Homework 3 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

3.7.3 We need to interpolate $(0,3),(1,1),(2,-5)$. Since we have 3 points, we use a degree 2 polynomial $f(x)=a_0+a_1x+a_2x^2$. This gives the linear system

$$\begin{aligned} a_0 &= 3 \\ a_0+a_1+a_2 &= 1 \\ a_0+2a_1+4a_2 &= -5 \end{aligned}$$

The first equation says $a_0=3$, so substituting we get the 2×2 system

$$\begin{aligned} a_1+a_2 &= -2 \\ 2a_1+4a_2 &= -8 \end{aligned}$$

We solve by row reduction:

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 4 & -8 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}$$

So $a_2=-2$ and thus $a_1=0$. So our polynomial is $f(x)=3-2x^2$.

4.1.1 We are given $\vec{a}=(2,5,-4)$ and $\vec{b}=(1,-2,-3)$. We calculate

$$\begin{aligned} \left|\vec{a}-\vec{b}\right| &= |(1,7,-1)| \\ &= \sqrt{1+49+1} \\ &= \sqrt{51} \\ 2\vec{a}+\vec{b} &= (4,10,-8)+(1,-2,-3) \\ &= (5,8,-11) \\ 3\vec{a}-4\vec{b} &= (6,15-12)-(4,-8,-12) \\ &= (2,23,0) \end{aligned}$$

4.1.19 We are given $\vec{u}=(2,0,1),\vec{v}=(-3,1,-1),\vec{w}=(0,-2,-1)$. We are asked to use row reduction in this case.

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$
$$\xrightarrow{R_2+R_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$
$$\xrightarrow{R_2+R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

We have a free variable, so these vectors are linearly dependent. To find a particular linear combination, we choose a value for c, say $c = 2$. Then we solve to get $b = 4, a = 6$. So we have $6\vec{u} + 4\vec{v} + 2\vec{w} = \vec{0}$.

4.1.23 We are given $\vec{u} = (2, 0, 3), \vec{v} = (5, 4, -2), \vec{w} = (2, -1, 1)$. We do the row reduction:

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix} \xrightarrow[2R_3]{3R_1} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 4 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$
$$\xrightarrow{-R_1+R_3} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 4 & -1 \\ 0 & -19 & -4 \end{bmatrix}$$
$$\xrightarrow{4R_2+R_3} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 4 & -1 \\ 0 & -3 & 0 \end{bmatrix}$$
$$\xrightarrow{R_3+R_2} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 1 & -1 \\ 0 & -3 & 0 \end{bmatrix}$$
$$\xrightarrow{3R_2+R_3} \begin{bmatrix} 6 & 15 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

We have an echelon form matrix with a leading entry in every row, so the homogeneous system has only the trivial solution. Hence the vectors are linearly independent.

4.1.27 We want to write $\vec{t} = a\vec{u} + b\vec{v} + c\vec{w}$, so we set up and reduce an augmented matrix:

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 4 & -2 & 4 & 0 \\ 3 & 2 & 1 & 19 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-4R_1+R_2} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 2 & -12 & 0 \\ 0 & 5 & -11 & 19 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 2 & -12 & 0 \\ 0 & 1 & 13 & 19 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 0 & -38 & -38 \\ 0 & 1 & 13 & 19 \end{bmatrix}$$

$$\xrightarrow[\frac{-1}{38}R_3]{(R_2,R_3)} \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & 13 & 19 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now back substitution gives us $c = 1$, $b = 6$, and $a = 2$. So $\vec{t} = 2\vec{u} + 6\vec{v} + \vec{w}$.

4.2.2 W is a subspace. To see why, suppose that both $\vec{x}$ and $\vec{y}$ are in W. Then for any scalar c, $c\vec{x} = (cx_1, cx_2, cx_3)$. Since we know $x_1 = 5x_2$, we have $cx_1 = 5(cx_2)$. So $c\vec{x}$ is in W. Also, $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$. We know $x_1 = 5x_2$ and $y_1 = 5y_2$, so $x_1+y_1 = 5(x_2+y_2)$. Hence $\vec{x}+\vec{y}$ is in W. This shows closure under scalar multiplication and under addition.

4.2.4 W is not a subspace. It fails everything pretty badly, but an easy way to see it is not a subspace is that it does not contain $\vec{0}$ since $0 + 0 + 0 \neq 1$.

4.2.17 *There is a typo in this problem in the book. The second equation is meant to have x_4 in place of x_5 and I have solved that version. If you solved it correctly as written, you received full points as well.*

We do row reduction to our system:

$$\begin{bmatrix} 1 & 3 & 8 & -1 \\ 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -6 & -18 & 6 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow[R_2+R_3]{\frac{1}{6}R_2} \begin{bmatrix} 1 & 3 & 8 & -1 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[-R_2]{3R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have two free variables. We set $x_3 = s$ and $x_4 = t$ and then use back substitution to find $x_2 = -3s + t$ and $x_1 = s - 2t$. So our solution vectors look like

$$\vec{x} = \begin{bmatrix} s - 2t \\ -3s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

4.2.21 We do row reduction to our system:

$$\begin{bmatrix} 1 & 7 & 2 & -3 \\ 2 & 7 & 1 & -4 \\ 3 & 5 & -1 & -5 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 7 & 2 & -3 \\ 0 & -7 & -3 & 2 \\ 0 & -16 & -7 & 4 \end{bmatrix}$$

$$\xrightarrow[-2R_2+R_3]{R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -7 & -3 & 2 \\ 0 & -2 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{-4R_3+R_2} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[-R_3+R_2]{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

We have one free variable, so we set $x_4 = t$. Back substitution gives us $x_3 = -4t, x_2 = 2t, x_1 = -3t$. So a typical solution looks like

$$\vec{x} = \begin{bmatrix} -3t \\ 2t \\ -4t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 2 \\ -4 \\ 1 \end{bmatrix}$$

Additional Problems:

1. We need to interpolate the points $(1, 10), (2, 15), (3, 10)$ with a quadratic $f(t) = a+bt+ct^2$. This sets up the linear system

$$\begin{aligned} a + b + c &= 10 \\ a + 2b + 4c &= 15 \\ a + 3b + 9c &= 10 \end{aligned}$$

The polynomial we get after solving is $f(t) = -5 + 20t - 5t^2$.

It is a *very bad* idea to use this polynomial to predict the price on day 4. For one thing, $f(4) = -5$ and a price of -\$5 is absurd. It also doesn't make very much sense that stock prices would behave like a parabola, where the value either always increases after a certain time or always decreases after a certain time. I'm not an enconomist, but I imagine that most stock prices would go up for a while, then down for a while, then up for a while, then down for a while, and so on. Using this polynomial to determine your investing strategy would be a great way to lose all your money.

2. Proper subspaces of $\mathbb{R}^2$ look like lines through $(0, 0)$. There is also the subspace that is all of $\mathbb{R}^2$ and the subspace that is just $\vec{0}$.

3. The solution set to $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$ if and only if $\vec{b} = \vec{0}$. On the one hand, we know that the solution set for a homogeneous linear system is always a subspace. On the other hand, if the solutions to $A\vec{x} = \vec{b}$ forms a subspace, then for any solution $\vec{x}_0$ we know by closure under scalar multiplication that $2\vec{x}_0$ is also a solution. So $\vec{b} = A(2\vec{x}_0) = 2A\vec{x}_0 = 2\vec{b}$, which only works when $\vec{b} = \vec{0}$.