HW 12

Math 2243
Linear Algebra and Differential Equations

Fall 2020
University of Minnesota, Twin Cities

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December 6, 2020 Compiled on December 6, 2020 at 5:18am

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1 Problem 4, section 1.4

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

$$(1+x)\frac{dy}{dx} = 4y$$

Solution

This is separable as it can be written as

$$\frac{dy}{dx} = F(x)G(y)$$

Where in this case $G(y) = y$ and $F(x) = \frac{4}{1+x}$. Assuming $x \neq 1$. Therefore we can now separate and write

$$\begin{aligned}\frac{dy}{dx}\frac{1}{G(y)} &= F(x)\\ \frac{dy}{G(y)} &= F(x)dx\end{aligned}$$

Integrating both sides gives

$$\int \frac{dy}{G(y)} = \int F(x)dx$$

Replacing $G(y) = y$ and $F(x) = \frac{4}{1+x}$, the above becomes

$$\begin{aligned}\int \frac{dy}{y} &= \int \frac{4}{1+x}dx\\ \ln|y| &= \ln\left|(1+x)^4\right| + c\end{aligned}$$

Taking the exponential of both sides

$$\begin{aligned}|y| &= e^{\ln\left|(1+x)^4\right|+c}\\ &= e^c e^{\ln\left|(1+x)^4\right|}\end{aligned}$$

Let $e^c = c_1$ and since $(1+x)^4$ can not be negative, therefore the above simplifies to

$$\begin{aligned}|y| &= c_1 e^{\ln(1+x)^4}\\ &= c_1(1+x)^4\end{aligned}$$

Let the sign $\pm$ be absorbed into the constant of integration. The above simplifies to

$$y(x) = c_1(1+x)^4 \qquad x \neq 1$$

2 Problem 17, section 1.4

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

$$\frac{dy}{dx} = 1 + x + y + xy$$

<u>Solution</u>

Writing the above as

$$\frac{dy}{dx} = (1+x)(1+y)$$

This is separable. It can be written as

$$\frac{dy}{dx} = F(x)G(y)$$

Where in this case $G(y) = (1+y)$ and $F(x) = (1+x)$. Therefore we can now separate and write

$$\begin{aligned}\frac{dy}{dx}\frac{1}{G(y)} &= F(x)\\ \frac{dy}{G(y)} &= F(x)dx\end{aligned}$$

Integrating both sides gives

$$\int \frac{dy}{G(y)} = \int F(x)dx$$

Replacing $G(y) = (1+y)$ and $F(x) = (1+x)$, the above becomes

$$\begin{aligned}\int \frac{dy}{(1+y)} &= \int (1+x)dx\\ \ln|1+y| &= x + \frac{x^2}{2} + c\end{aligned}$$

Taking the exponential of both sides

$$\begin{aligned}|1+y| &= e^{x+\frac{x^2}{2}+c}\\ &= e^c e^{x+\frac{x^2}{2}}\end{aligned}$$

Let $e^c = c_1$ the above becomes

$$|1+y| = c_1 e^{x+\frac{x^2}{2}}$$

Let the sign $\pm$ be absorbed into the constant of integration. The above simplifies to

$$\begin{aligned}1+y &= c_1 e^{x+\frac{x^2}{2}}\\ y &= c_1 e^{x+\frac{x^2}{2}} - 1\end{aligned}$$

3 Problem 19, section 1.4

Find explicit particular solutions of the initial value problems in Problems 19 through 28.

