0$, $a_2 = 1$ and $b_1 = 1$, $b_2 = 0$. Therefore $a_1a_2 = 0$ and $b_1b_2 = 0$. But we know that if $a_1a_2 \ge 0$ and $b_1b_2 \ge 0$, then $\lambda > 0$ is only possible eigenvalues. Let $\lambda_n = \alpha_n^2$. $\alpha > 0$. Hence the solution to the ODE is

$$X_n(x) = A\cos(\alpha_n x) + B\sin(\alpha_n x)$$

$$X'_n(x) = -A\alpha_n \sin(\alpha_n x) + B\alpha_n \cos(\alpha_n x)$$

First B.C X'(0) = 0 gives

 $0 = B\alpha_n$

Which implies B = 0. Hence the solution now becomes $X_n(x) = A \cos(\alpha_n x)$. For the second BC

$$0 = A \cos (\alpha_n c)$$
$$0 = \cos (\alpha_n c)$$

Which implies

$$\alpha_n c = \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}, \cdots$$

= $(2n-1)\frac{\pi}{2}$ $n = 1, 2, 3, \cdots$

Hence

$$\alpha_n = \frac{(2n-1)}{c} \frac{\pi}{2}$$
 $n = 1, 2, 3, \cdots$

And the corresponding eigenfunctions are

$$X_n(x) = \cos(\alpha_n x)$$
$$= \cos\left(\frac{(2n-1)}{c}\frac{\pi}{2}x\right)$$

To find the normalized $X_n(x)$ which we call it $\phi_n(x)$, then by definition

$$\phi_n\left(x\right) = \frac{X_n\left(x\right)}{\|X_n\left(x\right)\|}$$

But

$$||X_n(x)||^2 = \int_0^c p(x) X_n^2(x) dx$$

Comparing the ODE $X'' + \lambda X = 0$ to $(rX')' + (\lambda p + q)X = 0$, we see that r(x) = 1 and q = 0 and p = 1. Hence the above becomes

$$||X_n(x)||^2 = \int_0^c \cos^2(\alpha_n x) dx$$
$$= \frac{c}{2}$$

Therefore $||X_n(x)|| = \sqrt{\frac{c}{2}}$ which shows that

$$\phi_n(x) = \frac{X_n(x)}{\sqrt{\frac{c}{2}}}$$
$$= \sqrt{\frac{2}{c}} \cos(\alpha_n x)$$

where

$$\alpha_n = \frac{(2n-1)}{c} \frac{\pi}{2}$$
 $n = 1, 2, 3, \cdots$

Which is what required to show.

6. In Problem 1(a), Sec. 69, the eigenvalues and eigenfunctions of the Sturm-Liouville problem $(xX')' + \frac{\lambda}{x}X = 0, \qquad X(1) = 0, \qquad X(b) = 0$ were found to be $\lambda_n = \alpha_n^2, \qquad X_n(x) = \sin(\alpha_n \ln x) \qquad (n = 1, 2, ...),$ where $\alpha_n = n\pi/\ln b$. Show that the *normalized* eigenfunctions are $\phi_n(x) = \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln x) \qquad (n = 1, 2, ...),$ Suggestion: The integral that arises can be evaluated by making the substitution $s = \pi \frac{\ln x}{\ln b}$ and then referring to the integration formula established in Problem 9, Sec. 5.

Figure 3: Problem statement

Solution

$$X_n(x) = \sin(\alpha_n \ln x)$$
$$\alpha_n = \frac{n\pi}{\ln b} \qquad n = 1, 2, 3, \cdots$$

The normalized eigenfunction is given by

$$\phi_n\left(x\right) = \frac{X_n\left(x\right)}{\|X_n\left(x\right)\|}$$

But

$$||X_n(x)||^2 = \int_1^b p(x) X_n^2(x) dx$$

Comparing the ODE $(xX')' + \frac{\lambda}{x}X = 0$ to $(rX')' + (\lambda p + q)X = 0$, we see that r(x) = x and q = 0 and $p = \frac{1}{x}$. Hence the above becomes

