Homework 11 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

1.2.6 We are given $y' = x\sqrt{x^2+9}$; $y(-4) = 0$. We integrate to solve the differential equation, using a u-substitution of $u = x^2 + 9$ so that $du = 2x\,dx$.

$$\begin{aligned} y &= \int x\sqrt{x^2+9}\,dx \\ &= \frac{1}{2}\int \sqrt{u}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}} + C \\ &= \frac{1}{3}(x^2+9)^{\frac{3}{2}} + C \end{aligned}$$

To solve for C, we plug in our initial condition:

$$\begin{aligned} y(-4) = 0 &= \frac{1}{3}\left((-4)^2+9\right)^{\frac{3}{2}} + C \\ &= \frac{1}{3}(25)^{\frac{3}{2}} + C \\ &= \frac{125}{3} + C \end{aligned}$$

So $C = -\frac{125}{3}$ and our particular solution is $y(x) = \frac{1}{3}(x^2+9)^{\frac{3}{2}} - \frac{125}{3}$.

1.2.8 We are given $y' = \cos(2x)$; $y(0) = 1$. We integrate to solve the differential equation using $u = 2x$ so that $du = 2\,dx$.

$$\begin{aligned} y &= \int \cos(2x)\,dx \\ &= \frac{1}{2}\int \cos u\,du \\ &= \frac{1}{2}\sin u + C \\ &= \frac{1}{2}\sin(2x) + C \end{aligned}$$

To solve for C, we plug in our intial condition:

$$\begin{aligned} y(0) = 1 &= \frac{1}{2}\sin(0) + C \\ &= C \end{aligned}$$

So $C = 1$ and our particular solution is $y(x) = \frac{1}{2}\sin(2x) + 1$.

1.2.24 The building is 400 ft high, so $y_0 = 400$. We drop the ball, so $v_0 = 0$. The acceleration is $a(t) = -32$ ft/s^2 due to gravity. Here, we are using the convention that movement downward is a negative velocity. This is consistent with the ball falling down from $y(0) = 400$ to $y(t) = 0$.

Since we have constant acceleration, we can use the formulas given in the textbook. So the velocity and position functions are

$$\begin{aligned} v(t) &= -32t \\ y(t) &= -16t^2 + 400 \end{aligned}$$

We want to know when $y(t) = 0$. So, we solve

$$\begin{aligned} -16t^2 + 400 &= 0 \\ 16t^2 &= 400 \\ t^2 &= 25 \end{aligned}$$

There are two solutions here, $t = \pm 5$. Since we are modeling a process that is going forward in time, we choose the solution $t = 5$. At $t = 5$, we have $v(t) = -160$.

So the ball hits the ground after 5 seconds, at which point it is traveling 160 ft/s downwards.

1.2.26 We start on top of a building 20 m high, so $y_0 = 20$. We fire the projectile upwards at 100 m/s, so $v_0 = 100$. The acceleration is constant $a(t) = -9.8$ m/s^2. We are using the same convention about positive and negative velocities as in the previous problem. Our velocity and position functions are

$$\begin{aligned} v(t) &= -9.8t + 100 \\ y(t) &= -4.9t^2 + 100t + 20 \end{aligned}$$

(a) To find the maximum height above the ground, we maximize the function $y(t) = -4.9t^2 + 100t + 20$. This is a downward-facing parabola, with maximum value when $y'(t) = v(t) = 0$. So we solve $-9.8t + 100 = 0$ to get $t = 10.2$. At this time, $y(10.2) = 530.2$ m.

(b) The projectile passes the top of the building again when $y(t) = 20$. So we solve

$$\begin{aligned} -4.9t^2 + 100t + 20 &= 20 \\ t(-4.9t + 100) &= 0 \end{aligned}$$

The solutions here are $t = 0, 20.4$. The $t = 0$ is the moment we shoot the projectile up, and we already knew it was at the top of the building at that point. So the time we are looking for is $t = 20.4$ s.

