Homework 2 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

3.4.3

$$(-2)\begin{bmatrix} 5 & 0 \\ 0 & 7 \\ 3 & -1 \end{bmatrix} + (4)\begin{bmatrix} -4 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} -26 & 20 \\ 12 & -6 \\ 22 & 18 \end{bmatrix}$$

3.4.5

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -9 & 1 \\ -10 & 12 \end{bmatrix} \\ BA &= \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 8 \\ 11 & 5 \end{bmatrix} \end{aligned}$$

3.4.8

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 15 \\ 35 & 0 \end{bmatrix} \\ BA &= \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{bmatrix} \end{aligned}$$

3.4.11

$$\begin{aligned} AB &= \begin{bmatrix} 3 & -5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 1 & 5 & 3 \end{bmatrix} \end{aligned}$$

The product BA is not defined because B is 2×4 and A is 1×2, so the dimensions do not match.

3.5.3 For $A = \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}$, our formula for 2×2 matrices tells us that $A^{-1} = \frac{1}{36-35} \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}$. So if $A\vec{x} = \vec{b}$, we have that

$$\begin{aligned} \vec{x} &= A^{-1}\vec{b} \\ &= \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 33 \\ -28 \end{bmatrix} \end{aligned}$$

3.5.10 We adjoin an identity matrix and row reduce:

$$\begin{aligned} \begin{bmatrix} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix} &\xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 4 & 6 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-4R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \end{bmatrix} \\ &\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix} \\ &\xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{bmatrix} \end{aligned}$$

So the inverse matrix is $\begin{bmatrix} 3 & -\frac{7}{2} \\ -2 & \frac{5}{2} \end{bmatrix}$.

3.5.16 We adjoin an identity matrix and row reduce:

$$\begin{aligned} \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{bmatrix} &\xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow[3R_2+R_1]{-3R_2+R_3} \begin{bmatrix} 1 & 0 & 3 & -2 & 3 & 3 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & -3 & 1 & -3 & -2 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 3 & -2 & 3 & 3 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix} \\ &\xrightarrow[-3R_3+R_1]{-2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix} \end{aligned}$$

So the inverse is $\begin{bmatrix} -1 & 0 & 1 \\ -\frac{1}{3} & -1 & -\frac{1}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$

3.6.4 First, we expand along the third row:

$$\det \begin{bmatrix} 5 & 11 & 8 & 7 \\ 3 & -2 & 6 & 23 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 0 & 17 \end{bmatrix} = (-1)(-3)\det \begin{bmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{bmatrix}$$

Next, we expand along the third row:

$$(3)\det \begin{bmatrix} 5 & 11 & 8 \\ 3 & -2 & 6 \\ 0 & 4 & 0 \end{bmatrix} = (3)(-1)(4)\det \begin{bmatrix} 5 & 8 \\ 3 & 6 \end{bmatrix}$$

Finally, we evaluate using our formula for 2×2 determinants:

$$(-12)(5^*6 - 8^*3) = (-12)(6) = -72$$

3.6.9 Before proceeding with our calculation, we do the row operation $(-2)R_1 + R_3$ which does not change the determinant:

$$\det \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 6 & -4 & 12 \end{bmatrix} = \det \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{bmatrix}$$

We expand this along the new and improved first column:

$$\begin{aligned} \det \begin{bmatrix} 3 & -2 & 5 \\ 0 & 5 & 17 \\ 0 & 0 & 2 \end{bmatrix} &= (+1)(3)\det \begin{bmatrix} 5 & 17 \\ 0 & 2 \end{bmatrix} \\ &= (3)(5^*2 - 17^*0) \\ &= 30 \end{aligned}$$

3.6.21 Our system is

$$\begin{aligned} 3x + 4y &= 2 \\ 5x + 7y &= 1 \end{aligned}$$

The coefficient matrix has determinant $\det \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = 3^*7 - 4^*5 = 1$, so since this nonzero we can proceed with Cramer's Rule. Cramer's Rule tells us that the unique solution to this system is

$$
\begin{aligned}
x &= \frac{\det \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}}{\det \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}} \\
&= \frac{2*7 - 4*1}{3*7 - 4*5} \\
&= 10 \\
y &= \frac{\det \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}}{\det \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}} \\
&= \frac{3*1 - 2*5}{3*7 - 4*5} \\
&= -7
\end{aligned}
$$

Additional Problems:

1. Some possible easy choices here include taking $A = B$, taking $B = A^{-1}$, taking one of the matrices to be the identity, or choosing 1×1 matrices.

2. One choice that works here is $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We compute

$$
\begin{aligned}
CD &= \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \\
DC &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}
\end{aligned}
$$

   If you pick random entries for your matrices, odds are that they will work for this problem. Another easy choice would be picking C to be $n \times m$ and D to be $m \times n$ where $m \neq n$. Then CD and DC are different sizes, so are certainly not equal!

3. The product ABC is invertible, and the inverse is $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. To see why this is the inverse, we compute:

$$
\begin{aligned}
(ABC)(C^{-1}B^{-1}A^{-1}) &= AB(CC^{-1})B^{-1}A^{-1} \\
&= ABIB^{-1}A^{-1} \\
&= A(BB^{-1})A^{-1} \\
&= AIA^{-1} \\
&= AA^{-1} \\
&= I
\end{aligned}
$$

   You can calulate similarly that $(C^{-1}B^{-1}A^{-1})(ABC) = I$.

4. The determinant of $T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ is $t_1 t_2 t_3$. The cofactor expansion is straightforward here, no matter which row or column you choose to use.

   For an $n \times n$ diagonal matrix T, the determinant is the product $t_1 t_2 \cdots t_n$. In mathematics, we usually write a product like this as $\prod_{i=1}^{n} t_i$. This kind of notation is called "product notation" and works very similarly to summation notation that you may be already familiar with.