HW 10
MATH 4567 Applied Fourier Analysis
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Nasser M. Abbasi

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1 Section 69, Problem 1

1. (*a*) After writing the differential equation in the regular Sturm-Liouville problem

$$[xX'(x)]' + \frac{\lambda}{x}X(x) = 0 \qquad (1 < x < b),$$

$$X(1) = 0, \qquad X(b) = 0$$

in Cauchy-Euler form (see Problem 1, Sec. 44), use the substitution $x = \exp s$ to transform the problem into one consisting of the differential equation

$$\frac{d^2X}{ds^2} + \lambda X = 0 \qquad (0 < s < \ln b)$$

and the boundary conditions

$$X = 0 \quad \text{when} \quad s = 0 \qquad \text{and} \qquad X = 0 \quad \text{when} \quad s = \ln b.$$

Then, by simply referring to the solutions of the Sturm-Liouville problem (4) in Sec. 35, show that the eigenvalues and eigenfunctions of the original problem here are

$$\lambda_n = \alpha_n^2, \qquad X_n(x) = \sin(\alpha_n \ln x) \qquad (n = 1, 2, \ldots),$$

where $\alpha_n = n\pi / \ln b$.

(*b*) By making the substitution

$$s = \pi \frac{\ln x}{\ln b}$$

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in the integral involved and then referring to Problem 9, Sec. 5, give a direct verification that the set of eigenfunctions $X_n(x)$ obtained in part (*a*) is orthogonal on the interval $1 < x < b$ with weight function $p(x) = 1/x$, as ensured by Theorem 1 in Sec. 69.

Figure 1: Problem statement

Solution

1.1 Part (a)

$$X'(x) + xX''(x) + \frac{\lambda}{x}X(x) = 0$$
$$x^2X''(x) + xX'(x) + \lambda X(x) = 0 \tag{1}$$

To transform the above to $X''(s) + \lambda X(s) = 0$, let $x = e^s$. Therefore $\frac{dx}{ds} = e^s$ or $\frac{ds}{dx} = e^{-s}$. Now

$$\frac{dX}{dx} = \frac{dX}{ds}\frac{ds}{dx}$$
$$= \frac{dX}{ds}e^{-s} \tag{2}$$

And

$$\frac{d^2X}{dx^2} = \frac{d}{dx}\left(\frac{dX}{dx}\right)$$
$$= \frac{d}{dx}\left(\frac{dX}{ds}e^{-s}\right)$$

Hence, by product rule

$$\frac{d^2X}{dx^2} = \frac{d^2X}{ds^2}\frac{ds}{dx}e^{-s} + \frac{dX}{ds}\frac{d}{dx}\left(e^{-s}\right)$$
$$= \frac{d^2X}{ds^2}e^{-s}e^{-s} + \frac{dX}{ds}\frac{d}{ds}\left(e^{-s}\right)\frac{ds}{dx}$$
$$= \frac{d^2X}{ds^2}e^{-2s} + \frac{dX}{ds}\left(-e^{-s}\right)\left(e^{-s}\right)$$
$$= e^{-2s}\frac{d^2X}{ds^2} - e^{-2s}\frac{dX}{ds} \tag{3}$$

Substituting (2,3) back into (1) gives

$$x^2\left(e^{-2s}\frac{d^2X}{ds^2} - e^{-2s}\frac{dX}{ds}\right) + x\left(\frac{dX}{ds}e^{-s}\right) + \lambda X = 0$$

But $x = e^s$ and the above simplifies to

$$e^{2s}\left(e^{-2s}\frac{d^2X}{ds^2} - e^{-2s}\frac{dX}{ds}\right) + e^s\left(\frac{dX}{ds}e^{-s}\right) + \lambda X = 0$$
$$\frac{d^2X}{ds^2} - \frac{dX}{ds} + \frac{dX}{ds} + \lambda X = 0$$
$$\frac{d^2X(s)}{ds^2} + \lambda X(s) = 0$$