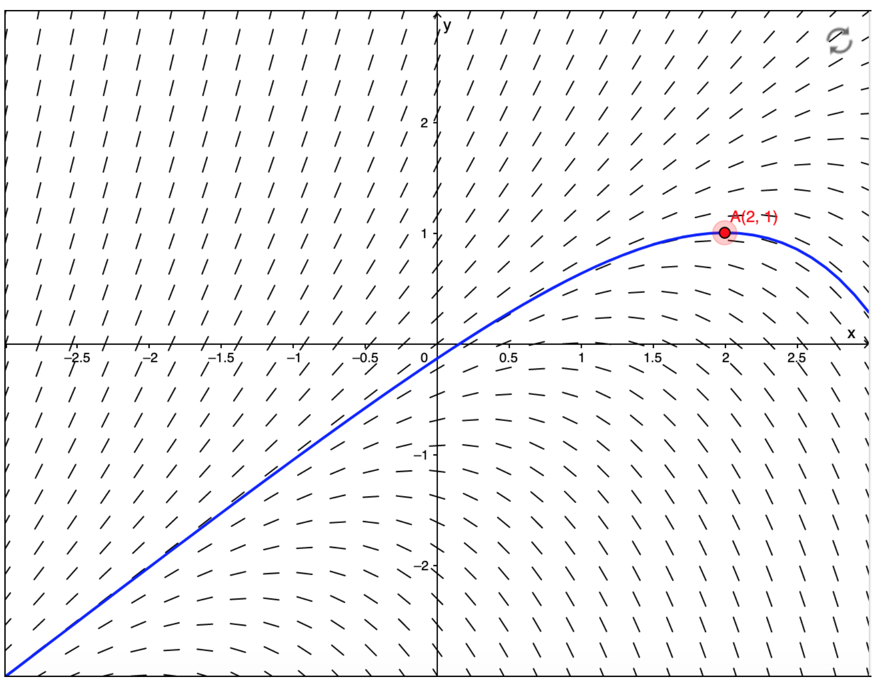
(c) The total time in the air is the time from $t = 0$ until $y(t) = 0$. So we solve