$$\frac{dy}{dx} = ye^x$$
$$y(0) = 2e$$

Solution

This is separable because it can be written as

$$\frac{dy}{dx} = F(x)G(y)$$

Where in this case $G(y) = y$ and $F(x) = e^x$. Therefore we can now separate and write

$$\frac{dy}{dx}\frac{1}{G(y)} = F(x)$$
$$\frac{dy}{G(y)} = F(x)dx$$

Integrating both sides gives

$$\int \frac{dy}{G(y)} = \int F(x)dx$$

Replacing $G(y) = y$ and $F(x) = e^x$, the above becomes

$$\int \frac{dy}{y} = \int e^x dx$$
$$\ln|y| = e^x + c$$

Taking the exponential of both sides

$$|y| = e^{e^x + c}$$
$$= e^c e^{e^x}$$

Let $e^c = c_1$, therefore the above simplifies to

$$|y| = c_1 e^{e^x}$$

Let the sign $\pm$ be absorbed into the constant of integration. The above simplifies to

$$y(x) = c_1 e^{e^x} \tag{1}$$

Now we apply initial conditions to find c_1. Since $y(0) = 2e$ then the above solution becomes

$$2e = c_1 e$$
$$c_1 = 2$$

Hence the general solution (1) now becomes

$$y(x) = 2e^{e^x}$$

4 Problem 33, section 1.4

A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

Solution

The differential equation model is

$$\frac{dP}{dt} = kP$$

Where $P(t)$ is the population at time t. The initial conditions are $P(0) = 25000$ where $t = 0$ is taken as the year 1960. We are also given that $P(10) = 30000$. We are asked to determine $P(40)$ which is the year 2000. First we solve the ode. This is both linear and separable. Using the separable method, it can be written as

$$\frac{dP}{dt} = F(t)G(P)$$

Where in this case $G(P) = P$ and $F(t) = k$. Therefore we can now separate and write

$$\frac{dP}{dt}\frac{1}{G(P)} = F(t)$$
$$\frac{dP}{G(P)} = F(t)dt$$

Integrating both sides gives

$$\int \frac{dP}{G(P)} = \int F(t)dt$$

Replacing $G(P) = P$ and $F(t) = k$, the above becomes

$$\int \frac{dP}{P} = \int k dt$$
$$\ln P = kt + c$$

No need for absolute sign here, since P can not be negative. Taking exponential of both sides gives

$$P(t) = ce^{kt} \tag{1}$$

Applying initial conditions $P(0) = 25000$ the above gives

$$25000 = c$$

Hence (1) now becomes

$$P(t) = 25000e^{kt} \tag{2}$$

Applying second condition $P(10) = 30000$ to the above gives

$$30000 = 25000e^{10k}$$
$$\frac{30000}{25000} = e^{10k}$$
$$\frac{6}{5} = e^{10k}$$

Taking natural log of both sides

$$\ln\left(\frac{6}{5}\right) = 10k$$
$$k = \frac{1}{10}\ln\left(\frac{6}{5}\right)$$

Hence (2) becomes

$$P(t) = 25000e^{\left(\frac{1}{10}\ln\left(\frac{6}{5}\right)\right)t}$$

At $t = 40$

$$P(40) = 25000e^{\left(\frac{1}{10}\ln\left(\frac{6}{5}\right)\right)40}$$

Using calculator it gives

$$P(40) = 51840$$

Hence the population in year 2000 is 51840.

5 Problem 43, section 1.4

Cooling. A pitcher of buttermilk initially at 25 C is to be cooled by setting it on the front porch, where the temperature is 0 C. Suppose that the temperature of the buttermilk has dropped to 15 C after 20 min. When will it be at 5 C?

Solution

Cooling of object is governed by the Newton's law cooling

$$\frac{dT}{dt} = k(T_{out} - T)$$

Where T_{out} is the ambient temperature, which is 0 C in this problem and k is positive constant. Hence the above becomes

$$\frac{dT}{dt} = -kT$$

This is separable (and also linear in T). Solving it as separable, it can be written as

$$\frac{dT}{dt} = F(t)G(T)$$

Where in this case $G(T) = T$ and $F(t) = -k$. Therefore we can now separate and write

$$\begin{aligned} \frac{dT}{dt}\frac{1}{G(T)} &= F(t) \\ \frac{dT}{G(T)} &= F(t)dt \end{aligned}$$

Integrating both sides gives

$$\int \frac{dT}{G(T)} = \int F(t)dt$$

Replacing $G(T) = T$ and $F(t) = -k$, the above becomes

$$\begin{aligned} \int \frac{dy}{T} &= -k \int dt \\ \ln|T| &= -kt + c \end{aligned}$$

