HW 5

Math 2243
Linear Algebra and Differential Equations

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1 Problem 7 section 4.7

In Problems 5–8, determine whether or not each indicated set of functions is a subspace of the space F of all real-valued functions on $\mathbb{R}$.

The set of all f such that $f(0) = 0$ and $f(1) = 1$

Solution

Let f, g be two functions such that $f(0) = 0, g(0) = 0$ and $f(1) = 1, g(1) = 1$ in F. Let us check if it is closed under addition

$$f(0) + g(0) = 0 + 0 = 0$$

OK.

$$f(1) + g(1) = 1 + 1 = 2 \neq 1$$

Hence not closed under addition. Therefore not a subspace.

2 Problem 10 section 4.7

In Problems 9–12, a condition on the coefficients of a polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ is given. Determine whether or not the set of all such polynomials satisfying this condition is a subspace of the space P of all polynomials

$$a_0 = a_1 = 0$$

Solution

Let

$$\begin{aligned} p_1(x) &= a_2x^2 + a_3x^3 \\ p_2(x) &= b_2x^2 + b_3x^3 \end{aligned}$$

Checking if closed under scalar multiplication. Let c be some scalar. Hence

$$\begin{aligned} cp_1(x) &= c\left(a_2x^2 + a_3x^3\right) \\ &= (ca_2)x^2 + (ca_3)x^3 \\ &= A_2x^2 + A_3x^3 \end{aligned}$$

Therefore closed. Now checking if closed under addition.

$$\begin{aligned} p_1(x) + p_2(x) &= a_2x^2 + a_3x^3 + b_2x^2 + b_3x^3 \\ &= (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \\ &= A_2x^2 + A_3x^3 \end{aligned}$$

Therefore Closed under addition. Also the zero polynomial in included when $a_2 = a_3 = 0$.

Therefore this is a subspace.

3 Problem 5 section 1.1

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

$$\begin{aligned} y' &= y + 2e^{-x} \qquad \text{(A)} \\ y &= e^{x} - e^{-x} \end{aligned}$$

<u>Solution</u>

Using the solution given, we see that

$$\begin{aligned} y' &= e^{x} - (-e^{-x}) \\ &= e^{x} + e^{-x} \qquad (1) \end{aligned}$$

Substituting (1) into EQ. (A) gives

$$\begin{aligned} e^{x} + e^{-x} &= (e^{x} - e^{-x}) + 2e^{-x} \\ e^{x} + e^{-x} &= e^{x} + e^{-x} \\ 0 &= 0 \end{aligned}$$

Hence the solution given <u>satisfies the ODE</u>.

4 Problem 17 section 1.1

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

$$\begin{aligned} y' + y &= 0 \\ y(x) &= Ce^{-x} \\ y(0) &= 2 \end{aligned} \tag{A}$$

Solution

Using the solution given, we see that

$$y' = -Ce^{-x} \tag{1}$$

Substituting (1) into EQ. (A) gives

$$\begin{aligned} -Ce^{-x} + Ce^{-x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence the solution gives satisfies the ODE.