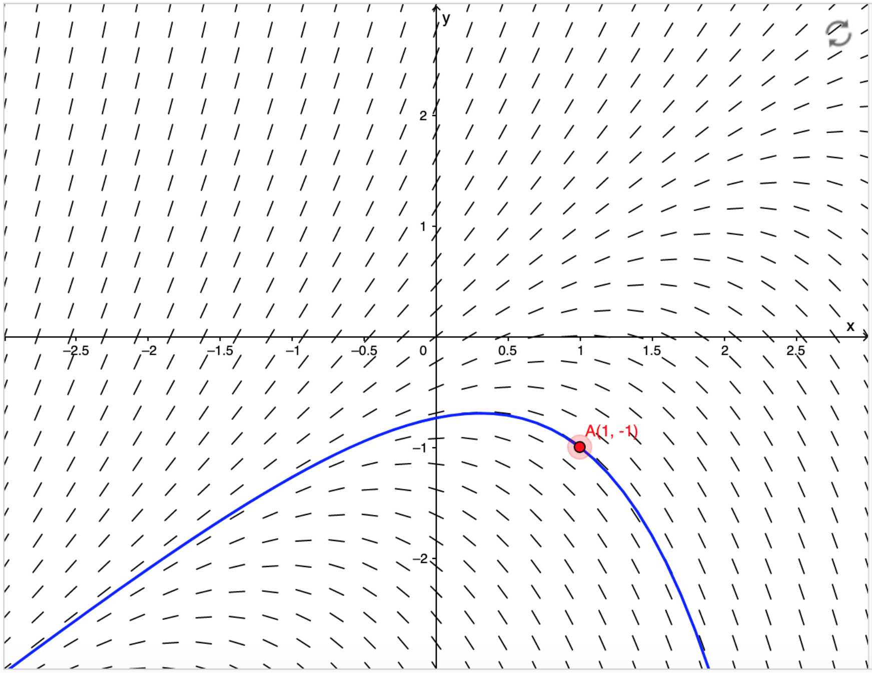
$$-4.9t^2 + 100t + 20 = 0$$

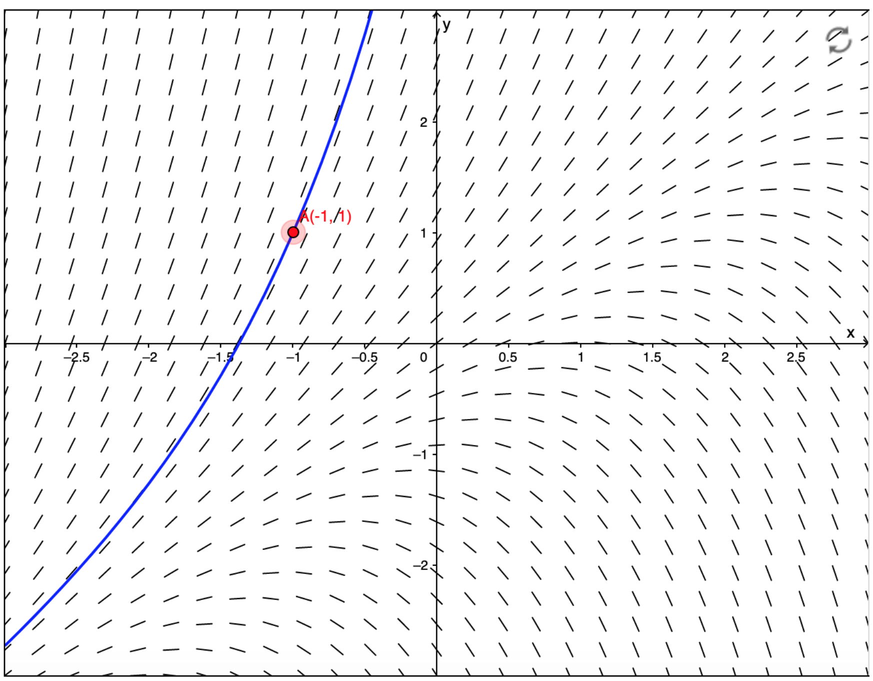
This quadratic has roots $t = 20.6, -0.2$. As usual, we choose the positive time $t = 20.6$ s. So the projectile is in the air for 20.6 seconds.

1.3.5 In order to give you the most accurate picture, I've given computer-generated solution curves. Your sketches don't need to be perfect, but they should look somewhat similar to the real thing.

The three solution curves I've given started from the initial points $(2, 1)$, $(1, -1)$, and $(-1, 1)$.







1.3.9 In order to give you the most accurate picture, I've given computer-generated solution curves. Your sketches don't need to be perfect, but they should look somewhat similar to the real thing.

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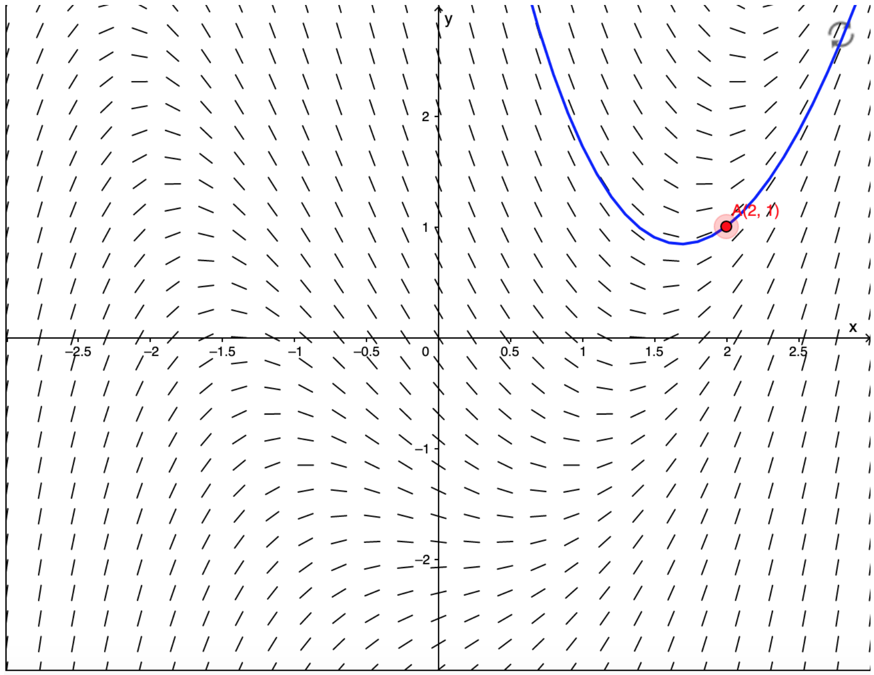

