Practice Final

1.1: #1 Write a differential equation that is a mathematical model of the following situation.

The time rate of change of the velocity v of a spaceship

is proportional to the square of v.

Differential Equation:
$v' = kv^2$ or $\frac{dv}{dt} = kv^2$ or $D_t(v) = kv^2$

1.2: #2 Find the position function $x(t)$ of a moving particle

with the given acceleration $a(t) = \frac{1}{(t+2)^3}$, initial position $x_0 = x(0) = 3$

and initial velocity $v_0 = v(0) = 0$.

$v(t) = \int (t+2)^{-3} dt = -\frac{1}{2}(t+2)^{-2} + C,$

$(v(0) = 0)$...

$0 = -\frac{1}{2}(0+2)^{-2} + C = -\frac{1}{8} + C, \qquad C = \frac{1}{8}.$

Hence $x(t) = \int \left[-\frac{1}{2}(t+2)^{-2} + \frac{1}{8}\right] dt = \frac{1}{2}(t+2)^{-1} + \frac{1}{8}t + C$

$(x(0) = 3)$…

$3 = \frac{1}{2}(0+2)^{-1} + \left(\frac{1}{8} \cdot 0\right) + C = \frac{1}{4} + C, \qquad C = \frac{11}{4}.$

$x(t) = \quad \frac{1}{2}(t+2)^{-1} + \frac{1}{8}t + \frac{11}{4}.$

1.3: #3 Find explicit particular solutions to the initial value problem: $x\frac{dy}{dx} - y = 2x^2 y, \qquad y(1) = 1.$

$x\frac{dy}{dx} = 2x^2y + y = y(2x^2+1) \qquad \frac{1}{y}dy = \frac{2x^2+1}{x} = 2x + \frac{1}{x}$

$\int \frac{dy}{y} = \int (\frac{1}{x} + 2x)dx; \qquad \ln|y| = \ln|x| + x^2 + \ln C; \qquad y = Cxe^{x^2}$

$y(1) = 1$ implies $1 = Ce^1$ and $C = e^{-1}$ so $y(x) = xe^{\left(x^2-1\right)}$.

$y(x) = xe^{\left(x^2-1\right)} = \frac{x}{e}e^{x^2}$

1.4: #4 An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 su (**safe units**)**, and 5 months later it is still 10 su. What amount of radioactive material will remain after one year? Note that any radioactive material has a decay rate proportional to the amount of radioactive material P(t) present at that time,** $\frac{dP(t)}{dt} = -kP(t), \quad k > 0.$

$\int \frac{1}{P}dP = -k\int dt,$ when $P \neq 0.$ (but if $P = 0,$ we see that this is a singular solution to the DEQ)

$\Rightarrow \quad \ln P = -kt + C \quad \Rightarrow \quad P = e^C e^{-kt}.$

Substituting in the first initial condition:

$15 = e^C e^0 \quad \Rightarrow \quad P = 15e^{-kt},$

Substituting in the 2nd initial condition:

$10 = 15e^{-5k} \quad \Rightarrow \quad \ln\frac{2}{3} = -5k$

$\quad \Rightarrow \quad k = -\frac{1}{5}\ln\frac{2}{3}.$

$\Rightarrow \quad P(12) = 15e^{-12\left(-\frac{1}{5}\ln\frac{2}{3}\right)} = 15e^{\frac{12}{5}\ln\frac{2}{3}} \approx 5.6686$ **su**.

1.4: **#5** **Find the general solution of the differential equation**: $(1+x)^2\frac{dy}{dx} = (1+y)^2$.

$\frac{1}{(1+y)^2}dy = \frac{1}{(1+x)^2}dx,$ when $y \neq -1.$ (although note that when $y \equiv -1,$ $\frac{dy}{dx} = 0,$ $(1+y)^2 = 0,$ and our differential equation is satisfied for all x).

Continuing on with the case $y \neq -1,$ we have: $\int\frac{1}{(1+y)^2}dy = \int\frac{1}{(1+x)^2}dx$

$$-(1+y)^{-1} = -(1+x)^{-1} + c \quad \Rightarrow \quad 1+x = (1+y) + (1+x)(1+y)c$$

$$1+x = 1+y+(c+cx+cy+cxy) \quad \Rightarrow \quad x = c+cx+(c+cx+1)y$$

$$-(c+cx+1)y = c+cx-x \quad \Rightarrow \quad y = -\frac{c+cx-x}{c+cx+1}. \quad \blacksquare$$

1.5: **#6** **Solve linear first order differential equation**: $y' + y = \sin x.$

$$\rho = e^{\int 1dx} = e^x$$

$$\Rightarrow \quad (ye^x)' = e^x\sin x$$

$$\Rightarrow \quad e^x y = \int e^x \sin x dx$$

Now we have to use integration by parts, twice!

