Homework 6 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

5.2.9 We compute

$$\begin{aligned}
W(x) &= \det \begin{bmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{bmatrix} \\
&= e^x \det \begin{bmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{bmatrix} - \cos x \det \begin{bmatrix} e^x & \cos x \\ e^x & -\sin x \end{bmatrix} + \sin x \det \begin{bmatrix} e^x & -\sin x \\ e^x & -\cos x \end{bmatrix} \\
&= e^x(\sin^2 x + \cos^2 x) - \cos x(-e^x \sin x - e^x \cos x) + \sin x(-e^x \cos x + e^x \sin x) \\
&= e^x + e^x(\cos x \sin x + \cos^2 x - \sin x \cos x + \sin^2 x) \\
&= 2e^x
\end{aligned}$$

Now $W(x) = 2e^x$ is not identically 0, so the functions are linearly independent.

5.2.16 Our general solution is $y = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$. The given initial conditions give us the following equations:

$$\begin{aligned}
y &= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \\
y(0) &= c_1 + c_2 = 1 \\
y' &= c_1 e^x + 2c_2 e^{2x} + c_3 e^{2x} + 2c_3 x e^{2x} \\
y'(0) &= c_1 + 2c_2 + c_3 = 4 \\
y'' &= c_1 e^x + 4c_2 e^{2x} + 4c_3 e^{2x} + 4c_3 x e^{2x} \\
y''(0) &= c_1 + 4c_2 + 4c_3 = 0
\end{aligned}$$

We have a system of three equations in three variables which we solve by row reducing the augmented matrix.

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 0 \end{bmatrix} &\xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & -1 \end{bmatrix} \\
&\xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -10 \end{bmatrix}
\end{aligned}$$

Now we back substitute to get $c_3 = -10, c_2 = 13, c_1 - 12$. So, the particular solution here is $y = -12e^x + 13e^{2x} - 10xe^{2x}$.

5.2.24 The general form of a solution is $y = c_1e^x \cos x + c_2e^x \sin x + x + 1$. The initial conditions give us the following:

$$\begin{aligned} y &= c_1e^x \cos x + c_2e^x \sin x + x + 1 \\ y(0) &= c_1 + 1 = 4 \\ y' &= c_1e^x \cos x - c_1e^x \sin x + c_2e^x \sin x + c_2e^x \cos x + 1 \\ y'(0) &= c_1 + c_2 + 1 = 8 \end{aligned}$$

We can solve immediately to get $c_1 = 3, c_2 = 4$. So the solution is $y = 3e^x \cos x + 4e^x \sin x + x + 1$.

5.3.8 We have $y'' - 6y' + 13y = 0$ with characteristic equation $r^2 - 6r + 13$. This doesn't factor obviously, so we use the quadratic equation:

$$\begin{aligned} r &= \frac{6 \pm \sqrt{36 - 52}}{2} \\ &= 3 \pm 2i \end{aligned}$$

In this case, we have complex conjugate roots $a \pm bi$ where $a = 3$ and $b = 2$. So our general solution is $y = c_1e^{3x} \cos(2x) + c_2e^{3x} \sin(2x)$.

5.3.11 We have $y^{(4)} - 8y^{(3)} + 16y'' = 0$ with characteristic equation $r^4 - 8r^3 + 16r^2 = r^2(r^2 - 8r + 16)$. We can factor the remaining quadratic easily as $(r - 4)^2$. So we have roots $r = 0, 4$ each of multiplicity 2.

Our general solution is $y = c_1 + c_2x + c_3e^{4x} + c_4xe^{4x}$.

5.3.14 We have characteristic equation $r^4 + 3r^2 - 4$. Mentally making the substitution $x = r^2$, we can see that there is a factorization $(r^2 + 4)(r^2 - 1) = (r^2 + 4)(r + 1)(r - 1)$. The roots are thus $\pm 2i, \pm 1$.

Our general solution is $y = c_1e^x + c_2e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)$.

5.3.18 We rewrite the differential equation as $y^{(4)} - 16y = 0$ so the characteristic equation is $r^4 - 16$. This is a difference of squares, so factors as $(r^2 - 4)(r^2 + 4)$. We can factor further using difference of squares to get $(r-2)(r+2)(r^2+4)$. So the roots are $\pm 2, \pm 2i$.

The general solution is $y = c_1e^{2x} + c_2e^{-2x} + c_3 \cos(2x) + c_3 \sin(2x)$.

Additional Problems:

1. We have the differential equation

$$y^{(7)} - 2y^{(6)} + 9y^{(5)} - 16y^{(4)} + 24y^{(3)} - 32y'' + 16y' = 0$$

   (a) The characteristic equation is $r^7 - 2r^6 + 9r^5 - 16r^4 + 24r^3 - 32r^2 + 16r$ which factors as $r(r^6 - 2r^5 + 9r^4 - 16r^3 + 24r^2 - 32r + 16)$

(b) 1 is a root as $1 - 2 + 9 - 16 + 24 - 32 + 16 = 0$. Polynomial long division (steps omitted) gives us the factorization $r(r-1)(r^5 - r^4 + 8r^3 - 8r^2 + 16r - 16)$.

1 is still a root of the degree 5 factor as $1 - 1 + 8 - 8 + 16 - 16 = 0$, so we do polynomial long division (steps omitted again) again to get the factorization $r(r-1)^2(r^4 + 8r^2 + 16)$.

1 is no longer a root, since $1 + 8 + 16 = 25$.

(c) We have $x^2 + 8x + 16 = (x+4)(x+4)$.

(d) Our characteristic polynomial factors as $r(r-1)^2(r^2+4)^2$. The roots of the r^2+4 factor are $\pm 2i$.

(e) Our roots are 0 (mult. 1), 1 (mult. 2), and $\pm 2i$ (each of mult. 2). This gives us the following general solution

$$y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 \cos(2x) + c_5 \sin(2x) + c_6 x \cos(2x) + c_7 x \sin(2x)$$

As expected, we have seven terms in the general solution.