4.1.4

The solution is

$$u(t,x) = \sum_{n=1}^{\infty} d_n \exp\left[-\left(n+\tfrac{1}{2}\right)^2 \pi^2 t\right] \sin\left(n+\tfrac{1}{2}\right)\pi x$$

where

$$d_n = 2\int_0^1 f(x) \sin\left(n+\tfrac{1}{2}\right)\pi x \, dx$$

are the "mixed" Fourier coefficients of the initial temperature $u(0,x) = f(x)$. All solutions decay exponentially fast to zero: $u(t,x) \to 0$ as $t \to \infty$. For most initial conditions, i.e., those for which $d_1 \neq 0$, the decay rate is $e^{-\pi^2 t/4} \approx e^{-2.4674\,t}$. The solution profile eventually looks like a rapidly decaying version of the first eigenmode $\sin \frac{1}{2}\pi x$.

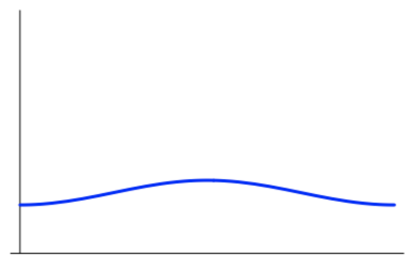
4.1.7

(a) $u(t,x) = \dfrac{1}{4} - \dfrac{2}{\pi^2} \displaystyle\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \exp\left(-(4j+2)^2 \pi^2 t\right) \cos(4j+2)\pi x$; (b) $\frac{1}{4}$;

(c) At an exponential rate of $e^{-4\pi^2 t}$;

(d) As $t \to \infty$, the solution becomes a vanishingly small cosine wave centered around $u = \frac{1}{4}$, namely

$$u(t,x) \approx \frac{1}{4} - \frac{2}{\pi^2} e^{-4\pi^2 t} \cos 2\pi x:$$



4.1.10c

(c) $u(t,x) = \frac{1}{2}\pi - \dfrac{4}{\pi} \displaystyle\sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t} \cos(2k+1)x}{(2k+1)^2}$; equilibrium temperature: $u(t,x) \to \frac{1}{2}\pi$.

4.1.16

(a) If $u(t,x) = e^{\alpha t}\, v(t,x)$, then

$$\frac{\partial u}{\partial t} = \alpha e^{\alpha t}\, v(t,x) + e^{\alpha t}\, \frac{\partial v}{\partial t}\,(t,x) = \gamma e^{\alpha t}\, \frac{\partial^2 v}{\partial x^2} = \gamma\, \frac{\partial^2 u}{\partial x^2}\,.$$

(b) $v(t,x) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n\, e^{-(\alpha+\gamma n^2\pi^2)t} \sin n\pi x$, where $b_n = 2\int_0^1 f(x) \sin n\pi x\, dx$ are the Fourier sine coefficients of the initial data. All solutions tend to the equilibrium value $u(t,x) \to 0$ as $t \to \infty$ at an exponential rate. For most initial data, i.e., those with $b_1 \neq 0$, the decay rate is $e^{-a t}$, where $a = \alpha + \gamma\pi^2$; other solutions decay at a faster rate.

other solutions decay at a faster rate.

(c)

$$v(t,x) = \tfrac{1}{2}\, a_0\, e^{-\alpha t} + e^{-\alpha t} \sum_{n=1}^{\infty} a_n\, e^{-(\alpha+\gamma n^2\pi^2)t} \sin n\pi x,$$

where

$$a_n = 2\int_0^1 f(x) \cos n\pi x\, dx,$$

are the Fourier cosine coefficients of the initial data. All solutions tend to zero equilibrium value $u(t,x) \to 0$ as $t \to \infty$ at an exponential rate. For most initial data, i.e., those of non-zero mean, $\tfrac{1}{2}\, a_0 = \int_0^1 f(x)\, dx \neq 0$, the decay rate is $e^{-\alpha t}$; other solutions decay at a faster rate.