When $x = 0$ the solution becomes

$$\begin{aligned} 2 &= Ce^{-(0)} \\ &= C \end{aligned}$$

Hence $C = 2$ and the particular solution becomes

$$y(x) = 2e^{-x}$$

The following are some solutions plots for different C

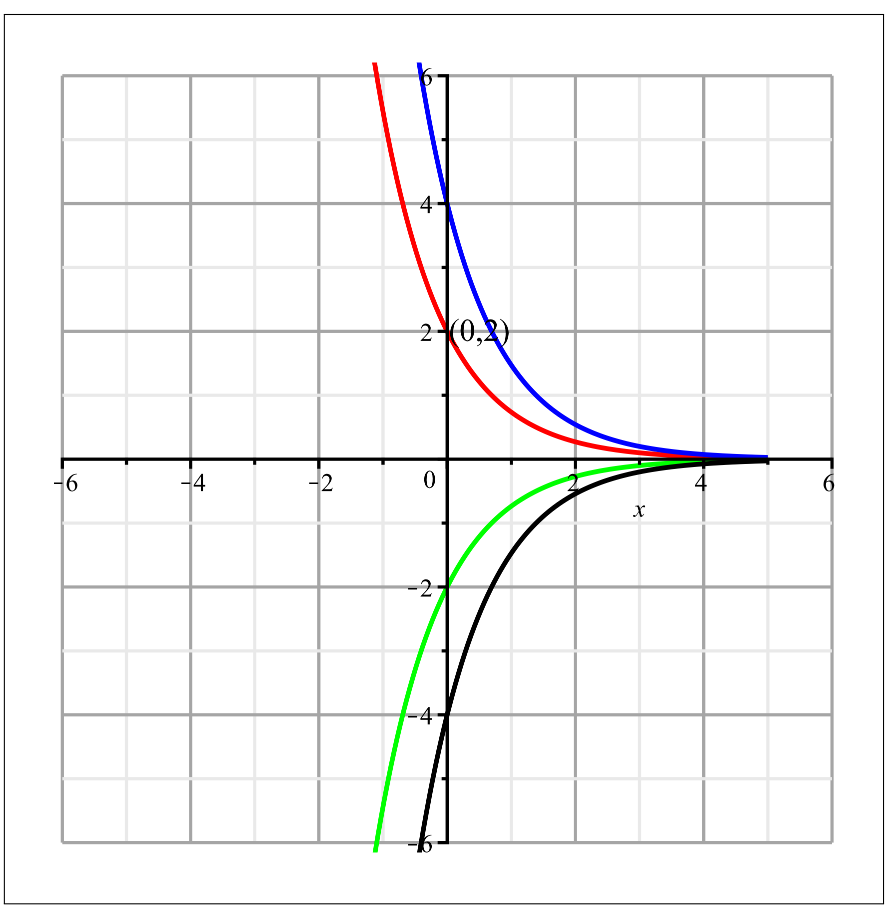

Figure 1: Plot of serveral solution with different c. Red solution is one given in problem.

```
restart;
f:=(x,c)->c*exp(-x)
p1:=plot(f(x,2),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=red):
p2:=plot(f(x,4),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=blue):
p3:=plot(f(x,-2),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=green):
p4:=plot(f(x,-4),x=-5..5,gridlines=true,view=[-6..6, -6..6],color=black):
T:=plots:-textplot([[.5,2,"(0,2)"]], font=[times,16],tickmarks=NULL):
plots:-display([p1,p2,p3,p4,T]);
```

5 Problem 3 section 5.1

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$
\begin{aligned}
y'' + 4y &= 0 \qquad (1)\\
y_1 &= \cos 2x\\
y_2 &= \sin 2x\\
y(0) &= 3\\
y'(0) &= 8
\end{aligned}
$$

<u>Solution</u>

Checking if $y_1(x)$ is a solution. Since

$$
\begin{aligned}
y_1' &= -2\sin 2x \qquad (2)\\
y_1'' &= -4\cos 2x \qquad (3)
\end{aligned}
$$

Substituting the above equations back into (1) gives

$$
\begin{aligned}
(-4\cos 2x) + 4\cos 2x &= 0\\
0 &= 0
\end{aligned}
$$

Hence y_1 is a solution. We do the same for y_2

$$
\begin{aligned}
y_2' &= 2\cos 2x \qquad (4)\\
y_2'' &= -4\sin 2x \qquad (5)
\end{aligned}
$$

Substituting (4,5) back into (1) gives

$$
\begin{aligned}
(-4\sin 2x) + 4(\sin 2x) &= 0\\
0 &= 0
\end{aligned}
$$

Hence y_2 is a solution. Let general solution be

$$
\begin{aligned}
y(x) &= c_1y_1(x) + c_2y_2(x)\\
&= c_1\cos 2x + c_2\sin 2x \qquad (6)
\end{aligned}
$$

Applying the first initial conditions $y(0) = 3$ in (6) gives

$$3 = c_1$$

Hence (6) now becomes

$$y(x) = 3\cos 2x + c_2\sin 2x \qquad (7)$$

Taking derivative of the above gives

$$y'(x) = -6\sin 2x + 2c_2\cos 2x$$

Applying the second initial conditions $y'(0) = 8$ in the above gives

$$
\begin{aligned}
8 &= 2c_2\\
c_2 &= 4
\end{aligned}
$$

Therefore the general solution (6) becomes

$$\boxed{y(x) = 3\cos 2x + 4\sin 2x} \qquad (8)$$

6 Problem 5 section 5.1

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$\begin{aligned} y'' - 3y' + 2y &= 0 \qquad (1) \\ y_1 &= e^x \\ y_2 &= e^{2x} \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Solution

Checking if $y_1(x)$ is a solution. Since

$$\begin{aligned} y_1' &= e^x \qquad (2) \\ y_1'' &= e^x \qquad (3) \end{aligned}$$

Substituting the above equations back into (1) gives

$$\begin{aligned} e^x - 3e^x + 2e^x &= 0 \\ 0 &= 0 \end{aligned}$$

Hence y_1 is a solution. We do the same for y_2

$$\begin{aligned} y_2' &= 2e^{2x} \qquad (4) \\ y_2'' &= 4e^{2x} \qquad (5) \end{aligned}$$

