# Midterm 1 Practice Problems

**1. Determine for what values of $k$ the following system has $(a)$ a unique solution, $(b)$ no solution, $(c)$ infinitely many solutions.**

$$\mathbf{3x + 2y = 1}$$

$$\mathbf{7x + 5y = k.}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 7 & 5 & k \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 1 & k-2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & k-2 \\ 3 & 2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & k-2 \\ 0 & -1 & 7-3k \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & k-2 \\ 0 & 1 & 3k-7 \end{array}\right]$$

$y = 3k - 7$ and $x = -y + k - 2 = -(3k-7) + k - 2 = 5 - 2k$.

$\vec{v} = (5 - 2k, 3k - 7) = k(-2, 3) + (5, -7)$.

a) Unique solution for every value of $k$.

b) There are no values of $k$ that will give no solution.

c) There are no values of $k$ that will give infinitely many solution.

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**2. Consider the system:**

$$\mathbf{4x_1 + 5x_2 + 3x_3 = 6}$$

$$\mathbf{3x_1 + 6x_2 + 5x_3 = 12}$$

$$\mathbf{2x_1 + 3x_2 + 2x_3 = 18}$$

**a) Write down the augmented coefficient matrix $M$ of the system.**

$$\left[\begin{array}{ccc|c} 4 & 5 & 3 & 6 \\ 3 & 6 & 5 & 12 \\ 2 & 3 & 2 & 18 \end{array}\right]$$

**b) Use the method of Gaussian elimination to transform the augmented coefficient matrix $M$ to and echelon form matrix.**

$$\Rightarrow \left[\begin{array}{ccc|c} 4 & 5 & 3 & 6 \\ 1 & 3 & 3 & -6 \\ 2 & 3 & 2 & 18 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & -6 \\ 4 & 5 & 3 & 6 \\ 2 & 3 & 2 & 18 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & -6 \\ 0 & -7 & -9 & 30 \\ 0 & -3 & -4 & 30 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & -6 \\ 0 & -1 & -1 & -30 \\ 0 & -3 & -4 & 30 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & -6 \\ 0 & -1 & -1 & -30 \\ 0 & 0 & -1 & 120 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 3 & -6 \\ 0 & 1 & 1 & 30 \\ 0 & 0 & 1 & -120 \end{array}\right]$$

**c) Use the method Gauss Jordan elimination to transform the augmented coefficient matrix** $M$ **to the reduced echelon matrix**.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -96 \\ 0 & 1 & 0 & 150 \\ 0 & 0 & 1 & -120 \end{array}\right]$$

**d) Use either** $b$ **or** $c$ **to solve the system**.

$x_3 = -120, \qquad x_2 = 150, \qquad x_1 = -96.$

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**3. List all possible reduced row echelon forms of a 3 x 3 matrix**.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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**4. Consider the system**:

$x_1 + 2x_2 + 4x_3 = 1$

$x_1 + 3x_2 + 9x_3 = 2$

$x_1 + 4x_2 + 16x_3 = 3$

**a) Write down the coefficient matrix** $A$ **of the system and the corresponding matrix equation** $Ax = b$.

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

**b) Use the algorithm explained in the class (see p181 of the textbook) to find the inverse of $A$.**

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 2 & 12 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 3 & -2 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 0 & -\frac{7}{2} & 6 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right], \quad A^{-1} = \begin{bmatrix} 6 & -8 & 3 \\ -\frac{7}{2} & 6 & -\frac{5}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}.$$

**c) Compute the determinant $\det(A)$ and the cofactor matrix $[A_{ij}]$ of $A$, and use the formula of the inverse for matrices to find $A^{-1}$.**

$$\det(A) = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 12 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 2.$$

$$[A_{ij}] = \begin{bmatrix} 12 & -7 & 1 \\ -16 & 12 & -2 \\ 6 & -5 & 1 \end{bmatrix}, \quad [A_{ij}]^T = \begin{bmatrix} 12 & -16 & 6 \\ -7 & 12 & -5 \\ 1 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -16 & 6 \\ -7 & 12 & -5 \\ 1 & -2 & 1 \end{bmatrix}.$$

**d) Use the formula $x = A^{-1}b$ to solve the system.**

$$x = A^{-1}b = \frac{1}{2}\begin{bmatrix} 12 & -16 & 6 \\ -7 & 12 & -5 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

**e) Use Cramer's rule to solve the system.**

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \\ 3 & 4 & 16 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -1,$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 4 \\ 1 & 2 & 9 \\ 1 & 3 & 16 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 12 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 1,$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$x = (x_1, x_2, x_3) = (-1, 1, 0).$$

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**5**. **Consider the following 3 vectors in** $R^4$:

$\mathbf{v}_1=(1,1,2,1)$, $\mathbf{v}_2=(1,0,3,4)$, $\mathbf{v}_3=(2,2,4,8)$.

**If they are linearly independent**, **show this**. **Otherwise**, **find real numbers** $c_1, c_2, c_3$ **not all zero such that** $c_1v_1 + c_2v_2 + c_3v_3 = 0$.

Observe that:

$$c_1v_1 + c_2v_2 + c_3v_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 4 & 8 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}.$$

Gaussian Elimination:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

They are linearly independent.

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**6**. **Consider the following 4 vectors in** $R^3$:

$\mathbf{v}_1=(1,1,2)$, $\mathbf{v}_2=(1,3,4)$, $\mathbf{v}_3=(2,2,4)$, $\mathbf{v}_4=(0,0,1)$.