Taking the exponential of both sides

$$\begin{aligned} |T| &= e^{-kt+c} \\ &= e^{c}e^{-kt} \end{aligned}$$

Let $e^c = c_1$, therefore the above simplifies to

$$|T| = c_1 e^{-kt}$$

Let the sign $\pm$ be absorbed into the constant of integration. The above simplifies to

$$T(t) = c_1 e^{-kt} \tag{1}$$

Now initial conditions are used to determine c_1. At $t = 0$, we are given $T(0) = 25$. The above becomes

$$25 = c_1$$

Therefore (1) becomes

$$T(t) = 25e^{-kt} \tag{2}$$

Now the second condition $T(20) = 15$ is used to determine k. The above becomes

$$\begin{aligned} 15 &= 25e^{-20k} \\ \frac{15}{25} &= e^{-20k} \\ \frac{3}{5} &= e^{-20k} \end{aligned}$$

Taking natural log of both sides gives (using property $\ln e^{f(x)} = f(x)$)

$$\begin{aligned}\ln\left(\frac{3}{5}\right) &= -20k \\ k &= \frac{-1}{20}\ln\left(\frac{3}{5}\right) \\ &= \frac{1}{20}\ln\frac{5}{3}\end{aligned}$$

Substituting the above value of k back into (2) gives

$$\begin{aligned}T(t) &= 25e^{\left(\frac{-1}{20}\ln\frac{5}{3}\right)t} \\ &= 25e^{\left(\frac{1}{20}\ln\frac{3}{5}\right)t}\end{aligned}$$

To answer the final part, let $T(t) = 5$ and we need to solve for t from the above.

$$\begin{aligned}5 &= 25e^{\left(\frac{1}{20}\ln\frac{3}{5}\right)t} \\ \frac{1}{5} &= e^{\left(\frac{1}{20}\ln\frac{3}{5}\right)t}\end{aligned}$$

Taking natural log of both sides gives

$$\begin{aligned}\ln\left(\frac{1}{5}\right) &= \left(\frac{1}{20}\ln\frac{3}{5}\right)t \\ t &= \frac{\ln\left(\frac{1}{5}\right)}{\ln\left(\frac{3}{5}\right)^{\frac{1}{20}}}\end{aligned}$$

Using the calculator gives

$$t = 63.013 \text{ min}$$

6 Problem 3, section 1.5

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

$$y' + 3y = 2xe^{-3x} \tag{1}$$

Solution

This is of the form $y' + p(x)y = q(x)$. Hence it is linear in y. Where

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 2xe^{-3x} \end{aligned}$$

The integrating factor is

$$\begin{aligned} \rho &= e^{\int p dx} \\ &= e^{\int 3 dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of (1) by the integration factor gives

$$\begin{aligned} \frac{d}{dx}(y\rho) &= \rho\left(2xe^{-3x}\right) \\ \frac{d}{dx}\left(e^{3x}y\right) &= e^{3x}\left(2xe^{-3x}\right) \\ \frac{d}{dx}\left(e^{3x}y\right) &= 2x \end{aligned}$$

Integrating gives

$$\begin{aligned} \int d\left(e^{3x}y\right) &= \int 2x dx \\ e^{3x}y &= x^2 + c \\ y(x) &= e^{-3x}\left(x^2 + c\right) \end{aligned}$$

The above is the general solution.