Substituting (4,5) back into (1) gives

$$\begin{aligned} \left(4e^{2x}\right) - 3\left(2e^{2x}\right) + 2\left(e^{2x}\right) &= 0 \\ 4e^{2x} - 6e^{2x} + 2e^{2x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence y_2 is a solution. Let general solution be

$$\begin{aligned} y(x) &= c_1y_1 + c_2y_2 \\ &= c_1e^x + c_2e^{2x} \qquad (6) \end{aligned}$$

Applying the first initial conditions $y(0) = 1$ in (6) gives

$$1 = c_1 + c_2 \qquad (7)$$

Taking derivative of Eq. (6) gives

$$y'(x) = c_1e^x + 2c_2e^{2x}$$

Applying the second initial conditions $y'(0) = 0$ in the above gives

$$0 = c_1 + 2c_2 \qquad (8)$$

We have two equations (7,8) to solve for the 2 unknowns c_1, c_2. (7)-(8) gives

$$c_2 = -1$$

Hence from (7) $c_1 = 1 - c_2 = 1 + 1 = 2$. Therefore the solution (6) now becomes

$$y(x) = 2e^x - e^{2x}$$

7 Problem 7 section 5.1

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$\begin{aligned} y'' + y' &= 0 && (1)\\ y_1 &= 1\\ y_2 &= e^{-x}\\ y(0) &= -2\\ y'(0) &= 8 \end{aligned}$$

Solution

Checking if $y_1(x)$ is a solution. Since

$$\begin{aligned} y_1' &= 0 && (2)\\ y_1'' &= 0 && (3) \end{aligned}$$

Substituting the above equations back into (1) gives

$$\begin{aligned} 0 + 0 &= 0\\ 0 &= 0 \end{aligned}$$

Hence y_1 is a solution. We do the same for y_2

$$\begin{aligned} y_2' &= -e^{-x} && (4)\\ y_2'' &= e^{-x} && (5) \end{aligned}$$

Substituting (4,5) back into (1) gives

$$\begin{aligned} (e^{-x}) - e^{-x} &= 0\\ 0 &= 0 \end{aligned}$$

Hence y_2 is a solution. Let general solution be

$$\begin{aligned} y(x) &= c_1y_1 + c_2y_2\\ &= c_1 + c_2e^{-x} && (6) \end{aligned}$$

Applying the first initial conditions $y(0) = -2$ in (6) gives

$$-2 = c_1 + c_2 \qquad (7)$$

Taking derivative of Eq. (6) gives

$$y'(x) = -c_2e^{-x}$$

Applying the second initial conditions $y'(0) = 8$ in the above gives

$$\begin{aligned} 8 &= -c_2\\ c_2 &= -8 && (8) \end{aligned}$$

Hence from (7)

$$\begin{aligned} -2 &= c_1 + c_2\\ &= c_1 - 8\\ c_1 &= 6 \end{aligned}$$

Therefore the solution (6) now becomes

$$\begin{aligned} y(x) &= c_1 + c_2e^{-x}\\ &= 6 - 8e^{-x} \end{aligned}$$

8 Problem 33 section 5.1

Apply Theorems 5 and 6 to find general solutions of the differential equations given in Problems 33 through 42. Primes denote derivatives with respect to x.

$$y'' - 3y' + 2y = 0$$

Solution

The characteristic equation is

$$\begin{aligned} r^2 - 3r + 2 &= 0 \\ (r-1)(r-2) &= 0 \end{aligned}$$

Hence the roots are $r_1 = 1, r_2 = 2$. Therefore the general solution is

$$\begin{aligned} y(x) &= Ae^{r_1 x} + Be^{r_2 x} \\ &= Ae^{x} + Be^{2x} \end{aligned}$$

Where A, B are the constants of integrations which are found from initial conditions.

9 Problem 35 section 5.1

Apply Theorems 5 and 6 to find general solutions of the differential equations given in Problems 33 through 42. Primes denote derivatives with respect to x.

$$y'' + 5y' = 0$$

<u>Solution</u>

The characteristic equation is

$$\begin{aligned} r^2 + 5r &= 0 \\ (r+5)r &= 0 \end{aligned}$$

Hence the roots are $r_1 = 0, r_2 = -5$. Therefore the general solution is

$$\begin{aligned} y(x) &= Ae^{r_1 x} + Be^{r_2 x} \\ &= A + Be^{-5x} \end{aligned}$$

Where A, B are the constants of integrations which are found from initial conditions.