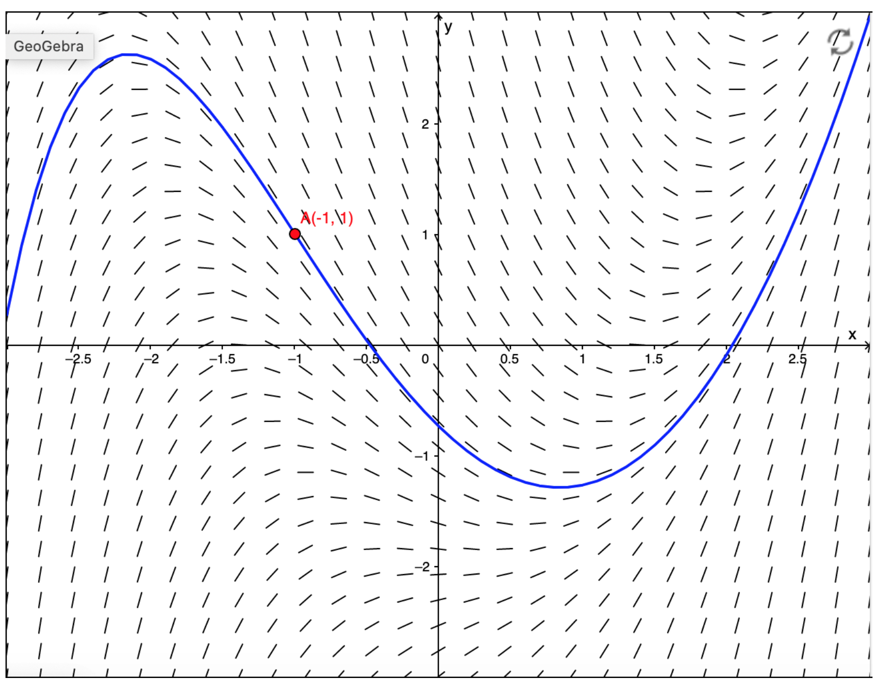

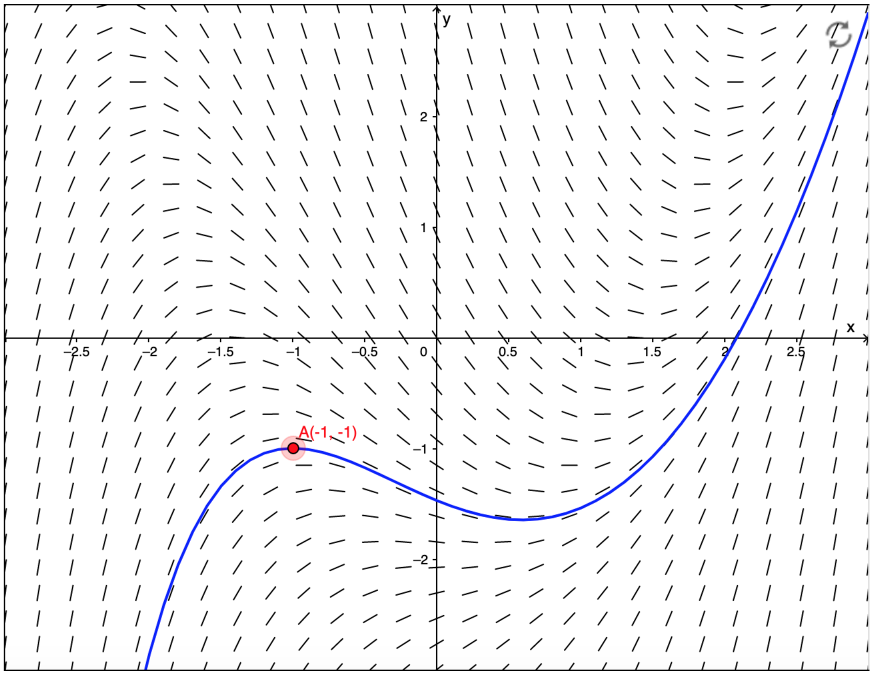

Additional Problems:

1. First, we convert our top speed to m/s:

$$\frac{300 \text{ km}}{1 \text{ h}} \cdot \frac{1000 \text{ m}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 83.33 \text{ m/s}$$

Now all of our units are consistent. Our acceleration is constant $a(t) = 14 \text{ m/s}^2$, our initial velocity is $v_0 = 0$ m/s and our initial position is $x_0 = 0$ m. So, our velocity and position functions are $v(t) = 14t$ and $x(t) = 7t^2$. We want the time when $v(t) = 83.33$, so we solve $14t = 83.33$ to get $t = 5.95$ s. At that time, the position is $x(5.95) = 7 \cdot 5.95^2 = 247.82$ m.

