Homework 10 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

7.3.4 We have $\vec{x}' = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \vec{x}$. We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{bmatrix} &= (4-\lambda)(-1-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda-5)(\lambda+2) \end{aligned}$$

For $\lambda_1 = -2$, we have

$$\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix} \to \begin{bmatrix} 6 & 1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (-1, 6)$.

For $\lambda_2 = 5$, we have

$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (1, 1)$.

Our general solution is

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 6 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7.3.6 We have $\vec{x}' = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \vec{x}$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{bmatrix} &= (9-\lambda)(-2-\lambda) + 30 \\ &= \lambda^2 - 7\lambda + 12 \\ &= (\lambda-3)(\lambda-4) \end{aligned}$$

For $\lambda_1 = 3$, we have

$$\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix} \to \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (-5, 6)$.

For $\lambda_2 = 4$, we have

$$\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (-1, 1)$.

Our general solution is

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} -5 \\ 6 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To apply our initial conditions, we set $t = 0$:

$$\vec{x}(0) = c_1 \begin{bmatrix} -5 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We solve the system by reducing the augmented matrix:

$$\begin{aligned} \begin{bmatrix} -5 & -1 & 1 \\ 6 & 1 & 0 \end{bmatrix} &\xrightarrow{R_1+R_2} \begin{bmatrix} -5 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{5R_2+R_1} \begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & 1 \end{bmatrix} \end{aligned}$$

So $c_1 = 1$ and $c_2 = -6$. Our particular solution is

$$\vec{x}(t) = e^{3t} \begin{bmatrix} -5 \\ 6 \end{bmatrix} + e^{4t} \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

7.3.8 We have $\vec{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \vec{x}$. We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix} &= (1-\lambda)(-1-\lambda) + 5 \\ &= \lambda^2 + 4 \end{aligned}$$

Our eigenvalues are $\pm 2i$. For $\lambda_1 = 2i$, we have

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \xrightarrow{\frac{1}{1-2i}R_1+R_2} \begin{bmatrix} 1-2i & -5 \\ 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (5, 1-2i)$. The corresponding complex-valued solution is

$$\begin{aligned} \vec{v}_1 e^{\lambda_1 t} &= \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} e^{2it} \\ &= \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} (\cos 2t + i \sin 2t) \\ &= \begin{bmatrix} 5\cos 2t + 5i \sin 2t \\ \cos 2t + i \sin 2t - 2i \cos 2t + 2 \sin 2t \end{bmatrix} \\ &= \begin{bmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + \begin{bmatrix} 5 \sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix} i \end{aligned}$$

Having separated this into real and imaginary components, we now have two linearly independent real-valued solutions. So our general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} 5\sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix}$$

7.3.18 We have $\vec{x}' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \vec{x}$. We need eigenvalues and eigenvectors.

$$\begin{aligned}
\det \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{bmatrix} &= (1-\lambda)\det \begin{bmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{bmatrix} - 2\det \begin{bmatrix} 2 & 2 \\ 1 & 7-\lambda \end{bmatrix} + 2\det \begin{bmatrix} 2 & 2 \\ 7-\lambda & 1 \end{bmatrix} \\
&= (1-\lambda)\left[(7-\lambda)^2 - 1\right] - 2(14 - 2\lambda - 2) + 2(2 - 14 + 2\lambda) \\
&= (1-\lambda)(\lambda^2 - 14\lambda + 48) + 8\lambda - 48 \\
&= (1-\lambda)(\lambda - 6)(\lambda - 8) + 8(\lambda - 6) \\
&= (\lambda - 6)\left[(1-\lambda)(\lambda - 8) + 8\right] \\
&= (\lambda - 6)(-\lambda^2 + 9\lambda) \\
&= -\lambda(\lambda - 6)(\lambda - 9)
\end{aligned}$$

For $\lambda_1 = 0$, we have

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -3 \\ -3 & 3 & \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (-4, 1, 1)$.

For $\lambda_2 = 6$, we have

$$\begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[-R_2+R_3]{3R_2+R_1} \begin{bmatrix} 1 & 5 & 5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 5 & 5 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (0, 1, -1)$.

For $\lambda_3 = 9$, we have

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \xrightarrow[4R_2+R_1]{-R_2+R_3} \begin{bmatrix} 0 & -6 & 6 \\ 2 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{2R_3+R_1} \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix}$$

An eigenvector is $\vec{v}_3 = (1, 2, 2)$.

Our general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

7.3.38 We have the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$. We need eigenvalues and eigenvectors.

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 0 & 3 & 3-\lambda & 0 \\ 0 & 0 & 4 & 4-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)$$

Here we used that the determinant of a lower triangular matrix is the product of the diagonal entries. Alternatively, you can expand along the first row several times.