7 Problem 17, section 1.5

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

$$\begin{aligned}(1+x)y' + y &= \cos x \\ y(0) &= 1\end{aligned} \tag{1}$$

Solution

Dividing both sides of (1) by $(1+x)$ where $x \neq -1$ gives

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x} \tag{2}$$

This is now in the form $y' + p(x)y = q(x)$. Hence it is linear in y. Where

$$\begin{aligned}p(x) &= \frac{1}{1+x} \\ q(x) &= \frac{\cos x}{1+x}\end{aligned}$$

The integrating factor is

$$\begin{aligned}\rho &= e^{\int p dx} \\ &= e^{\int \frac{1}{1+x} dx} \\ &= e^{\ln(1+x)} \\ &= 1+x\end{aligned}$$

Multiplying both sides of (2) by the above integration factor gives

$$\begin{aligned}\frac{d}{dx}(y\rho) &= \rho\left(\frac{\cos x}{1+x}\right) \\ \frac{d}{dx}\big((1+x)y\big) &= (1+x)\left(\frac{\cos x}{1+x}\right) \\ \frac{d}{dx}\big((1+x)y\big) &= \cos x\end{aligned}$$

Integrating gives

$$\begin{aligned}\int d\big((1+x)y\big) &= \int \cos x dx \\ (1+x)y &= \sin x + c \\ y(x) &= \frac{1}{1+x}(\sin x + c) \qquad x \neq -1\end{aligned} \tag{3}$$

The above is the general solution. Now we use initial conditions to determine c. Since we are given that $y(0) = 1$ then (3) becomes

$$\begin{aligned}1 &= (\sin 0 + c) \\ c &= 1\end{aligned}$$

Therefore (3) becomes

$$y(x) = \frac{1}{1+x}(1 + \sin x) \qquad x \neq -1$$

8 Problem 37, section 1.5

A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Solution

Let $x(t)$ be mass of salt in lb at time t in the tank. The differential equation that describes how the mass of salt changes in time is therefore

$$\frac{dx}{dt} = (5)(1) - (3)\frac{x}{V(t)} \tag{1}$$

But

$$\begin{aligned} V(t) &= 100 + (5t - 3t) \\ &= 100 + 2t \end{aligned}$$

Therefore (1) becomes

$$\frac{dx}{dt} = 5 - 3\frac{x}{100 + 2t}$$

$$\frac{dx}{dt} + \frac{3}{100 + 2t}x = 5 \tag{2}$$

This is now in the form $x' + p(t)x = q(t)$. Hence it is linear in x. Where

$$\begin{aligned} p(t) &= \frac{3}{100 + 2t} \\ q(t) &= 5 \end{aligned}$$

The integrating factor is

$$\begin{aligned} \rho &= e^{\int p dt} \\ &= e^{3\int \frac{1}{100+2t} dt} \end{aligned}$$

Let $100 + 2t = u$. Hence $\frac{du}{dt} = 2$. The integral becomes $\int \frac{1}{100+2t} dt$ becomes $\int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(100 + 2t)$. The above becomes

$$\begin{aligned} \rho &= e^{\frac{3}{2}\ln(100+2t)} \\ &= (100 + 2t)^{\frac{3}{2}} \end{aligned}$$

Multiplying both sides of (2) by the above integration factor gives

$$\frac{d}{dt}\left(x\rho\right) = 5\rho$$

$$\frac{d}{dt}\left((100 + 2t)^{\frac{3}{2}}x\right) = 5(100 + 2t)^{\frac{3}{2}}$$

Integrating gives

$$(100 + 2t)^{\frac{3}{2}}x = 5\int (100 + 2t)^{\frac{3}{2}} dt$$

Let $100 + 2t = u$ hence $\frac{du}{dt} = 2$ and the integral on the right becomes $\int \frac{u^{\frac{3}{2}}}{2} du = \frac{1}{2}\frac{u^{\frac{5}{2}}}{\frac{5}{2}} = \frac{1}{5}u^{\frac{5}{2}}$. Hence the above now becomes

$$\begin{aligned} (100 + 2t)^{\frac{3}{2}}x &= 5\left(\frac{1}{5}u^{\frac{5}{2}}\right) + c \\ &= u^{\frac{5}{2}} + c \\ &= (100 + 2t)^{\frac{5}{2}} + c \end{aligned}$$

Solving for $x(t)$ gives

$$\begin{aligned} x &= (100+2t)^{\frac{5}{2}-\frac{3}{2}} + c(100+2t)^{\frac{-3}{2}} \\ &= (100+2t) + c(100+2t)^{\frac{-3}{2}} \end{aligned} \tag{3}$$