$$||X_n(x)||^2 = \int_1^b \frac{1}{x} \sin^2(\alpha_n \ln x) \, dx$$

Let $s = \frac{\ln x}{\ln b}\pi$. Then $\frac{ds}{dx} = \frac{1}{x}\frac{\pi}{\ln b}$ or $dx = \frac{x}{\pi}\ln(b) ds$. When x = 1 then s = 0 and when x = b then $s = \pi$. Hence the above integral becomes

$$\left\|X_{n}\left(x\right)\right\|^{2} = \int_{s=0}^{s=\pi} \frac{1}{x} \sin^{2}\left(\alpha_{n} \frac{s \ln b}{\pi}\right) \left(\frac{x}{\pi} \ln\left(b\right) ds\right)$$
$$= \frac{1}{\pi} \ln\left(b\right) \int_{0}^{\pi} \sin^{2}\left(\alpha_{n} \frac{s \ln b}{\pi}\right) ds$$

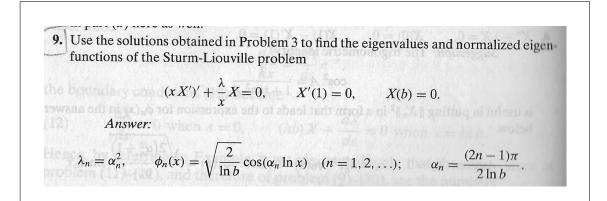
But $\alpha_n = \frac{n\pi}{\ln(b)}$ therefore the above becomes

$$\begin{split} \|X_n(x)\|^2 &= \frac{1}{\pi} \ln{(b)} \int_0^\pi \sin^2\left(\frac{n\pi}{\ln{(b)}} \frac{s\ln{b}}{\pi}\right) ds \\ &= \frac{1}{\pi} \ln{(b)} \int_0^\pi \sin^2{(ns)} \, ds \\ &= \frac{1}{\pi} \ln{(b)} \int_0^\pi \frac{1}{2} - \frac{1}{2} \cos{(2ns)} \, ds \\ &= \frac{1}{\pi} \ln{(b)} \left(\frac{\pi}{2} - \frac{1}{2} \sin\left(\frac{2ns}{2n}\right)_0^\pi\right) \\ &= \frac{1}{\pi} \ln{(b)} \left(\frac{\pi}{2} - \frac{1}{2} \sin{(s)}_0^\pi\right) \\ &= \frac{1}{2} \ln{(b)} \end{split}$$

Hence

$$\phi_n(x) = \frac{\sin(\alpha_n \ln x)}{\sqrt{\frac{1}{2} \ln(b)}}$$
$$= \sqrt{\frac{2}{\ln(b)}} \sin(\alpha_n \ln x)$$

Which is what required to show.





solution

From problem section 69 problem 1, we know that $(xX'(x))' + \frac{\lambda}{x}X(x) = 0$ can be transformed to $X''(s) + \lambda X(s) = 0$ using $x = e^s$. With boundary conditions in *s* found as follows. When x = 1 then s = 0 and when x = b then $s = \ln b$. Hence we obtain the SL problem

$$X''(s) + \lambda X(s) = 0$$
(1)

$$X'(0) = 0$$
(1)

$$X(\ln b) = 0$$

But problem 3 is

$$X'' + \lambda X = 0$$

$$X'(0) = 0$$

$$X(c) = 0$$
(2)

And it had the solution

$$\phi_n(x) = \sqrt{\frac{2}{c}} \cos\left(\alpha_n x\right)$$

where

$$\alpha_n = \frac{(2n-1)}{c} \frac{\pi}{2}$$
 $n = 1, 2, 3, \cdots$

By comparing (2) and (1) we see it is the same problem, except $c \rightarrow \ln b$. Hence the solution to (2) is the same as the solution in (1) but with *c* replaced by $\ln b$. Hence the solution is

$$\phi_n(s) = \sqrt{\frac{2}{\ln b}} \cos(\alpha_n s)$$
$$\alpha_n = \frac{(2n-1)}{\ln b} \frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

But $s = \ln x$, hence the above becomes

$$\phi_n(x) = \sqrt{\frac{2}{\ln b}} \cos(\alpha_n \ln x)$$
$$\alpha_n = \frac{(2n-1)}{\ln b} \frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

Which is what required to show.