Midterm 2 Practice Sheet Solutions

1. Find the general solution of $xy' + 3y = \frac{\sin x}{x^2}$.

Putting it in standard format: $y' + \frac{3}{x}y = \frac{\sin x}{x^3}$. Note this is first-order linear.

$\rho = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln|x|^3} = x^3$. Therefore, $y = \frac{1}{x^3}\int x^3 \frac{\sin x}{x^3}dx = -\frac{1}{x^3}\cos x + \frac{c}{x^3}$.

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2. Find the general solution of $y' = \frac{3x^2 - e^x}{2y-5}$.

Note this is separable. $(2y-5)dy = (3x^2 - e^x)dx \quad \Rightarrow \quad \int(2y-5)dy = \int(3x^2 - e^x)dx$

$\Rightarrow \quad y^2 - 5y = x^3 - e^x + c.$

You could solve explicitly for y, using the quadratic equation: $y = \dfrac{5 \pm \sqrt{25 - 4\left(-x^3 + e^x - c\right)}}{2}$.

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3. Use Euler's method w/step $h = 0.1$ to find a numeric solution of initial value problem at $x = 0.1,\ 0.2$.

$y' = x^2 + y^2, \quad y(0) = 1.$

$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.1)(0^2 + 1^2) = 1.1$ at $x_1 = 0.1$.

$y_2 = y_1 + hf(x_1, y_1) = 1.1 + (0.1)(0.1^2 + 1.1^2) = 1.222$ at $x_2 = 0.2$.

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4. Find the general solution of $y'' - 2y' + y = 0$.

$r^2 - 2r + 1 = (r-1)^2 \quad \Rightarrow \quad r \in \{1, 1\}$. Therefore, $y_g = c_1 e^x + c_2 x e^x$.

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5. Solve the initial value problem: $y'' + y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$.

$r^2 + r + 1 = 0 \quad \Rightarrow \quad r = \frac{-1 \pm \sqrt{1-4}}{2} \quad \Rightarrow \quad r \in \left\{-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right\}$.

$e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} = e^{-\frac{1}{2}x} e^{i\frac{\sqrt{3}}{2}x} = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right)$

$y_g = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$.

Initial condition: $1 = e^0(c_1) \quad \Rightarrow \quad c_1 = 1$.

Initial condition: $y_g' = -\frac{1}{2}e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}e^{-\frac{1}{2}x}\left(-c_1\sin\frac{\sqrt{3}}{2}x + c_2\cos\frac{\sqrt{3}}{2}x\right)$

$$0 = -\frac{1}{2}e^0(c_1) + \frac{\sqrt{3}}{2}e^0(c_2) = -\frac{c_1}{2} + \frac{\sqrt{3}}{2}c_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}c_2 \quad \Rightarrow \quad c_2 = \frac{1}{\sqrt{3}}.$$

Therefore, $y_p = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right)$.

6. Find the general solution to the inhomogeneous equation: $y'' + 2y' + y = 2e^t + \cos t + t$.

$r^2 + 2r + 1 = (r+1)^2 \quad \Rightarrow \quad r \in \{-1,-1\} \quad \Rightarrow \quad y_c = c_1 e^{-t} + c_2 t e^{-t}$.

$y_0 = Ae^t + (B\cos t + C\sin t) + (D + Et)$

Observe that the pretrialsolution is linearly independent from the complementary solution, therefore $y_0 = y_{trial} = Ae^t + (B\cos t + C\sin t) + (D + Et)$.

Taking derivatives: $y'_{trial} = Ae^t + (-B\sin t + C\cos t) + E$,

$y''_{trial} = Ae^t - (B\cos t + C\sin t)$.

Therefore: $y''_{trial} + 2y'_{trial} + y_{trial}$

$= Ae^t - (B\cos t + C\sin t) + 2(Ae^t + (-B\sin t + C\cos t) + E) + Ae^t + (B\cos t + C\sin t) + (D + Et)$

$= (A + 2A + A)e^t + ((B + 2C - B)\cos t + (C - C - 2B)\sin t) + Et + 2E + D$

$= 4Ae^t + 2(C\cos t - B\sin t) + Et + 2E + D$.

Comparing this to the right hand side of our given equation: $2e^t + \cos t + t$, we have $4A = 2, \quad 2C = 1, \quad -2B = 0, \quad E = 1, \quad 2E + D = 0$.

From this, we see that $A = \frac{1}{2}, \ B = 0, \ C = \frac{1}{2}, \ E = 1$, and $D = -2$.

Therefore, $y_p = \frac{1}{2}e^t + \frac{1}{2}\sin t - 2 + t$.

Finally, the general solution is $y_g = y_c + y_p = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2}e^t + \frac{1}{2}\sin t - 2 + t$.

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7. Given an example of a 2nd order nonlinear differential equation.