Now we find c from initial conditions. At $t = 0$ we are told that $x = 50$. Hence

$$\begin{aligned} 50 &= (100) + c(100)^{\frac{-3}{2}} \\ -50 &= \frac{c}{100^{\frac{3}{2}}} \\ c &= (-50)\left(100^{\frac{3}{2}}\right) \\ &= -50000 \end{aligned}$$

Therefore (3) becomes

$$x(t) = (100+2t) - \frac{50000}{(100+2t)^{\frac{3}{2}}} \tag{4}$$

The above gives the mass of salt as function of time. We now to find the time when the tank is full. From the volume function we know that $V(t) = 100 + 2t$. Since the tank size is 400 gal, then we solve for t from

$$\begin{aligned} 400 &= 100 + 2t \\ t &= \frac{300}{2} \\ &= 150 \text{ sec} \end{aligned}$$

So the tank fills up after 150 seconds. Substituting this value of time in (4) gives

$$\begin{aligned} x(t) &= (100 + 2(150)) - \frac{50000}{(100+2(150))^{\frac{3}{2}}} \\ &= (100+300) - \frac{50000}{(100+300)^{\frac{3}{2}}} \\ &= 400 - \frac{50000}{400^{\frac{3}{2}}} \\ &= \frac{1575}{4} \\ &= 393.75 \text{ lb} \end{aligned}$$

9 Problem 15, section 2.1

Consider a population $P(t)$ satisfying the logistic equation $\frac{dP}{dt} = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur. If the initial population $P(0) = P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is $M = \frac{B_0 P_0}{D_0}$

Solution

We are given the logistic equation in he form

$$\begin{aligned}\frac{dP}{dt} &= aP - bP^2\\ &= a\left(P - \frac{b}{a}P^2\right)\\ &= aP\left(1 - \frac{b}{a}P\right) \qquad (1)\end{aligned}$$

Comparing (1) to the other standard form given in textbook which is

$$\frac{dP}{dt} = kP(M - P) \qquad (2)$$

Where in this form M is the limiting population. Factoring M out from (2) gives

$$\frac{dP}{dt} = (kM)P\left(1 - \frac{P}{M}\right) \qquad (3)$$

Comparing (1) and (3) shows that, by inspection that

$$\begin{aligned}a &= kM\\ M &= \frac{a}{b} \qquad (4)\end{aligned}$$

But we are told that $a = \frac{B}{P}$. At time $t = 0$ this gives

$$a = \frac{B_0}{P_0} \qquad (5)$$

And we are told that $b = \frac{D}{P^2}$ which at $t = 0$ gives

$$b = \frac{D_0}{P_0^2} \qquad (6)$$

Substituting (5,6) back in (4) gives

$$\begin{aligned}M &= \frac{\frac{B_0}{P_0}}{\frac{D_0}{P_0^2}}\\ &= \frac{B_0 P_0^2}{P_0 D_0}\end{aligned}$$

Or

$$M = \frac{B_0 P_0}{D_0}$$

Which is what we are asked to show.

10 Problem 16, section 2.1

Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 15. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M ?

Solution

We have $P(0) = 120$ and $B = aP = 8$ per month and $D = bP^2 = 6$ per month. Hence

$$a = \frac{B}{P} = \frac{8}{P(0)} = \frac{8}{120} = \frac{1}{15}$$

The limiting population is

$$\begin{aligned} M &= \frac{B_0 P_0}{D_0} \\ &= \frac{(8)(120)}{6} \\ &= 160 \end{aligned}$$

Therefore, we need to find the time the population reaches 95% of the above value, or $\frac{95}{100}(160) = 152$ rabbits. The solution to the logistic equation is given in equation (7) page 77 as

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

This was derived from the form $\frac{dP}{dt} = kp(M - P)$. But as we found in the last problem, $k = \frac{a}{M}$ and $a = \frac{1}{15}$ in this problem. Hence $k = \frac{1}{15M}$. The above solution now becomes