$\int e^x\sin x dx = -e^x\cos x + \int e^x\cos x dx = -e^x\cos x + \left(e^x\sin x - \int e^x\sin x dx\right)$

$\quad \Rightarrow \quad 2\int e^x\sin x dx = e^x(\sin x - \cos x) + c$

$\quad \Rightarrow \quad y = e^{-x}e^x\,(\sin x - \cos x) = \frac{1}{2}\sin x - \frac{1}{2}\cos x + \frac{c}{e^x}.$

1.5: #7 Solve the initial value problem: $xy' - 2y = 2x^2 \ln x;\; y(1) = 3.$

$y' - \frac{2}{x}y = 2x,$ when $x \neq 0.$ However, note that we already preclude $x = 0,$ as this is not defined for $\ln x.$

Integrating factor: $\rho = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}.$

$x^{-2}(y' - \frac{2}{x}y) = 2x \cdot x^{-2}$

$(y \cdot x^{-2})' = \frac{2}{x}$

$yx^{-2} = 2\int \frac{1}{x}dx + c = 2\ln|x| + c$

$y = 2x^2 \ln|x| + cx^2$

$3 = 2 \cdot 1 \ln(1) + c \cdot 1 \quad \Rightarrow \quad c = 3,$ so

$y = 2x^2 \ln|x| + 3x^2.$

2.1: #8 During the period from 1790 to 1930, the U.S. population $P(t)$ (t **in years**) **grew from 4 million to** 124 **million. Throughout this period,** $P(t)$ **remained close to the solution of the initial value problem**:

$\frac{dP}{dt} = 0.03P - 0.00015P^2, \quad P(0) = 4.$

a) What 1930 population does this logistic equation predict?

$\frac{dP}{dt} = P\left(\frac{3}{100} - \frac{15}{100000}P\right) = \frac{15}{100,000}P\left(\frac{3}{100}\frac{100,000}{15} - P\right) = \frac{3}{20,000}P(200 - P).$

$\int \frac{1}{P(200-P)}dP = \frac{3}{20,000}\int dt \quad \Rightarrow \quad \frac{1}{P(200-P)} = \frac{A}{P} + \frac{B}{200-P} \quad \Rightarrow \quad 1 = A(200-P) + BP$

$\Rightarrow \quad 1 = (B-A)P + 200A \quad \Rightarrow \quad A = 1/200$ and $B = 1/200$

$\Rightarrow \quad \int \frac{1}{P} + \frac{1}{200-P}dP = \frac{3}{100}t + C \quad \Rightarrow \quad \ln P - \ln(200-P) = \frac{3}{100}t + C$

$\ln\frac{P}{200-P} = \frac{3}{100}t + C \quad \Rightarrow \quad \frac{P}{200-P} = De^{\frac{3}{100}t} \quad \Rightarrow \quad \frac{1}{49} = D$

$\frac{P}{200-P} = \frac{1}{49}e^{\frac{3}{100}\cdot 140} \approx 1.361 \quad \Rightarrow \quad P = 1.361\left(200 - P\right) \quad \Rightarrow \quad 2.361P \approx 272.2 \Rightarrow P \approx 115.3$ million.

b) What limiting population does it predict?

Limiting population of 200 million since in the logistic form of the equation ($P' = kP(M-P) = \frac{3}{20,000}P(200-P)$), $M = 200$ is in the spot for the limiting population.

2.2: #9 The equation $y' = -y^2(y^2 - 4)$ **has** …

A) A stable critical point at $0.$ **B) A stable critical point at** $2.$

C) If $y(0) = -1,$ **then** $y(t) \to 0$ **as** $t \to \infty.$ **D) If** $y(5) = -1,$ **then** $y(t) \to 0$ **as** $t \to \infty.$

E) If $y(0) = -1,$ **then** $y(t) \to 0$ **as** $t \to -\infty.$ **F) If** $y(0) = 6,$ **then** $y(t) \to 2$ **as** $t \to -\infty.$

Answers: B,C,D

2.3: #10 **A skydiver drops from an airplane at an altitude of 5000 m and dives freely, with negligible air resistance, for 30 seconds. He then opens his parachute and falls such that the acceleration due to air resistance is proportional to his velocity,** $a_R = -2v$.

You may take the acceleration of gravity to be $9.8 m/s^2$.

(**a**) **Find the altitude at which the skydiver opened the parachute**.

Prior to opening the parachute, since a resistance is negligible, we have $a = \frac{dv}{dt} = 9.8$. Integrating to find velocity, we have: $\int \frac{dv}{dt} dt = \int -9.8 dt \quad \Rightarrow \quad v = -9.8t + v_0 \quad \Rightarrow \quad v = -9.8t$ (because we set $t = 0$ to be the time when they jump out of the plane, and observe that at that time $v = 0$, so $v_0 = 0$). Integrating again to get the position function:

$\int \frac{dx}{dt} dt = \int -9.8t\, dt \quad \Rightarrow \quad x = -4.9t^2 + c$. Observing that $x(0) = 5000$, we have:

$5000 = -4.9 \cdot 0^2 + c \quad \Rightarrow \quad c = 5000$. So $x = -4.9t^2 + 5000$.