So we reach top speed after 5.95 s, at which point we have traveled 247.82 m.

Cultural Aside: These acceleration and velocity numbers are as close as I could find to the real figures for modern Formula One cars. However, the fastest F1 cars take over 8 seconds to reach 300 kph. There are two primary factors that explain the difference between our calculation and the real-world data. First of all, F1 cars do not have enough traction to convert all of their power to forward motion at low speeds. You will often see cars spinning their wheels at the start of a race for just this reason. The second factor is that once the car gets above about 100 kph, there is a significant amount of drag due to air resistance. The aerodynamics of F1 cars slightly reduce top speed, but make the cars faster over the course of a lap by providing incredible grip through the corners.

2. We already calculated that 300 km/h is 83.33 m/s. We similarly calculate that

$$\frac{80 \text{ km}}{1 \text{ h}} \cdot \frac{1000 \text{ m}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 22.22 \text{ m/s}$$

Our units are now consistent. We are applying brakes which decreases velocity, so the acceleration is a negative constant $a(t) = -39$ m/s^2. The initial speed is $v_0 = 83.33$ m/s. At time $t = 0$ when we start braking, we set our position as $x_0 = 0$ m. So our velocity and position functions are $v(t) = -39t + 83.33$ and $x(t) = -\frac{39}{2}t^2 + 83.33t$. We want to know the time when $v(t) = 22.22$, so we solve $-39t + 83.33 = 22.22$ to get $t = 1.57$ s. At that time, the position is $x(1.57) = -\frac{39}{2}(1.57)^2 + 83.33 \cdot 1.57 = 82.76$ m.

So 82.76 m after we start braking, the car will reach the target speed of 80 km/h. The driver must brake 82.76 m before the corner entry.

3. To take care of units, we calculate

$$\frac{100 \text{ km}}{1 \text{ h}} \cdot \frac{1000 \text{ m}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 27.78 \text{ m/s}$$

We have two different cars' positions to model in this case. We have accelerations $a_A(t) = 14$ and $a_B(t) = 13$. Both cars have the same initial velocity $v_A(0) = v_B(0) = 27.78$. At time $t = 0$, we will set car B's position as $x_B(0) = 0$ and car A 10 m behind at $x_A(0) = -10$. We are looking for the time t where $x_A(t) = x_B(t)$. Our velocity and position functions are

$$\begin{aligned} v_A(t) &= 14t + 27.78 & \qquad v_B(t) &= 13t + 27.78 \\ x_A(t) &= 7t^2 + 27.78t - 10 & \qquad x_B(t) &= 6.5t^2 + 27.78t \end{aligned}$$

We want $x_A(t) = x_B(t)$, so we solve

$$\begin{aligned} 7t^2 + 27.78t - 10 &= 6.5t^2 + 27.78t \\ 0.5t^2 &= 10 \end{aligned}$$

There are two solutions here, one with t positive and one with t negative. It is reasonable to assume that time only moves forwards, so we choose the solution with $t > 0$, namely $t = 4.47$ s. At this time, we have

$$\begin{aligned} v_A(4.47) &= 14 \cdot 4.47 + 27.78 \\ &= 90.36 \\ v_B(4.47) &= 13 \cdot 4.47 + 27.78 \\ &= 85.89 \\ x_A(4.47) = x_B(4.47) &= 7 \cdot 4.47^2 + 27.78 \cdot 4.47 - 10 \\ &= 254.05 \end{aligned}$$

So after 4.47 s and 254.05 m from the corner exit, the two cars will be side-by-side. At that moment, car A is traveling 90.36 m/s (about 325 km/h) and car B is traveling 85.89 m/s (about 309 km/h).

If there is a long enough straight following the corner, we would expect car A to use its superior acceleration to move ahead of car B. However, if car B has a higher top speed or can brake much later into the next corner, it may be able to stay ahead.