When $X(1) = 0$, which means when $x = 1$, and since $x = e^s$, then when $s = 0$. Hence $X(1) = 0$ becomes $X(0) = 0$. And when $x = b$, then $s = \ln(b)$. Hence the second condition becomes $X(\ln(b)) = 0$. Therefore the new B.C. are

$$X(0) = 0$$
$$X(\ln(b)) = 0$$

By referring to problem (4) in section 35 we see that the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{c}\right)^2$$

Where here $c = \ln(b)$. Hence

$$\lambda_n = \left(\frac{n\pi}{\ln(b)}\right)^2 \qquad n = 1, 2, 3, \cdots$$
$$= \alpha_n^2$$

Where $\alpha_n = \frac{n\pi}{\ln(b)}$. And the eigenfunctions are, per section 35

$$X_n(s) = \sin(\alpha_n s)$$

In terms of x, the eigenfunctions become

$$X_n(s) = \sin(\alpha_n \ln x)$$

1.2 Part (b)

$$\langle X_n(x), X_m(x)\rangle = \int_1^b \sin(\alpha_n \ln x)\sin(\alpha_m \ln x)\, p(x)\, dx$$

But from $(xX'(x))' + \frac{\lambda}{x}X(x) = 0$ and comparing this to $(rX')' + (\lambda p + q)X = 0$, we see that $r(x) = x$ and $q = 0$ and $p = \frac{1}{x}$. Hence the above integral becomes

$$\langle X_n(x), X_m(x)\rangle = \int_1^b \frac{1}{x}\sin(\alpha_n \ln x)\sin(\alpha_m \ln x)\, dx$$

Let $s = \frac{\ln x}{\ln b}\pi$. Then $\frac{ds}{dx} = \frac{1}{x}\frac{\pi}{\ln b}$ or $dx = \frac{x}{\pi}\ln(b)\, ds$. When $x = 1$ then $s = 0$ and when $x = b$ then $s = \pi$. Hence the above integral becomes

$$\begin{aligned}
\langle X_n(x), X_m(x)\rangle &= \int_{s=0}^{s=\pi} \frac{1}{x}\sin\left(\alpha_n \frac{s\ln b}{\pi}\right)\sin\left(\alpha_m \frac{s\ln b}{\pi}\right)\left(\frac{x}{\pi}\ln(b)\, ds\right)\\
&= \frac{1}{\pi}\ln(b)\int_0^\pi \sin\left(\alpha_n \frac{s\ln b}{\pi}\right)\sin\left(\alpha_m \frac{s\ln b}{\pi}\right) ds
\end{aligned}$$

But $\alpha_n = \frac{n\pi}{\ln(b)}$ and $\alpha_m = \frac{m\pi}{\ln(b)}$, therefore the above becomes

$$\begin{aligned}
\langle X_n(x), X_m(x)\rangle &= \frac{1}{\pi}\ln(b)\int_0^\pi \sin\left(\frac{n\pi}{\ln(b)}\frac{s\ln b}{\pi}\right)\sin\left(\frac{m\pi}{\ln(b)}\frac{s\ln b}{\pi}\right) ds\\
&= \frac{1}{\pi}\ln(b)\int_0^\pi \sin(ns)\sin(ms)\, ds \qquad (1)
\end{aligned}$$

Referring to Problem 9., section 5 which says that

$$\int_0^\pi \sin(nx)\sin(mx)\, dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = 0 \end{cases}$$

Applying this to (1) shows that

$$\langle X_n(x), X_m(x)\rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = 0 \end{cases}$$

Hence $X_n(x)$ and $X_m(x)$ are orthogonal, since this is the definition of orthogonality.

2 Section 72, Problem 3

3. $X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(c) = 0.$

Answer: $\lambda_n = \alpha_n^2, \quad \phi_n(x) = \sqrt{\frac{2}{c}} \cos \alpha_n x \quad (n = 1, 2, \ldots); \quad \alpha_n = \frac{(2n-1)\pi}{2c}.$

Figure 2: Problem statement

Solution

Solve for eigenvalues and normalized eigenfunctions.