Evaluating this at 30 sec, we have: $x(30) = -4.9(30)^2 + 5000 \approx 590\ m$.

(**b**) **Find the velocity of the skydiver 1 minute after he jumped**.

Let $\overline{v}$ represent a new velocity function with initial condition $\overline{v}(0) = v(30)$. In other words, it represents the velocity function after the skydiver deployed the parachute. Observe that the new acceleration function is: $\frac{d\overline{v}}{dt} = -9.8 - 2\overline{v}$. In other words, we are now taking into account the fact that air resistance is slowing the fall (also note that since velocity is in the opposite direction from our position function, that $-2\overline{v}$ is a positive number!). Using separation of variables (although it is also possible to use an integrating factor), we have:

$$-\int \frac{1}{9.8+2\overline{v}} d\overline{v} = \int dt \quad \Rightarrow \quad -\frac{1}{2} \ln|9.8 + 2\overline{v}| = t + c$$

$$\Rightarrow \quad \ln|9.8 + 2\overline{v}| = -2t - c \quad \Rightarrow \quad |9.8 + 2\overline{v}| = e^{-2t-c}$$

Observe from above we have: $v = -9.8t$, so $\overline{v}(0) = v(30) = -9.8 \cdot 30 = -294$. So plugging in this initial condition, we have:

$|9.8 + 2 \cdot (-294)| = e^{-2 \cdot 0 - c} \quad \Rightarrow \quad e^{-c} = 578.2$. Plugging this in:

$\Rightarrow \quad |9.8 + 2\overline{v}| = 578.2e^{-2t} \quad \Rightarrow \quad \overline{v} = -289.1e^{-2t} - 4.9$ (0ur choice of sign is once again due to the fact that since $\overline{v}(0) = -294$.)

Therefore, the velocity at 60 seconds is: $\overline{v}(30) = -289.1e^{-2 \cdot 30} - 4.9 \approx -4.9\ m/s$.

(**c**) **Find his terminal velocity, that is, find** $\lim_{t \to \infty} v(t)$.

$$\overline{v}(\infty) = -289.1e^{-2 \cdot \infty} - 4.9 = -4.9\ m/s.$$

2.3: #11 **You just bought a new car, and its acceleration is proportional to the difference between 250 km per hour and its velocity. If your new car can accelerate from rest to 100 km/hr in 10 seconds, how long will it take for the car to accelerate from rest to 200 km/hr?**

$a(t) = \frac{dv}{dt} = k(250 - v), \qquad v(10) = 100, \qquad v(0) = 0, \qquad v(?) = 200.$

$\frac{dv}{250-v} = kdt \qquad \Rightarrow \qquad \int \frac{dv}{250-v} = \int kdt \qquad \Rightarrow \qquad -\ln|250 - v| = kt + c.$

$250 - v = e^{-kt-c} \qquad \Rightarrow \qquad v = 250 - Ce^{-kt}.$

$0 = 250 - Ce^0$, so $C = 250$.

$100 = 250 - 250e^{-10k} \qquad \Rightarrow \qquad e^{-10k} = \frac{150}{250} = \frac{3}{5}$

$-10k = \ln(\frac{3}{5}) \qquad \Rightarrow \qquad k = -\frac{1}{10}\ln(\frac{3}{5}) \approx 0.0511.$

$200 = 250 - 250e^{-kt} \qquad \Rightarrow \qquad e^{-0.0511t} = \frac{50}{250} = \frac{1}{5}$

$-0.0511t = \ln(\frac{1}{5}) = -\ln(5) \qquad \Rightarrow \qquad t = \frac{\ln(5)}{0.0511} \approx 31.5$ sec. ■

2.4: #12 **Use the Euler method (NOT the improved Euler method) with step size $h = 0.2$ to approximate $y(0.4)$ where $y(x)$ is the solution of the differential equation $y' = -2xy$ with initial value $y(0) = 2$.**

$y(0.2) \approx 2 + 0.2(0) = 2$

$y(0.4) \approx 2 + 0.2(-2 \cdot 0.2 \cdot 2) = 1.84.$ ■

3.1: #13 a) **Write down the augmented matrix of the system:**

$$\begin{aligned} x + 3y + 2z &= 2 \\ 2x + 7y + 7z &= -1 \\ 2x + 5y + 2z &= 7 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 2 & 7 & 7 & -1 \\ 2 & 5 & 2 & 7 \end{array}\right]$$

b) **Row reduce the matrix to echelon form (just echelon, not reduced echelon).**

$$\Rightarrow r_2 - 2r_1 \text{ and } r_3 - 2r_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -2 & 3 \end{array}\right]$$

$$\Rightarrow r_3 + r_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

c) **How many solutions does the system have? Justify your answer.**

The system has only one solution.

