Midterm 2, Lecture 20

1. **Solve the initial value problem**: $xy' + 2y = 4x^2,\ y(1) = 2.$

This is a linear first order nonhomogeneous differential equation.

Putting it in standard form: $y' + \frac{2}{x}y = 4x \quad \Rightarrow \quad \rho = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = x^2.$

Therefore, $y = x^{-2}\int x^2(4x)dx = 4x^{-2}\int x^3 dx = 4x^{-2}\left(\frac{1}{4}x^4 + c\right).$

Applying initial condition: $2 = 4\left(\frac{1}{4} + c\right), \quad \Rightarrow \quad c = \frac{1}{4}.$

Therefore, $y = 4x^{-2}\left(\frac{1}{4}x^4 + \frac{1}{4}\right) = x^2 + x^{-2}.$

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2. **Find the initial value problem**: $y'' + y' + y = 0, \quad y(0) = 1, \quad y'(0) = 3.$

Characteristic equation: $r^2 + r + 1 \quad \Rightarrow \quad r = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Euler Formula: $e^{rx} = e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} = e^{-\frac{1}{2}x}e^{i\frac{\sqrt{3}}{2}x} = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right).$

Therefore: $y_g = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right).$

The first initial condition gives: $1 = c_1.$

Taking the derivative for the next initial condition:
$y_g' = -\frac{1}{2}e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}e^{-\frac{1}{2}x}\left(-c_1\sin\frac{\sqrt{3}}{2}x + c_2\cos\frac{\sqrt{3}}{2}x\right).$

Applying the initial condition: $3 = -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = -\frac{1}{2}(1) + \frac{\sqrt{3}}{2}c_2 \quad \Rightarrow \quad c_2 = \left(3 + \frac{1}{2}\right)\frac{2}{\sqrt{3}} = \frac{7}{\sqrt{3}}.$

Therefore: $y_p = e^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + \frac{7}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right).$

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3. **Find a particular solution to the nonhomogeneous equation**: $y'' + 4y' + 4y = e^{-2x}.$

Characteristic equation: $r^2 + 4r + 4 = (r+2)^2 \quad \Rightarrow \quad r \in \{-2, -2\}.$

Complementary solution: $y_c = c_1e^{-2x} + c_2xe^{-2x}.$

Pre-trial solution: $y_i = Ce^{-2x}.$

Clearing the dependencies, we get the trial solution: $y_{trial} = Cx^2e^{-2x}.$

Taking derivatives: $y'_{trial} = 2C(1-x)xe^{-2x}, \quad y''_{trial} = 2C(2x^2 - 4x + 1)e^{-2x}.$

Substituting this into our differential equation: $y''_{trial} + 4y'_{trial} + 4y_{trial}$

$= 2C(2x^2 - 4x + 1)e^{-2x} + 8C(1-x)xe^{-2x} + 4Cx^2e^{-2x} = 2Ce^{-2x}.$

Setting this equal to $f(x)$: $2Ce^{-2x} = e^{-2x}.$

Comparing sides of the equation gives us: $2C = 1$ or $C = \frac{1}{2}.$

Therefore: $y_p = \frac{1}{2}x^2e^{-2x}.$

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4. **Consider the differential equation**: $xy' = 6y$.

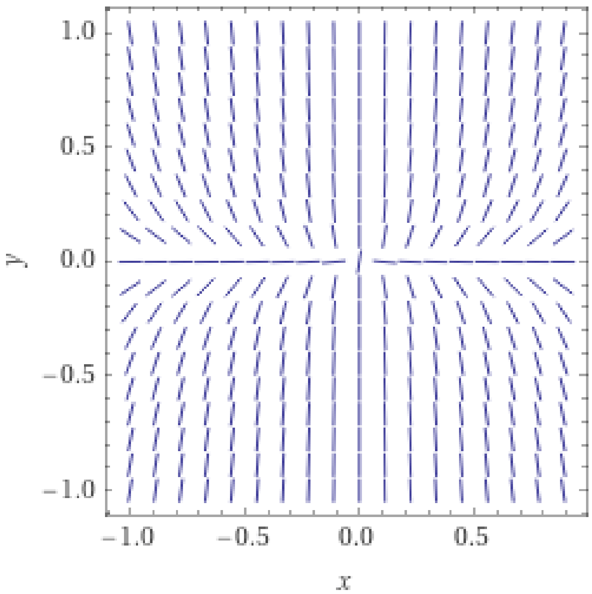
1) Find the singular solutions and the general solutions.

It is separable, so: $\int \frac{1}{y}dy = \int \frac{6}{x}dx$, when $y \neq 0$.

$\ln|y| = 6\ln|x| + c \quad \Rightarrow \quad |y| = e^{\ln x^6 + c} \quad \Rightarrow \quad y = Cx^6$, where $C \neq 0$.

But what about when $y \equiv 0$? Note that $xy' = 6y$ is satisfied for the function $y(x) = 0$. So, to include this singular solution, we say $y = Cx^6$, where $C \in \mathbb{R}$.

2) Sketch the direction field of the differential equation.



3) Show that there are infinitely many solutions of the differential equation with initial value $y(0) = 0$.

Note that with this initial value we have $0 = C0^6$ satisfied for any value of C. This gives us an infinite family of solutions going thru this initial value.

4) Explain why part 3 does not contradict the uniqueness theorem for differential equations.

Note that the uniqueness theorem requires that the coefficient functions be continuous around the initial value. However, in this case, in standard form our differential equation is $y' - \frac{6}{x}y = 0$, and the coefficient function $-\frac{6}{x}$ is not continuous around $(x, y) = (0, 0)$. Therefore, the uniqueness theorem does not apply, so there is no contradiction.

5. **Consider the functions** $f_1 = e^x$ **and** $f_2 = xe^x$ **on the real line.**

a) Compute the Wronskian of f_1 **and** f_2.

$$W(f_1,f_2) = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^x \begin{vmatrix} 1 & xe^x \\ 1 & e^x(1+x) \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = e^{2x}(1+x-x) = e^{2x} \neq 0$$

b) Are the functions f_1,f_2 **linearly independent? If your answer is yes, please explain why. If your answer is no, please find constants** $c_1, c_2,$ **not all zero, such that** $c_1 f_1 + c_2 f_2 = 0$.

Using part a, since there is no interval on the real line in which the Wronskian e^{2x} is equivalent to the zero function, the functions f_1,f_2 are linearly independent

Alternatively, observe that f_1,f_2 are linearly independent if they are not scalar multiples of each other.

Using proof by contradiction, assume they are scalar multiples of each other, we get $f_1 = cf_2$ $\Rightarrow \quad e^x = cx\,e^x \quad \Rightarrow \quad \frac{1}{c} = x.$ However, x is not a constant. So our assumption must be wrong, and it must be that the two functions are not scalar multiples of each other, and therefore linearly independent.