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-\frac{1}{15}t}}$$

But $M = 160$ and $P_0 = 120$. The above becomes

$$\begin{aligned} P(t) &= \frac{(160)(120)}{120 + (160 - 120)e^{-\frac{1}{15}t}} \\ &= \frac{19200}{120 + 40e^{-\frac{1}{15}t}} \end{aligned}$$

We want to find t when $P(t) = 152$. Hence

$$152 = \frac{19200}{120 + 40e^{-\frac{1}{15}t}}$$

We need to solve the above for t.

$$\begin{aligned} 152\left(120 + 40e^{-\frac{1}{15}t}\right) &= 19200 \\ 6080e^{\frac{-1}{15}t} + 18240 &= 19200 \\ e^{\frac{-1}{15}t} &= \frac{19200 - 18240}{6080} \\ &= \frac{3}{19} \end{aligned}$$

Taking natural log gives

$$\begin{aligned} \frac{-1}{15}t &= \ln\left(\frac{3}{19}\right) \\ t &= -15\ln\left(\frac{3}{19}\right) \end{aligned}$$

Using the calculator the above gives

$$t = 27.687 \text{ months}$$

11 Problem 17, section 2.1

Consider a rabbit population $P(t)$satisfying the logistic equation as in Problem 15. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$. How many months does it take for $P(t)$ to reach 105% of the limiting population M ?

Solution

This is similar to the above problem. We have $P(0) = 240$ and $B = aP = 9$ per month and $D = bP^2 = 12$ per month. Hence

$$a = \frac{B}{P} = \frac{9}{P(0)} = \frac{9}{240} = \frac{3}{80}$$

The limiting population is

$$\begin{aligned} M &= \frac{B_0 P_0}{D_0} \\ &= \frac{(9)(240)}{12} \\ &= 180 \end{aligned}$$

Therefore, we need to find the time the population reaches 105% of the above value, or $\frac{105}{100}(180) = 189$ rabbits. The solution to the logistic equation is given in equation (7) page 77 as

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

This was derived from the form $\frac{dP}{dt} = kp(M - P)$. But as we found in the last problem, $k = \frac{a}{M}$ and $a = \frac{3}{80}$ in this problem. Hence $k = \frac{3}{80M}$. The above solution now becomes

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-\frac{3}{80}t}}$$

But $M = 180$ and $P_0 = 240$. The above becomes

$$\begin{aligned} P(t) &= \frac{(180)(240)}{240 + (180 - 240)e^{-\frac{3}{80}t}} \\ &= \frac{43200}{240 - 60e^{-\frac{3}{80}t}} \end{aligned}$$

We want to find t when $P(t) = 189$. Hence

$$189 = \frac{43\,200}{240 - 60e^{-\frac{9}{140}t}}$$

We need to solve the above for t.

$$\begin{aligned} 189\left(240 - 60e^{-\frac{3}{80}t}\right) &= 43200 \\ 45360 - 11\,340e^{-\frac{3}{80}t} &= 43200 \\ e^{-\frac{3}{80}t} &= -\frac{43200 - 45360}{11\,340} \\ &= \frac{4}{21} \end{aligned}$$

Taking natural log gives

$$\begin{aligned} -\frac{3}{80}t &= \ln\left(\frac{4}{21}\right) \\ t &= -\frac{80}{3}\ln\left(\frac{4}{21}\right) \end{aligned}$$

Using the calculator the above gives

$$t = 44.219 \text{ months}$$

12 Additional problem 1

Solution

12.1 Part a

$$y' + y = 0$$

The characteristic equation is

$$r + 1 = 0$$

The root is $r = -1$. Therefore the general solution is given by

$$\begin{aligned} y_h(x) &= Ce^{rx} \\ &= Ce^{-x} \end{aligned}$$

Where C is arbitrary constant.