This is because each column representing a variable has a leading "1", this means we do not have an arbitrary parameter, and therefore there are not an infinite number of solutions. Also, the reduction did not give us a contradictory $0x + 0y + 0z = c \neq 0$ row (which would indicate an inconsistent matrix with no solution), but instead the reduction left us in the desirable position of being able to solve for z by just reading it off the matrix, and then using back substitution to determine the exact values of x and y.

The question didn't ask for the actual answer, but that would be:

$z = -2,$

$y = -3z - 5 = 1,$

$x = -3y - 2z + 2 = 3,$

So, $\vec{x} = (3, 1, -2)$. ■

3.2: #14 **Use the method of Gaussian elimination to solve the system of equations:**

$$\begin{cases} x_1 - x_2 - x_4 = 1, \\ 2x_1 + x_2 + 3x_3 + 7x_4 = -1, \\ 3x_1 - 2x_2 + x_3 = 2. \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 2 & 1 & 3 & 7 & -1 \\ 3 & -2 & 1 & 0 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 0 & 3 & 3 & 9 & -3 \\ 0 & 1 & 1 & 3 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = t_4,\ x_3 = t_3,\ x_2 = -t_3 - 3t_4 - 1,\ x_1 = -t_3 - 2t_4$

$$(x_1, x_2, x_3, x_4) = \left(-t_3 - 2t_4, \quad -t_3 - 3t_4 - 1, \quad t_3, \quad t_4\right)$$

$$= (0, -1, 0, 0) + t_3(-1, -1, 1, 0) + t_4(-2, -3, 0, 1), \text{ for any } t_3, t_4 \in \mathbb{R}.$$

3.2: #15 **Consider the matrices**: $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

Calculate, if possible, $(\mathbf{A} - 2\mathbf{B})C$, $(\mathbf{AB} - 3\mathbf{A})C$, **and** $AB - BA$. **If one (or more) of these expressions is not defined, state so and give the reason.**

Case: $(\mathbf{A} - 2\mathbf{B})\mathbf{C}$, The matrices **A** and 2**B** are not compatible for subtraction, as they have different dimensions.

Case: $(\mathbf{AB} - 3\mathbf{A})\mathbf{C}$, $(\mathbf{AB} - 3\mathbf{A})\mathbf{C} = \mathbf{ABC} - 3\mathbf{AC} = \mathbf{A}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} - 3\begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = \begin{bmatrix} -18 \\ 9 \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 18 \\ 9 \end{bmatrix} = \begin{bmatrix} 27 \\ 27 \\ 9 \end{bmatrix}.$$

Case: $\mathbf{AB} - \mathbf{BA}$, The matrix **B** cannot be multiplied by **A**, because the number of columns of **B** do not equal the number of rows for **A**. ■

3.3 #16 **Find the reduced echelon form of the following matrix**:

$$\begin{bmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{bmatrix}$$

$$\Rightarrow \tfrac{1}{4}r_3 \text{ and } \tfrac{1}{5}r_1 \Rightarrow \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 1 & \frac{1}{4} & 3 \end{bmatrix} \Rightarrow r_3 - r_1 \Rightarrow \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & \frac{1}{4} - \frac{2}{5} & 3 - \frac{18}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & -\frac{3}{20} & -\frac{3}{5} \end{bmatrix} \Rightarrow -\tfrac{20}{3}r_3 \Rightarrow \begin{bmatrix} 1 & \frac{2}{5} & \frac{18}{5} \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix}.$$

Reduced Echelon Form: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

3.5: **#17** **Find the inverse of the matrix** $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$. (**show all your work**!).

We would like to calculate $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{ij}]^T$. So first to calculate the determinant:

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{vmatrix} \overset{c_3+c_2}{=} \begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = 1.$$

$$\text{Therefore, } \mathbf{A}^{-1} = \frac{1}{1}[A_{ij}]^T = \begin{bmatrix} +0 & +1 & +1 \\ -1 & -1 & -0 \\ +3 & +1 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

$$\text{Alternatively, one could calculate this with: } \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{r_3-r_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \overset{r_2-2r_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \overset{-r_2 \text{ and } r_1+r_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{r_2+r_3 \text{ and } r_1+3r_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \overset{-r_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

b) **Then use this inverse to solve the system** $\mathbf{A}\vec{x} = \vec{b}$, **where** $\vec{b} = \begin{bmatrix} 15 \\ -9 \\ -3 \end{bmatrix}$. **You will not receive credit if you solve the system by any other method**.

