MIDTERM EXAM I, MATH 2243 (030), FALL 2020

This exam contains 6 problems. To receive full credit on a problem, you must show and explain your work.

1. Determine for what values of k the following system

$$3x + 2y = 1$$

$$6x + ky = 3$$

has

(a) (6 points) a unique solution

(b) (6 points) no solution

(c) (6 points) infinitely many solutions

2. Consider the system

$$4x_1 + 3x_2 + 2x_3 = 6$$

$$3x_1 + 5x_2 + 2x_3 = 10$$

$$5x_1 + 6x_2 + 3x_3 = 9$$

(a) (6 points) Write down the augmented coefficient matrix $\mathbf{M}$ of the system

(b) (6 points) Use the method of Gauss-Jordan elimination to transform the augmented coefficient matrix $\mathbf{M}$ to the reduced echelon form.

(c) (6 points) Use (b) to solve the system.

3. Consider the system

$$2x_1 + 3x_2 + 4x_3 = 2$$

$$4x_1 + 9x_2 + 16x_3 = 1$$

$$x_1 + x_2 + x_3 = 3$$

(a) (6 points) Write down the coefficient matrix $\mathbf{A}$ of the system and the corresponding matrix equation $\mathbf{Ax} = \mathbf{b}$.

(b) (10 points) Compute the determinant $\det(\mathbf{A})$ and the cofactor matrix $[\mathbf{A}_{ij}]$ of $\mathbf{A}$, and use the formula of the inverse for matrices to find $\mathbf{A}^{-1}$.

(c) (6 points) Use the formula $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ to solve the system.

4. (12 points) Consider the following four vectors in $\mathbf{R}^3$:

$$\mathbf{v}_1 = (2, 1, 3), \mathbf{v}_2 = (1, 3, 4), \mathbf{v}_3 = (2, 5, 4), \mathbf{v}_4 = (1, 1, 1)$$

If they are linearly independent, show this; otherwise find real numbers c_1, c_2, c_3, c_4 not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = 0$.

5. (12 points) Find a basis of the solution space of the homogenous linear system

$$\begin{aligned} 3x_1 + x_2 + 4x_3 + 18x_4 &= 0 \\ x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

6. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & 3 & 4 \\ 3 & -2 & 7 & 0 \\ 3 & -1 & 8 & 6 \\ 0 & 1 & 1 & 7 \end{bmatrix}$$

(a) (6 points) Find a basis of the row space of $\mathbf{A}$ and use it to find the rank of $\mathbf{A}$.

(b) (6 points) Find a basis of the column space of $\mathbf{A}$

(c) (6 points) Find a subset of the vectors $\mathbf{v}_1 = (1, 1, 3, 3, 0), \mathbf{v}_2 = (-1, 0, -2, -1, 1), \mathbf{v}_3 = (2, 3, 7, 8, 1), \mathbf{v}_4 = (-2, 4, 0, 6, 7)$ that forms a basis for the subspace $\mathbf{W}$ of $\mathbf{R}^5$ spanned by these four vectors.

1) $\begin{bmatrix} 3 & 2 & 1 \\ 6 & k & 3 \end{bmatrix}$

$\xrightarrow{1/3 R_1} \begin{bmatrix} 1 & 2/3 & 1/3 \\ 6 & k & 3 \end{bmatrix}$

$\xrightarrow{R_2 - 6R_1} \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & k-4 & 2 \end{bmatrix}$

$k - 4 = 2$

$\Rightarrow k = 6$

SO THERE IS

a) A UNIQUE SOLUTION FOR $k = 6$

b) NO SOLUTION FOR $k \neq 6$

c) INFINITELY MANY SOLUTIONS FOR NO VALUES OF k

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2)

a) $M = \left[\begin{array}{ccc|c} 4 & 3 & 2 & 6 \\ 3 & 5 & 2 & 10 \\ 5 & 6 & 3 & 9 \end{array}\right]$

b) $\mathrm{rref}(M) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -33 \end{array}\right]$

c) So $x_1 = 12$, $x_2 = 8$, $x_3 = -33$

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3) a) $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 9 & 16 \\ 1 & 1 & 1 \end{bmatrix}$ $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

b) $\det A = +1 \cdot \begin{vmatrix} 3 & 4 \\ 9 & 16 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 4 \\ 4 & 16 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}$

$= +2$

$[A_{ij}] = \begin{bmatrix} -7 & 12 & -5 \\ 1 & -2 & 1 \\ 12 & -16 & 6 \end{bmatrix}$

$$A^{-1} = \frac{[A_{ij}]^T}{|A|} = \frac{1}{2}\begin{bmatrix} -7 & 12 & -5 \\ 1 & -2 & 1 \\ 12 & -16 & 6 \end{bmatrix}$$

c) $$\frac{1}{2}\begin{bmatrix} -7 & 12 & -5 \\ 1 & -2 & 1 \\ 12 & -16 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 23 \\ -26 \\ 9 \end{bmatrix}$$

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4) $v_1 = (2,1,3)$, $v_2 = (1,3,4)$, $v_3 = (2,5,4)$
$v_4 = (1,1,1)$

FOUR VECTORS IN $\mathbb{R}^3$ ARE ALWAYS LINEARLY DEPENDENT $\quad \frac{2}{15}v_1 - \frac{1}{15}v_2 + \frac{4}{15}v_3 - v_4 = 0$

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5 $$\begin{bmatrix} 3 & 1 & 4 & 18 & 0 \\ 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SO LETTING $x_3 = s, x_4 = t$, WE SEE

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-5t \\ -s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

SO A BASIS FOR THE SOLUTION SPACE IS

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

6)

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & 3 & 4 \\ 3 & -2 & 7 & 0 \\ 3 & -1 & 8 & 6 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) THE ROW SPACE HAS BASIS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

SO $\operatorname{rk} A = 3$

b) THE COLUMN SPACE HAS BASIS $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 7 \end{bmatrix} \right\}$

SINCE THOSE ARE THE COLS. CORRESPONDING TO PIVOTS.

c) SINCE THE COLUMN SPACE IS THE SPACE WE ARE SPANNING, THE BASIS FOR W IS THE SAME AS THE BASIS FOR COLUMN SPACE. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 7 \end{bmatrix} \right\}$