In general, linear 2nd order is of the form: $a(x)(y'')^i + b(x)(y')^j + c(x)y^k = d(x)$, where $i = j = k = 1$ and $a(x) \neq 0$.

So, if i, j, or k is not equal to one, we have nonlinear. An example is when $a(x) = 1, \ i = 2$, and $b(x) = c(x) = d(x) = 0$.

In other words: $(y'')^2 = 0$. Another way to have nonlinear is if $a(x, y, y')$, for example if $a(x, y, y') = xy$. If we use this in the previous example, but let $i = 1$, we have $xyy'' = 0$ as a nonlinear equation.

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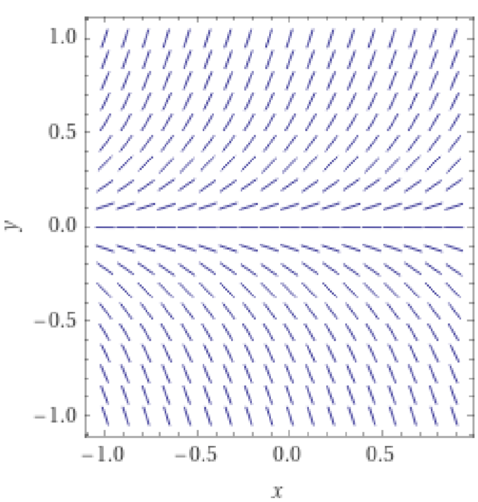
8. Sketch the slope fields of the differential equation: $y' = 3y$.



9. Find the general solution of $y^{(4)} + 2y^{(2)} + y = 0$.

$r^4 + 2r^2 + 1 = (r^2+1)^2 \quad \Rightarrow \quad r \in \{\pm i, \pm i\}$

$e^i = \cos x + i \sin x$

$y_g = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$

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10. Consider the logistic equation: $y' = y(3-y)$.

a) Find the critical points and the corresponding equilibrium solutions.

$f(x) = y(3-y) = 0$ when $y \in \{0, 3\}$.

b) Determine whether each critical point is stable or unstable.

Checking x-values on either side of the critical points, we find $f(-1) = -1(4) = -4 < 0$, $f(1) = 2 > 0$, $f(4) = 4(-1) = -4 < 0$. This gives us the phase diagram: $\leftarrow \; 0 \; \rightarrow \; 3 \; \leftarrow$

Therefore, $x = 0$ is an unstable critical point, and $x = 3$ is a stable critical point.

c) Find the general solution.

$\int \frac{1}{y(3-y)} dy = \int dx \qquad (*)$

Partial fractions: $\frac{1}{y(3-y)} = \frac{A}{y} + \frac{B}{3-y}$ when $1 = A(3-y) + By = (B-A)y + 3A$, and comparing powers of y, we have: $3A = 1$ and $B - A = 0$. Therefore, $A = \frac{1}{3}$ and $B = \frac{1}{3}$.

$\int \frac{1}{y(3-y)} dy = \frac{1}{3} \int \frac{1}{y} + \frac{1}{3-y} dy = \frac{1}{3}(\ln|y| - \ln|3-y|) + C_0 = \frac{1}{3} \ln\left|\frac{y}{3-y}\right| + C_0.$

Therefore from $(*)$, we have: $\frac{1}{3} \ln\left|\frac{y}{3-y}\right| = x + C_1$ or $\left|\frac{y}{3-y}\right| = e^{3x+3C_1} = C_2 e^{3x}$ where $C_2 > 0$.

Removing the absolute value: $\frac{y}{3-y} = Ce^{3x}$, where $C \neq 0$.

Solving explicitly: $y = Ce^{3x}(3-y) = 3Ce^{3x} - yCe^{3x} \quad \Rightarrow \quad y(1 + Ce^{3x}) = 3Ce^{3x} \quad \Rightarrow \quad y = \frac{3Ce^{3x}}{1+Ce^{3x}}.$

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11. Computer the Wronskian of the functions: $y_1 = e^{3x}$, $y_2 = \sin x$, $y_3 = \cos x$, and use it to show that y_1, y_2, and y_3 are linearly independent.

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{3x} & \sin x & \cos x \\ 3e^{3x} & \cos x & -\sin x \\ 9e^{3x} & -\sin x & -\cos x \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & \sin x & \cos x \\ 3 & \cos x & -\sin x \\ 9 & -\sin x & -\cos x \end{vmatrix} \overset{R_3+R_1}{=} e^{3x} \begin{vmatrix} 1 & \sin x & \cos x \\ 3 & \cos x & -\sin x \\ 10 & 0 & 0 \end{vmatrix}$$

$= 10e^{3x} \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = 10e^{3x}(-\sin^2 x - \cos^2 x) = -10e^{3x} \neq 0$, therefore y_1, y_2, and y_3 are linearly independent.