$$\vec{x} = \mathbf{A}^{-1}\vec{b} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 15 \\ -9 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 21 \\ 18 \end{bmatrix}. \quad \blacksquare$$

3.6: #18 Use cofactors (not the identity matrix) to evaluate the inverse of…

$$A = \begin{bmatrix} 3 & 5 & 2 \\ -2 & 3 & -4 \\ -5 & 0 & -5 \end{bmatrix}.$$

Using the last row to calculate my determinant…

$\det A = -5 \cdot (5 \cdot (-4) - 2 \cdot 3) + 0 - 5(3 \cdot 3 - (5)(-2)) = -5 \cdot (-26) - 5 \cdot 19$

$= 130 - 95 = 35.$ So, $\frac{1}{\det A} = \frac{1}{35}.$

$$[c_{mn}] = \begin{bmatrix} -15 & 10 & 15 \\ 25 & -5 & -25 \\ -26 & 8 & 19 \end{bmatrix}, \qquad [c_{mn}]^T = \begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{35}\begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix} \text{ or } A^{-1} = \begin{bmatrix} -\frac{3}{7} & \frac{5}{7} & -\frac{26}{35} \\ \frac{2}{7} & -\frac{1}{7} & \frac{8}{35} \\ \frac{3}{7} & -\frac{5}{7} & \frac{19}{35} \end{bmatrix}}$$

3.6: #19 Use Cramer's Rule to solve the following system. First, construct a matrix A to be the matrix associated with the system:

$5x + 8y = 3$, $8x + 13y = 5$.

$$A = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}, \qquad \det A = 1, \qquad \frac{1}{\det A} = 1.$$

$$\boxed{\begin{aligned} &\det A = 1. \\ &x = \frac{1}{\det A}\det\begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix} = -1. \\ &y = \frac{1}{\det A}\det\begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix} = 1. \end{aligned}}$$

3.7: #20 **With the data points given, find the** *nth* **degree polynomial** $y = f(x)$ **that fits these points:** $(-1,1), (1,5),$ **and** $(2,16)$.

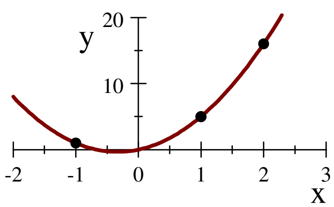
$ax^2 + bx + c = y$

$a(-1)^2 + b(-1) + c = 1, \qquad a(1)^2 + b(1) + c = 5, \qquad a(2)^2 + b(2) + c = 16$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 5 \\ 4 & 2 & 1 & 16 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 6 & -3 & 12 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \quad c = 0, \quad b = 2, \text{ and } d = 3.$$

Therefore, $y = 3x^2 + 2x$



4.1: #21 **Consider the matrix** $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & -1 \end{bmatrix}$. **Find a basis for each of the subspaces:**

a) $Null(\mathbf{A}) = \{x \in \mathbb{R}^4 : \mathbf{A}x = \mathbf{0}\}$.

b) The row space $Row(\mathbf{A})$.

c) The column space $Col(\mathbf{A})$.

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$x_4 = s, \ x_3 = -s, \ x_2 = 2s, \ x_1 = 0.$

$Null(\mathbf{A}) = span\{(0,2,-1,1)\}$.

$Row(\mathbf{A}) = span\left\{(1,0,0,0), \ (0,1,0,-2), \ (0,0,1,1)\right\}$.

$Col(\mathbf{A}) = span\left\{(1,1,0), \ (-1,1,2), \ (0,2,3)\right\}$.

4.2: #22 **Let V be a vector space. Let W be a subset of V. There are three conditions you can check which are sufficient to show that W is a subspace of V. One of them is that W must not be empty. What are the other two conditions?**

Condition 1: $W \neq \emptyset$, (or some other condition which implies the existence of at least one vector)

Condition 2: For all $\vec{w}_1, \vec{w}_2$ in W, $\vec{w}_1 + \vec{w}_2$ is also in W.

Condition 3: For all $\vec{w}_1$ in W, and $c \in \mathbb{R}$; $c\vec{w}_1$ is also in W.

4.4: #23 **Find a basis for the solution space of the given homogeneous linear system:**

$x_1 - 4x_2 - 3x_3 - 7x_4 = 0$

$2x_1 - x_2 + x_3 + 7x_4 = 0$

$x_1 + 2x_2 + 3x_3 + 11x_4 = 0$

$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$, Gaussian elimination: $\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x_4 = s,\ x_3 = t,\ x_2 = -t - 3s,\ x_1 = -t - 5s$

$(x_1, x_2, x_3, x_4) = (-t - 5s, -t - 3s, t, s) = s(-5, -3, 0, 1) + t(-1, -1, 1, 0)$.