12.2 Part b

$$y' + y = e^x \tag{1}$$

From part (a) we found that e^{-x} is basis solution for the homogeneous ODE. The RHS in this ode is e^x. No duplication. Therefore we let

$$y_p = Ae^x$$

Substituting this in (1) gives

$$\begin{aligned} Ae^x + Ae^x &= e^x \\ 2A &= 1 \\ A &= \frac{1}{2} \end{aligned}$$

Therefore

$$y_p = \frac{1}{2}e^x$$

12.3 Part c

The general solution is the sum of the homogeneous solution (part a) and the particular solution (part b). Therefore

$$\begin{aligned} y &= y_h + y_p \\ &= Ce^{-x} + \frac{1}{2}e^x \end{aligned}$$

12.4 Part d

The ODE

$$y' + y = e^x \tag{1}$$

Has the form

$$y' + P(x)y = Q(x)$$

Which implies that

$$\begin{aligned} P(x) &= 1 \\ Q(x) &= e^x \end{aligned}$$

12.5 Part e

The integrating factor is therefore $\rho = e^{\int P(x)dx} = e^{\int dx} = e^x$. Multiplying both sides of (1) by ρ results in

$$\begin{aligned} \frac{d}{dx}\left(\rho y\right) &= \rho e^x \\ d\left(\rho y\right) &= \left(\rho e^x\right)dx \\ d\left(e^x y\right) &= e^{2x}dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int e^{2x}dx \\ e^x y &= \frac{1}{2}e^{2x} + C \end{aligned}$$

Therefore

$$y = \frac{1}{2}e^x + Ce^{-x}$$

12.6 Part f

Comparing the solution obtained in part (c) and (e) shows they are the same solution.

13 Additional problem 2

Solution

13.1 Part (a)

$$\frac{dP}{dt} = kP(M-P) \tag{1}$$

The solution $P(t)$, where $P(t)$ is number of positive cases at time t should satisfy the above ODE, with $P(0) = 5000$.

13.2 Part (b)

The ODE in part(a) is separable. It has the form

$$\frac{dP}{dt} = F(t)G(P)$$

Where

$$\begin{aligned} F(t) &= 1 \\ G(P) &= kP(M-P) \end{aligned}$$

Therefore the ODE (1) can be written as

$$\begin{aligned} \frac{dP}{G(P)} &= F(t)dt \\ \frac{dP}{kP(M-P)} &= dt \\ \int \frac{dP}{kP(M-P)} &= \int dt \end{aligned} \tag{2}$$

To integrate the left side will use partial fractions. Let

$$\frac{1}{kP(M-P)} = \frac{A}{kP} + \frac{B}{M-P}$$

Therefore

$$\begin{aligned} A &= \left. \frac{1}{M-P} \right|_{P=0} \\ &= \frac{1}{M} \end{aligned}$$

And

$$\begin{aligned} B &= \left. \frac{1}{kP} \right|_{P=M} \\ &= \frac{1}{kM} \end{aligned}$$

Hence (2) becomes

$$\begin{aligned} \int \frac{1}{M}\frac{1}{kP} + \frac{1}{kM}\frac{1}{M-P} dP &= t + C \\ \frac{1}{kM}\ln|P| - \frac{1}{kM}\ln|M-P| &= t + C \\ \frac{1}{kM}\ln\left|\frac{P}{M-P}\right| &= t + C \\ \ln\left|\frac{P}{M-P}\right| &= kMt + C_2 \end{aligned}$$

Where $C_2 = CkM$ a new constant. The above now can be written as

$$\frac{P}{M-P} = C_3 e^{kMt}$$

Where the ± sign it taken care of by the constant C_3. Hence

$$\begin{aligned}
P &= C_3 e^{kMt}(M-P) \\
P &= C_3 M e^{kMt} - C_3 P e^{kMt} \\
P + C_3 P e^{kMt} &= C_3 M e^{kMt} \\
P\left(1 + C_3 e^{kMt}\right) &= C_3 M e^{kMt} \\
P(t) &= \frac{C_3 M e^{kMt}}{1 + C_3 e^{kMt}} \\
&= \frac{C_3 M}{e^{-kMt} + C_3} \qquad (3)
\end{aligned}$$