For $\lambda_1 = 1$, we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

If we set $x_4 = 4$, we get $x_3 = -3$, $x_2 = 2$, and $x_1 = -1$. So an eigenvector is $\vec{v}_1 = (-1, 2, -3, 4)$.

For $\lambda_2 = 2$, we have

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (0, 1, -3, 6)$.

For $\lambda_3 = 3$, we have

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_3 = (0, 0, 1, -4)$.

For $\lambda_4 = 4$, we have

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_4 = (0, 0, 0, 1)$.

Our general solution is

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} + c_4 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Additional Problems:

1. (a) Let $x_1 = x, x_2 = x', x_3 = x''$. We get the system

$$x_3' + x_3 - 2x_2 = 0$$
$$x_2' = x_3$$
$$x_1' = x_2$$

(b) We have to rearrange some terms to write things in matrix form, but we get

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) We need eigenvalues and eigenvectors.

$$\begin{aligned} \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 2 & -1-\lambda \end{bmatrix} &= -\lambda \det \begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} \\ &= -\lambda(\lambda + \lambda^2 - 2) \\ &= -\lambda(\lambda+2)(\lambda-1) \end{aligned}$$

For $\lambda_1 = 0$, we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_1 = (1, 0, 0)$.
For $\lambda_2 = 1$, we have

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector is $\vec{v}_2 = (1, 1, 1)$.
For $\lambda_3 = -2$, we have

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

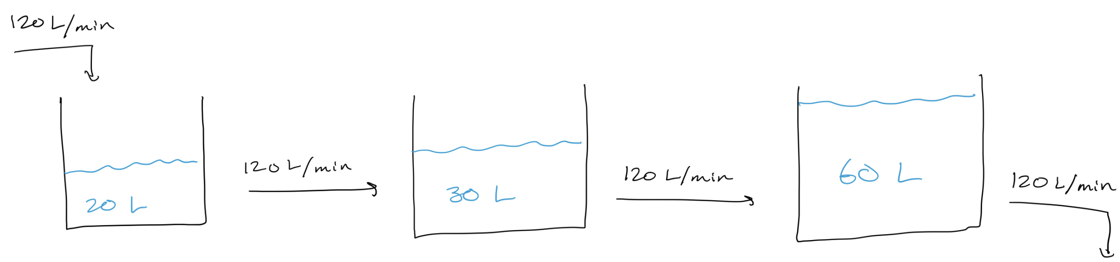
An eigenvector is $\vec{v}_3 = (1, -2, 4)$
Our general solution to this system is thus

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

(d) The first component of $\vec{x}$ is $x_1 = x$. So looking at that first component of our general solution we can say that the general solution of our original equation is

$$x(t) = c_1 + c_2 e^t + c_3 e^{-2t}$$

2. (a) Our system looks like:



(b) Note that each tank has constant volume. Our differential equations for the amount of salt in each tank are:

$$x_1' = -120 * \frac{x_1}{20}$$
$$x_2' = 120 * \frac{x_1}{20} - 120 * \frac{x_2}{30}$$
$$x_3' = 120 * \frac{x_2}{30} - 120 * \frac{x_3}{60}$$

Writing this as a matrix equation, we have

$$\vec{x}' = \begin{bmatrix} -6 & 0 & 0 \\ 6 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} \vec{x}$$

(c) To solve, we need eigenvalues and eigenvectors:

$$\det \begin{bmatrix} -6-\lambda & 0 & 0 \\ 6 & -4-\lambda & 0 \\ 0 & 4 & -2-\lambda \end{bmatrix} = (-6-\lambda)(-4-\lambda)(-2-\lambda)$$

For $\lambda_1 = -6$, we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 6 & 2 & 0 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have eigenvector $\vec{v}_1 = (1, -3, 3)$.

For $\lambda_2 = -4$, we have

$$\begin{bmatrix} -2 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have eigenvector $\vec{v}_1 = (0, 1, -2)$.
For $\lambda_3 = -2$, we have

$$\begin{bmatrix} -4 & 0 & 0 \\ 6 & -2 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have eigenvector $\vec{v}_1 = (0, 0, 1)$.
So our general solution is

$$\vec{x}(t) = c_1 e^{-6t} \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(d) We have the initial condition $\vec{x}(0) = \begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix}$. Plugging this in to our general solution, we have

$$\begin{bmatrix} 100 \\ 20 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ -3c_1 + c_2 \\ 3c_1 - 2c_2 + c_3 \end{bmatrix}$$

We can immediately back substitute to get $c_1 = 100$, $c_2 = 320$, and $c_3 = 340$. The particular solution is

$$\vec{x}(t) = e^{-6t} \begin{bmatrix} 100 \\ -300 \\ 300 \end{bmatrix} + e^{-4t} \begin{bmatrix} 0 \\ 320 \\ -640 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 340 \end{bmatrix}$$