Therefore, the basis for the solution space is $\{(-5, -3, 0, 1), (-1, -1, 1, 0)\}$

4.4: #24 **a) Determine (with justification) if the vectors $\vec{v}_1 = (1, 1, 1, 1)$, $\vec{v}_2 = (1, 2, 3, 0)$, $\vec{v}_3 = (3, 6, 0, 0)$, and $\vec{v}_4 = (-1, 0, 0, 0)$ form a basis of $\mathbb{R}^4$ or not.**

$\begin{vmatrix} 1 & 1 & 3 & -1 \\ 1 & 2 & 6 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 & 6 \\ 1 & 3 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = 0 - 12 \neq 0.$ Therefore the column vectors we used were linearly independent. And since we used four vectors in $\mathbb{R}^4$, which is a vector space of four dimensions, we have a basis for $\mathbb{R}^4$.

b) Determine (with justification!) If the subset $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 x_2 = x_3 + x_4 + x_5\}$ is a subspace of $\mathbb{R}^5$ or not.

Since $\vec{0}$ is an element of any subspace, this would require that:

$\vec{0} = \left(\frac{1}{x_2}(x_3 + x_4 + x_5),\ \frac{1}{x_1}(x_3 + x_4 + x_5),\ x_1x_2 - x_4 - x_5,\ x_1x_2 - x_3 - x_5,\ x_1x_2 - x_3 + x_4\right)$.

From the first two components, we see we need $x_3 + x_4 + x_5 = 0$, and neither x_1 or x_2 are equal to zero. Plugging this into the third component, we have $x_1x_2 + x_3 = 0$. Similarly in the fourth component, we have $x_1x_2 + x_4 = 0$. Combining these, we see that $x_3 = x_4$. However, the fourth component and gives us that $x_1x_2 = 0$, implying that either x_1 or x_2, or both must be equal to zero. But this contradicts an earlier conclusion. Therefore, this must not be a subspace.

5.1 #25 **Solve the following equation**: $y''' + 2y'' + y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1.$

$r^3 + 2r^2 + r + 2 = 0$, note that $r = -2$ solves this. So, $(r+2)$ is a factor. Dividing:

$r^3 + 2r^2 + r + 2 = (r+2)r^2 + (r+2) = (r+2)(r^2+1),$ so our roots are $r \in \{-2, \pm i\}$, and our general equation is $y_g = c_1 e^{-2x} + c_2 \cos x + c_3 \sin x.$

$0 = c_1 + c_2$ or $c_1 = -c_2$ and $y = -c_2 e^{-2x} + c_2 \cos x + c_3 \sin x.$

$y' = 2c_2 e^{-2x} - c_2 \sin x + c_3 \cos x.$

$0 = 2c_2 + c_3$ or $c_2 = -\frac{1}{2}c_3$ and $y' = -c_3 e^{-2x} + \frac{1}{2}c_3 \sin x + c_3 \cos x.$

$y'' = 2c_3 e^{-2x} + \frac{1}{2}c_3 \cos x - c_3 \sin x.$

$1 = 2c_3 + \frac{1}{2}c_3$ or $c_3 = \frac{2}{5}$, $c_2 = -\frac{1}{5}$ and $c_1 = \frac{1}{5}$.

So, $y = \frac{1}{5}e^{-2x} - \frac{1}{5}\cos x + \frac{2}{5}\sin x.$

5.2: #26 **Show that the functions** $y_1(x) = e^x,\ y_2(x) = e^{2x},\ y_3(x) = e^{3x}$ **are linearly independent on** $(-\infty, \infty)$.

$$\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2e^{6x} \neq 0 \text{ for any } x \in (-\infty, \infty)\ .$$

5.5: #27 **Find all solutions to**: $y' \cos x + y^2 \sin x = \sin x.$ **Write the solution(s) in explicit form**.

$y' = \frac{\sin x}{\cos x}(1 - y^2) \quad \Rightarrow \quad \int \frac{1}{1-y^2} dy = \int \frac{\sin x}{\cos x} dx,$ when $y \neq 1.$

However, observe that when $y = 1,$ we have $0 \bullet \cos x + \sin x = \sin x,$ which is true, so $y = 1$ is a solution. Continuing on with the case $y \neq 1\ldots$

$\Rightarrow \quad \frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y} \quad \Rightarrow \quad A(1+y) + B(1-y) = 1$

$\Rightarrow \quad y(A-B) + (A+B) = 1 \quad \Rightarrow \quad A = B,$ and $A = \frac{1}{2}.$

$\Rightarrow \quad \frac{1}{2} \int \frac{1}{1-y} + \frac{1}{1+y} dy = \int \frac{\sin x}{\cos x} dx \quad \Rightarrow \quad -\ln|1-y| + \ln|1+y| = -2\ln|\cos x|$