**If they are linearly independent**, **show this**. **Otherwise**, **find real numbers** $c_1, c_2, c_3, c_4$, **not all zero such that** $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$.

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \vec{0}.$$

Gaussian Elimination:

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 4 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So: $c_3 = t, \quad c_1 = -2t, \quad c_2 = 0, \quad c_4 = 0.$

And: $\vec{c} = (c_1, c_2, c_3, c_4) = (-2t, 0, t, 0) = t(-2, 0, 1, 0).$

And observe that $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = -2v_1 + 0v_2 + v_3 + 0v_4 = 0.$

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**7**. **Find a basis for the following vector spaces**:

**a**) **The set of all vectors of the form** $(a, b, c, d)$ **for which** $a + 2d = c + 3d = 0.$

So vectors have the form: $(-2d, b, -3d, d) = b(0,1,0,0) + d(-2,0,-3,1).$

From this, we discover the vectors $\{(0,1,0,0), (-2,0,-3,1)\}$ span the set of given vectors.

Now let's verify that they are linearly independent. Looking at the 3 x 3 submatrices, we calculate the sub determinants:

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 3 & 4 \end{vmatrix} = -1(4-6) = 2 \neq 0. \qquad \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 4 & 8 \end{vmatrix} = 1(24-16) + 1(0-6) = 2 \neq 0.$$

So we see that these vectors are linearly independent. Therefore, we have a basis $\{(0,1,0,0), (-2,0,-3,1)\}$ for our subspace of vectors.

**b**) **The solution space of the homogeneous linear system**:

$$x_1 - 3x_2 - 10x_3 + 5x_4 = 0$$
$$x_1 - 4x_2 + 11x_3 - 2x_4 = 0$$
$$x_1 - 3x_2 + 8x_3 - x_4 = 0.$$

Putting things into a matrix, and performing Gaussian reduction:

$$\begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & -4 & 11 & -2 \\ 1 & -3 & 8 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & -1 & 21 & -7 \\ 0 & 0 & 18 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & -10 & 5 \\ 0 & 1 & -21 & 7 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -73 & 26 \\ 0 & 1 & -21 & 7 \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

Applying an arbitrary parameter $t$ to our free column $x_4$, gives us $x_3 = \frac{1}{3}t,\ x_2 = 0,$ and $x_1 = -\frac{5}{3}t.$

Therefore, $\vec{x} = (x_1,\ x_2,\ x_3,\ x_4) = \left(-\frac{5}{3}t,\ 0,\ \frac{1}{3}t,\ t\right) = t\left(-\frac{5}{3}, 0, \frac{1}{3}, 1\right)$.

And finally, we see a basis for our solution subspace by setting $t$ to any value. To make our basis look simple, I will choose to set $t = 3$, so our basis becomes: $\{(-5, 0, 1, 3)\}$.

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**8**. **Consider the following matrix**:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 2 \\ -3 & 4 & 1 & 2 \\ 0 & 1 & 7 & 4 \\ -5 & 7 & 4 & -2 \end{bmatrix}$$

**a**) **Find a basis of the row space of** $A$.

Doing our Gaussian reduction this:

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ -3 & 4 & 1 & 2 \\ 0 & 1 & 7 & 4 \\ -5 & 7 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & 7 & 4 \\ 0 & 2 & 14 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbb{E}.$$

For the row space, we take at the nonzero rows of our reduced system:

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & 9 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 7 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

**b**) **Find a basis of the column space of** $A$.

For the column space, we look back at the original matrix $A$, and the basis consists of the columns in $A$ corresponding to the pivot columns in $\mathbb{E}$. Note that the columns in $\mathbb{E}$ that had the "leading ones" (the pivot columns) were columns $1, 2, 4$. So taking those columns from $A$, gives us:

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ -2 \end{bmatrix} \right\}.$$

9. **Find a subset of the vectors** $v_1 = (1,-3,0,5),\ v_2 = (-1,4,1,7),\ v_3 = (2,1,7,4),\ v_4 = (2,2,4,-2)$ **that forms a basis for the subspace** $W$ **of** $R^4$ **spanned by these 4 vectors**.

Placing the vectors into columns of a matrix, we see:

$$\begin{vmatrix} 1 & -1 & 2 & 2 \\ -3 & 4 & 1 & 2 \\ 0 & 1 & 7 & 4 \\ -5 & 7 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & 7 & 4 \\ 0 & 2 & 14 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 8 \\ 1 & 7 & 4 \\ 2 & 14 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 8 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Therefore, the four vectors are not linearly independent. From the previous problem, we see that the 1st, 2nd, and 4th columns are linearly independent. But only being 3 vectors, we conclude that $\{v_1, v_2,\ v_4\}$ form a basis for a three-dimensional subspace $W$ of $\mathbb{R}^4$.

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10. **Consider the following matrices**:

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

**Calculate the following number of matrices**: **a**) $\det(A^{-1})$, **b**) $A^T$, **c**) **BC**, **d**) **CB**, **e**) **AB**.

a) Recall that: $\det(A^{-1}) = \frac{1}{\det(A)}$. So let's calculate: $\det(A) = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} = -1 \cdot 1 \cdot 4 = -4.$

Therefore, $\det(A^{-1}) = -\frac{1}{4}$.

b) $A^T = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 4 \end{bmatrix}$

c) $\mathbf{BC} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{bmatrix}$

d) $\mathbf{CB} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 22$

e) $\mathbf{AB} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \\ 20 \end{bmatrix}$