When $t = 0, P = P_0$. Hence the above becomes

$$\begin{aligned}
P_0 &= \frac{C_3 M}{1 + C_3} \\
P_0 + P_0 C_3 &= C_3 M \\
C_3(P_0 - M) &= -P_0 \\
C_3 &= \frac{P_0}{M - P_0}
\end{aligned}$$

Substituting this back in (3) gives

$$\begin{aligned}
P(t) &= \frac{\frac{P_0}{M-P_0} M}{e^{-kMt} + \frac{P_0}{M-P_0}} \\
&= \frac{P_0 M}{e^{-kMt}(M - P_0) + P_0}
\end{aligned}$$

Or

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

Which is the solution given in the textbook. Now, using $P_0 = 5000$ given in this problem gives

$$P(t) = \frac{5000M}{5000 + (M - 5000)e^{-kMt}}$$

But $M = 100000$ which is the limiting capacity (total population). The above simplifies to

$$\begin{aligned}
P(t) &= \frac{(5000)(100000)}{5000 + (100000 - 5000)e^{-100000kt}} \\
&= \frac{(5000)(100000)}{5000 + (95\,000)e^{-100000kt}} \\
&= \frac{100000}{1 + \left(\frac{95\,000}{5000}\right)e^{-100000kt}} \\
&= \frac{100000}{1 + 19e^{-100000kt}} \qquad (4)
\end{aligned}$$

The above is the solution we will use for the rest of the problem.

13.3 Part (c)

We are told there are 500 new cases on first day. This means $P(1) = 5000 + 500 = 5500$. Using the solution found above we now solve for k. Let $t = 1$, we obtain

$$\begin{aligned}
5500 &= \frac{100000}{1 + 19e^{-100000k}} \\
e^{-100000k} &= 100000 \\
&= \frac{100000 - 5500}{(19)(5500)} \\
&= \frac{189}{209}
\end{aligned}$$

Hence

$$\begin{aligned} -k100000 &= \ln\left(\frac{189}{209}\right) \\ k &= -\frac{1}{100000}\ln\left(\frac{189}{209}\right) \\ &= 1\times 10^{-6} \end{aligned}$$

13.4 Part (d)

We now need to find the time t where $P(t) = 50000$. Therefore, using (4)

$$P(t) = \frac{100000}{1 + 19e^{-100000kt}}$$

And replacing k by value found in part(c) and $P(t)$ by 50000 gives

$$\begin{aligned} 50000 &= \frac{100000}{1 + 19e^{-100000\left(1\times 10^{-6}\right)t}} \\ 50000 &= \frac{100000}{1 + 19e^{-\frac{1}{10}t}} \\ 50000\left(1 + 19e^{-\frac{1}{10}t}\right) &= 100000 \\ 1 + 19e^{-\frac{1}{10}t} &= \frac{100000}{50000} \\ 1 + 19e^{-\frac{1}{10}t} &= 2 \\ e^{-\frac{1}{10}t} &= \frac{1}{19} \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{1}{10}t &= \ln\left(\frac{1}{19}\right) \\ t &= -10\ln\left(\frac{1}{19}\right) \\ &= 29.444 \end{aligned}$$

Therefore it will take about 29 days for the half the population to be infected.

13.5 Part (e)

The model

$$\frac{dP}{dt} = kP(M - P)$$

Says that the rate of infection depends on $M-P$ where P is current size of infected population and M is limiting size of the population that could become infected, which is assumed to be the total population, and this is assumed to remain constant all the time. Hence as more population is infected, the value $M - P$ becomes smaller and smaller, since $P(t)$ is increasing, but M is fixed. This means the rate at which people get infected becomes smaller as more people are infected. This is a good model, assuming people who get infected remain infected all the time, which is the case here, and assuming M remain constant. This model does not account for death or birth of the overall population and any migration from outside. A more accurate model would account for this.

This model gives useful information for predicting how many of the population will become infected in the future given initial conditions.