$\Rightarrow \quad \ln\left|\frac{1+y}{1-y}\right| = -2\ln|\cos x| \quad \Rightarrow \quad \frac{1+y}{1-y} = \frac{1}{\cos^2 x} \quad \Rightarrow \quad \cos^2 x(1+y) = 1 - y$

$\Rightarrow \quad y\ \cos^2 x + y = 1 - \cos^2 x \quad \Rightarrow \quad y = -\frac{\cos^2 x - 1}{\cos^2 x + 1}$ or $y = 1.$

5.5: #28 **The roots of equation** $r^2 - 10r + 74 = 0$ **are** $r = 5 \pm 7i.$

Write down the general form of a particular solution with undetermined coefficients for the differential equation $y'' - 10y' + 74y = xe^{5x} \sin 7x + \cos 5x.$ **Do not attempt to evaluate the coefficients**.

From the homogeneousversion of our equation,we get the characteristic equation: $r^2 - 10r + 74 = 0,$ which we are told has the solutions $r = 5 \pm 7i.$ Being complex conjugates, we get both linearly independent solutions from either one, so choosing $5 + 7i,$ and putting it into the form $e^{rt},$ we get…

$e^{(5+7i)x} = e^{5x}(\cos 7x + i\sin 7x)$

Taking the real and imaginary parts to be linearly independent solutions, we get the complementary solution:

$y_c = c_1 e^{5x}\cos 7x + c_2 e^{5x}\sin 7x.$

My pretrial solution is:

$y_0 = (A + Bx)e^{5x}\sin 7x + (C + Dx)e^{5x}\cos 7x + E\cos 5x + F\sin 5x$

Clearing up any linear dependence between y_0 and y_c (by multiplying terms by x as needed), I get:

$y_{trial} = (Ax + Bx^2)e^{5x}\sin 7x + (Cx + Dx^2)e^{5x}\cos 7x + E\cos 5x + F\sin 5x$

General form of a Particular Solution:

$$\begin{aligned} y_g &= y_c + y_{trial} \\ &= (c_1 e^{5x}\cos 7x + c_2 e^{5x}\sin 7x) + (Ax + Bx^2)e^{5x}\sin 7x + (Cx + Dx^2)e^{5x}\cos 7x + E\cos 5x + F\sin 5x \end{aligned}$$

6.1: #29 **Find an eigenvector associated to the eigenvalue** $\lambda_1 = 2 + 2i$ **of the matrix** $A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$.

$$A - \lambda_1 I = \begin{bmatrix} 1-\lambda_1 & -5 \\ 1 & 3-\lambda_1 \end{bmatrix} = \begin{bmatrix} 1-(2+2i) & -5 \\ 1 & 3-(2+2i) \end{bmatrix}$$

$$= \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \overset{r_1 \leftrightarrow r_2}{\Rightarrow} \begin{bmatrix} 1 & 1-2i \\ -1-2i & -5 \end{bmatrix} \overset{r_2+(1+2i)r_1}{\Rightarrow}$$

Observe that: $(1+2i)(1-2i) = 5$.

So we have: $\begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}$, and $y = s,\ x = -(1-2i)y = -s + 2is$. So, $\vec{v}_1 = \begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$, when $s = 1$.

6.2: #30 **The eigenvalues of the matrix** $A = \begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$ **are** $\lambda_1 = 1,\ \lambda_2 = 2,$ **and** $\lambda_3 = 3$. **Find a matrix** P **and a diagonal matrix** D **such that** $A = PDP^{-1}$.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\lambda_1 : \begin{bmatrix} 6-1 & -5 & 2 \\ 4 & -3-1 & 2 \\ 2 & -2 & 3-1 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 2 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $y = t$, $z = 0$, and $x = y = t$. So, $\vec{v}_1 = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ where $t = 1$.

$$\lambda_2 : \begin{bmatrix} 6-2 & -5 & 2 \\ 4 & -3-2 & 2 \\ 2 & -2 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 2 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$, $z = t$, $y = 0$, and $x = y - \frac{1}{2}z = -\frac{1}{2}t$. So,

$\vec{v}_2 = t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ where $t = 2$.

$$\lambda_3 : \begin{bmatrix} 6-3 & -5 & 2 \\ 4 & -3-3 & 2 \\ 2 & -2 & 3-3 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -5 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $z = t$, $y = z = t$, and $x = y = t$. So, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Therefore, $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$.

6.3: #31 Use the diagonalization method to compute A^5 where $A = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$.

Seeking eigenvalues, we calculate: $|A - \lambda I| = \begin{vmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{vmatrix}$

$= (6-\lambda)(-3-\lambda) + 20 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0.$

Therefore, $\lambda \in \{1, 2\}$.