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X(c) &= 0 \end{aligned}$$

Writing the boundary conditions in SL standard form

$$\begin{aligned} a_1 X(0) + a_2 X'(0) &= 0 \\ b_1 X(c) + b_2 X'(c) &= 0 \end{aligned}$$

Shows that $a_1 = 0, a_2 = 1$ and $b_1 = 1, b_2 = 0$. Therefore $a_1 a_2 = 0$ and $b_1 b_2 = 0$. But we know that if $a_1 a_2 \geq 0$ and $b_1 b_2 \geq 0$, then $\lambda > 0$ is only possible eigenvalues. Let $\lambda_n = \alpha_n^2$. $\alpha > 0$. Hence the solution to the ODE is

$$\begin{aligned} X_n(x) &= A\cos(\alpha_n x) + B\sin(\alpha_n x) \\ X_n'(x) &= -A\alpha_n \sin(\alpha_n x) + B\alpha_n \cos(\alpha_n x) \end{aligned}$$

First B.C $X'(0) = 0$ gives

$$0 = B\alpha_n$$

Which implies $B = 0$. Hence the solution now becomes $X_n(x) = A\cos(\alpha_n x)$. For the second BC

$$\begin{aligned} 0 &= A\cos(\alpha_n c) \\ 0 &= \cos(\alpha_n c) \end{aligned}$$

Which implies

$$\begin{aligned} \alpha_n c &= \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}, \cdots \\ &= (2n-1)\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots \end{aligned}$$

Hence

$$\alpha_n = \frac{(2n-1)}{c}\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

And the corresponding eigenfunctions are

$$\begin{aligned} X_n(x) &= \cos(\alpha_n x) \\ &= \cos\left(\frac{(2n-1)}{c}\frac{\pi}{2}x\right) \end{aligned}$$

To find the normalized $X_n(x)$ which we call it $\phi_n(x)$, then by definition

$$\phi_n(x) = \frac{X_n(x)}{\|X_n(x)\|}$$

But

$$\|X_n(x)\|^2 = \int_0^c p(x) X_n^2(x)\,dx$$

Comparing the ODE $X'' + \lambda X = 0$ to $(rX')' + \left(\lambda p + q\right) X = 0$, we see that $r(x) = 1$ and $q = 0$ and $p = 1$. Hence the above becomes

$$\begin{aligned} \|X_n(x)\|^2 &= \int_0^c \cos^2(\alpha_n x)\,dx \\ &= \frac{c}{2} \end{aligned}$$

Therefore $\|X_n(x)\| = \sqrt{\frac{c}{2}}$ which shows that

$$\begin{aligned} \phi_n(x) &= \frac{X_n(x)}{\sqrt{\frac{c}{2}}} \\ &= \sqrt{\frac{2}{c}}\cos(\alpha_n x) \end{aligned}$$

where

$$\alpha_n = \frac{(2n-1)}{c}\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

Which is what required to show.

3 Section 72, Problem 6

6. In Problem 1(a), Sec. 69, the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$(xX')' + \frac{\lambda}{x}X = 0, \qquad X(1) = 0, \qquad X(b) = 0$$

were found to be

$$\lambda_n = \alpha_n^2, \qquad X_n(x) = \sin(\alpha_n \ln x) \qquad (n = 1, 2, \ldots),$$

where $\alpha_n = n\pi/\ln b$. Show that the *normalized* eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{\ln b}}\sin(\alpha_n \ln x) \qquad (n = 1, 2, \ldots).$$

Suggestion: The integral that arises can be evaluated by making the substitution

$$s = \pi\frac{\ln x}{\ln b}$$

and then referring to the integration formula established in Problem 9, Sec. 5.

Figure 3: Problem statement

<u>Solution</u>

$$X_n(x) = \sin(\alpha_n \ln x)$$
$$\alpha_n = \frac{n\pi}{\ln b} \qquad n = 1, 2, 3, \cdots$$