10 Problem 39 section 5.1

Apply Theorems 5 and 6 to find general solutions of the differential equations given in Problems 33 through 42. Primes denote derivatives with respect to x.

$$4y'' + 4y' + y = 0$$

Solution

The characteristic equation is

$$\begin{aligned} 4r^2 + 4r + 1 &= 0 \\ r^2 + r + \frac{1}{4} &= 0 \\ \left(r + \frac{1}{2}\right)^2 &= 0 \end{aligned}$$

Hence the root is $r = -\frac{1}{2}$. A double root. Therefore the general solution is

$$\begin{aligned} y(x) &= Ae^{rx} + Bxe^{rx} \\ &= Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x} \end{aligned}$$

Where A, B are the constants of integrations which are found from initial conditions.

11 Additional problem 1

Let P_2 be subspace of polynomials of degree at most 2. So elements of P_2 look like $a_0 + a_1 x + a_2 x^2$. Show that $\left\{3+x, 1+x+x^2, x-2x^2\right\}$ is basis for P_2

Solution

Assuming these are basis, then we can write

$$a_0 + a_1 x + a_2 x^2 = c_1(3+x) + c_2\left(1+x+x^2\right) + c_3\left(x-2x^2\right)$$

For constants c_1, c_2, c_3. If we can find unique solution for the c_i then these are basis. The above becomes

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 &= 3c_1 + c_2 + xc_1 + xc_2 + xc_3 + x^2 c_2 - 2x^2 c_3 \\ &= (3c_1 + c_2) + x(c_1 + c_2 + c_3) + x^2(c_2 - 2c_3) \end{aligned}$$

Comparing coefficients gives the equations

$$\begin{aligned} a_0 &= 3c_1 + c_2 \\ a_1 &= c_1 + c_2 + c_3 \\ a_2 &= c_2 - 2c_3 \end{aligned}$$

In Matrix form the above becomes

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Augmented matrix is

$$\begin{bmatrix} 3 & 1 & 0 & a_0 \\ 1 & 1 & 1 & a_1 \\ 0 & 1 & -2 & a_2 \end{bmatrix}$$

Replacing row 2 with row 1 gives

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 3 & 1 & 0 & a_0 \\ 0 & 1 & -2 & a_2 \end{bmatrix}$$

$R_2 \rightarrow -3R_1 + R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 0 & -2 & -3 & a_0 - 3a_1 \\ 0 & 1 & -2 & a_2 \end{bmatrix}$$

$R_3 \rightarrow R_2 + 2R_3$ gives

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 0 & -2 & -3 & a_0 - 3a_1 \\ 0 & 0 & -7 & a_0 - 3a_1 + 2a_2 \end{bmatrix}$$

The matrix is now in Echelon form. We see that there are no free variables. Only leading variables c_1, c_2, c_3. This implies we have unique solution. Which means we can solve for c_1, c_2, c_3 in terms of a_1, a_2, a_3. We are not asked to complete the solution, only to say if these are basis. So we can stop here.

This shows that $\left\{3+x, 1+x+x^2, x-2x^2\right\}$ are basis for P_2.

12 Additional problem 2

Find the general solution for $y''-25y=0$. What is the particular solution for $y(0)=a, y'(0)=b$?

Solution

The characteristic equation is

$$\begin{aligned} r^2 - 25 &= 0 \\ r &= \pm 5 \end{aligned}$$

Two distinct real roots $r_1 = 5, r_2 = -5$. Therefore the general solution is

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{5x} + c_2 e^{-5x} \end{aligned} \tag{1}$$

Now we apply the initial conditions. The first one $y(0)=a$ applied to the above gives

$$a = c_1 + c_2 \tag{2}$$

Taking derivative of (1) gives

$$y' = 5c_1 e^{5x} - 5c_2 e^{-5x}$$

Applying second initial conditions $y'(0)=b$ to the above gives

$$b = 5c_1 - 5c_2 \tag{3}$$

Multiplying (2) by 5 and adding the result to Eq (3) gives

$$\begin{aligned} 5a + b &= (5c_1 + 5c_2) + (5c_1 - 5c_2) \\ 5a + b &= 10c_1 \end{aligned}$$

Hence

$$c_1 = \frac{5a+b}{10}$$

From (2) we now solve for c_2

$$\begin{aligned} a &= \frac{5a+b}{10} + c_2 \\ c_2 &= a - \frac{5a+b}{10} \\ &= \frac{a}{2} - \frac{b}{10} \end{aligned}$$

Now that we found both constants, the particular solution becomes

$$\begin{aligned} y(x) &= c_1 e^{5x} + c_2 e^{-5x} \\ &= \left(\frac{a}{2} + \frac{b}{10}\right) e^{5x} + \left(\frac{a}{2} - \frac{b}{10}\right) e^{-5x} \end{aligned}$$