So we are guaranteed that A is diagonalizable and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Seeking the eigenvectors, we calculate:

$$\lambda = 1 : (A - I) = \begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow y = b,\ x = 2b.$$

$\vec{v}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$.

$$\lambda = 2 : (A - 2I) = \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -5 \end{bmatrix} \Rightarrow y = b,\ x = \tfrac{5}{2}b.$$

$\vec{v}_2 = \begin{bmatrix} 5 & 2 \end{bmatrix}^T$.

Therefore, $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$ and $P^{-1} = \frac{1}{|A|}[A_{ij}]^T = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$.

$$A^5 = (PDP^{-1})(PDP^{-1})\ldots(PDP^{-1}) = PD(P^{-1}P)D(P^{-1}\ldots P)DP^{-1} = PDIDI\ldots IDP^{-1} = PD^5P^{-1}$$

$$= \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1^5 & 0 \\ 0 & 2^5 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 156 & -310 \\ 62 & -123 \end{bmatrix}.$$

7.1: #32 **Transform $x'' + 3x' + 7x = t^2$ into a system of first-order differential equations**.

$x_1 = x, \quad x_2 = x' = x_1'$. Below, we ideally want a system of first order equations (only in the new variables x_i), so that we are in a position to place them in a matrix equation $\vec{x}' = A\vec{x} + \vec{y}(t)$, for easy solving using our new techniques.

System of first order differential equations:
$x_1' = x_2$
$x_2' = t^2 - 3x_2 - 7x_1$

7.2: #33 **Write the given system in the vector/matrix form $\vec{x}' = A\vec{x} + \vec{f}(t)$. Then, find eigenvalues for A**.

$x' = x + 2y + 3e^t, \quad y' = x - 3y - t^2$

$$\vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}.$$

Eigenvalues for **A** : $\begin{vmatrix} 1-\lambda & 2 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 5. \qquad \lambda_{1,2} = \frac{-2\pm\sqrt{4+20}}{2} = -1 \pm \sqrt{6}$.

Observe that you could now solve for the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of $\mathbf{A}$, then come up with the complementary solution ($\vec{x}_c = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$) to the associated homogeneous differential equation $\vec{x}' = \mathbf{A}\vec{x}$.

7.3 #34 Apply the eigenvalue method to find a general solution to this system.

$$x_1' = 2x_1 - 5x_2, \qquad x_2' = 4x_1 - 2x_2$$

Hint: **The characteristic equation is** $\lambda^2 + 16 = 0$, **and the eigenvalues are** $\lambda = \pm 4i$.

With $+4i$: $(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix}$. Find $\vec{v} = \begin{bmatrix} 1+2i & 2 \end{bmatrix}^T$ from usual method, OR..

Alternative Method: Possible e-vector from first row (switch entries w/ sign change): $\begin{bmatrix} 5 & 2-4i \end{bmatrix}^T$.

Test on 2nd row…

$$\begin{bmatrix} 4 & -2-4i \end{bmatrix}\begin{bmatrix} 5 & 2-4i \end{bmatrix}^T = 4(5) + (-2-4i)(2-4i) = 20 - 4 - 8i + 8i - 16 = 0. ✓$$

Or with $-4i$: $(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 2+4i & -5 \\ 4 & -2+4i \end{bmatrix}$. Find $\vec{v} = \begin{bmatrix} 1-2i & 2 \end{bmatrix}^T$ from usual method, OR..

Alternative Method: Possible e-vector from first row (switch entries w/ sign change):

$\begin{bmatrix} 5 & 2+4i \end{bmatrix}^T$.

Using $\vec{v} = \begin{bmatrix} 5 & 2-4i \end{bmatrix}^T$: $\quad \vec{v}e^{4it} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}(\cos 4t + i\sin 4t) = \begin{bmatrix} 5(\cos 4t + i\sin 4t) \\ (2-4i)(\cos 4t + i\sin 4t) \end{bmatrix}$

$$= \begin{bmatrix} 5\cos 4t + 5i\sin 4t \\ 2(\cos 4t + i\sin 4t) - 4i(\cos 4t + i\sin 4t) \end{bmatrix} = \begin{bmatrix} 5\cos 4t + 5i\sin 4t \\ 2\cos 4t + 2i\sin 4t - 4i\cos 4t + 4\sin 4t \end{bmatrix}$$

$$= \begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + i\begin{bmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{bmatrix}.$$

The general solution (from $\begin{bmatrix} 5 & 2 \pm 4i \end{bmatrix}^T$)… $\vec{x} = c_1\begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + c_2\begin{bmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{bmatrix}$, or some constant multiple.
From $\begin{bmatrix} 1 \pm 2i & 2 \end{bmatrix}^T$ ...   OR $\vec{x} = c_1\begin{bmatrix} \cos 4t - 2\sin 4t \\ 2\cos 4t \end{bmatrix} + c_2\begin{bmatrix} 2\sin 4t + 2\cos 4t \\ 2\sin 4t \end{bmatrix}$, or some constant multiple.