The normalized eigenfunction is given by

$$\phi_n(x) = \frac{X_n(x)}{\|X_n(x)\|}$$

But

$$\|X_n(x)\|^2 = \int_1^b p(x)\, X_n^2(x)\, dx$$

Comparing the ODE $(xX')' + \frac{\lambda}{x}X = 0$ to $(rX')' + (\lambda p + q)X = 0$, we see that $r(x) = x$ and $q = 0$ and $p = \frac{1}{x}$. Hence the above becomes

$$\|X_n(x)\|^2 = \int_1^b \frac{1}{x}\sin^2(\alpha_n \ln x)\, dx$$

Let $s=\frac{\ln x}{\ln b}\pi$. Then $\frac{ds}{dx}=\frac{1}{x}\frac{\pi}{\ln b}$ or $dx=\frac{x}{\pi}\ln(b)\,ds$. When $x=1$ then $s=0$ and when $x=b$ then $s=\pi$. Hence the above integral becomes

$$\begin{aligned}\|X_n(x)\|^2 &= \int_{s=0}^{s=\pi}\frac{1}{x}\sin^2\left(\alpha_n\frac{s\ln b}{\pi}\right)\left(\frac{x}{\pi}\ln(b)\,ds\right)\\ &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\sin^2\left(\alpha_n\frac{s\ln b}{\pi}\right)ds\end{aligned}$$

But $\alpha_n=\frac{n\pi}{\ln(b)}$ therefore the above becomes

$$\begin{aligned}\|X_n(x)\|^2 &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\sin^2\left(\frac{n\pi}{\ln(b)}\frac{s\ln b}{\pi}\right)ds\\ &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\sin^2(ns)\,ds\\ &= \frac{1}{\pi}\ln(b)\int_0^{\pi}\frac{1}{2}-\frac{1}{2}\cos(2ns)\,ds\\ &= \frac{1}{\pi}\ln(b)\left(\frac{\pi}{2}-\frac{1}{2}\sin\left(\frac{2ns}{2n}\right)_0^{\pi}\right)\\ &= \frac{1}{\pi}\ln(b)\left(\frac{\pi}{2}-\frac{1}{2}\sin(s)_0^{\pi}\right)\\ &= \frac{1}{2}\ln(b)\end{aligned}$$

Hence

$$\begin{aligned}\phi_n(x) &= \frac{\sin(\alpha_n\ln x)}{\sqrt{\frac{1}{2}\ln(b)}}\\ &= \sqrt{\frac{2}{\ln(b)}}\sin(\alpha_n\ln x)\end{aligned}$$

Which is what required to show.

4 Section 72, Problem 9

9. Use the solutions obtained in Problem 3 to find the eigenvalues and normalized eigenfunctions of the Sturm-Liouville problem

$$(xX')' + \frac{\lambda}{x}X = 0, \qquad X'(1) = 0, \qquad X(b) = 0.$$

Answer:

$$\lambda_n = \alpha_n^2, \qquad \phi_n(x) = \sqrt{\frac{2}{\ln b}}\cos(\alpha_n \ln x) \quad (n = 1, 2, \ldots); \qquad \alpha_n = \frac{(2n-1)\pi}{2\ln b}.$$

Figure 4: Problem description

<u>solution</u>

From problem section 69 problem 1, we know that $(xX'(x))' + \frac{\lambda}{x}X(x) = 0$ can be transformed to $X''(s) + \lambda X(s) = 0$ using $x = e^s$. With boundary conditions in s found as follows. When $x = 1$ then $s = 0$ and when $x = b$ then $s = \ln b$. Hence we obtain the SL problem

$$\begin{aligned} X''(s) + \lambda X(s) &= 0 \\ X'(0) &= 0 \\ X(\ln b) &= 0 \end{aligned} \tag{1}$$

But problem 3 is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X(c) &= 0 \end{aligned} \tag{2}$$

And it had the solution

$$\phi_n(x) = \sqrt{\frac{2}{c}}\cos(\alpha_n x)$$

where

$$\alpha_n = \frac{(2n-1)}{c}\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

By comparing (2) and (1) we see it is the same problem, except $c \to \ln b$. Hence the solution to (2) is the same as the solution in (1) but with c replaced by $\ln b$. Hence the solution is

$$\begin{aligned} \phi_n(s) &= \sqrt{\frac{2}{\ln b}}\cos(\alpha_n s) \\ \alpha_n &= \frac{(2n-1)}{\ln b}\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots \end{aligned}$$

But $s = \ln x$, hence the above becomes

$$\phi_n(x) = \sqrt{\frac{2}{\ln b}} \cos(\alpha_n \ln x)$$

$$\alpha_n = \frac{(2n-1)}{\ln b} \frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

Which is what required to show.