4.2.3d

(d) $u(t,x) = \dfrac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\cos 2n\pi t + \dfrac{\sin 2n\pi t}{2n\pi} \right) \dfrac{\sin n\pi x}{n}$

4.2.4b

(b) $1,\ t,\ \cos nt \cos nx,\ \sin nt \cos nx,$ for $n = 0, 1, 2, \ldots.$

4.2.6

(a)
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(t,-\pi) = u(t,\pi), \quad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi), \quad \begin{array}{l} -\pi < x < \pi, \\ -\infty < t < \infty, \end{array}$$

subject to the initial conditions

$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \qquad -\pi < x < \pi.$$

(b) The series solution is

$$u(t,x) = \tfrac{1}{2} a_0 + \tfrac{1}{2} c_0 t + \sum_{n=1}^{\infty} \Big(a_n \cos nct \, \cos nx + b_n \cos nct \, \sin nx$$
$$+ \frac{c_n}{nc} \sin nct \, \cos nx + \frac{d_n}{nc} \sin nct \, \sin nx \Big),$$

where a_n, b_n are the Fourier coefficients of $f(x)$, while c_n, d_n are the Fourier coefficients of $g(x)$.

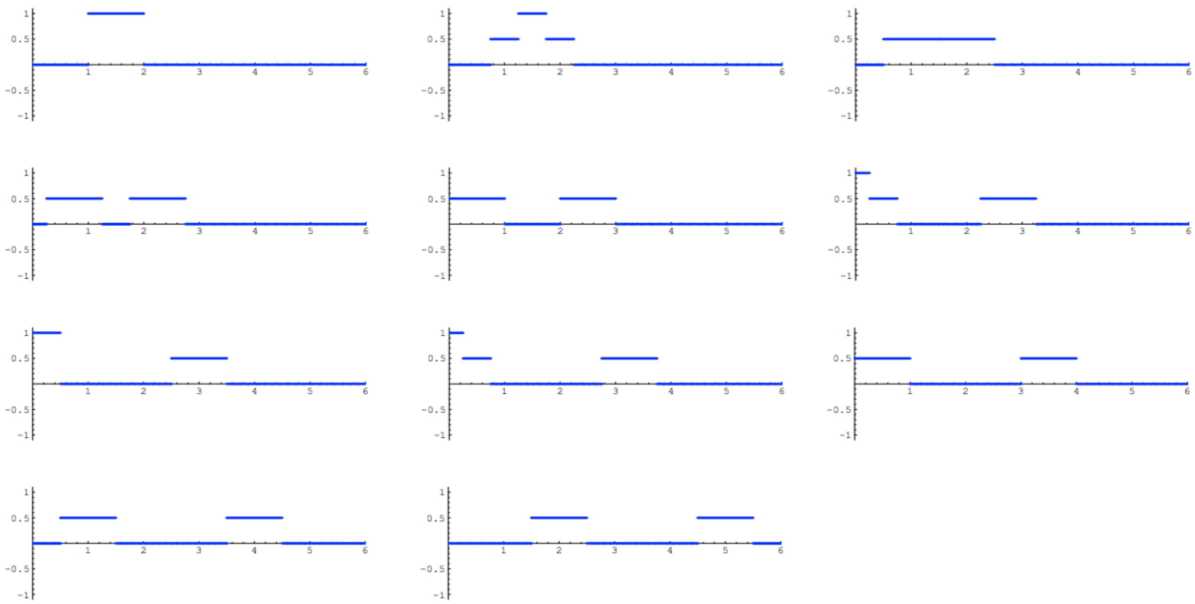
(c) The solution is periodic, with period $\dfrac{2\pi}{c}$, if and only if $c_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} g(x)\,dx = 0$, i.e., the average initial velocity is zero. Otherwise, it includes an unstable, linearly growing mode. *Note*: special solutions may have a shorter period. For example, if all odd coefficients vanish, $a_{2j+1} = b_{2j+1} = c_{2j+1} = d_{2j+1} = 0$, and $c_0 = 0$, then the solution has period π/c.

(d) The initial displacement breaks up into two half size replicas traveling with speed c in opposite directions. When the right moving wave arrives at the end point $-\pi$, it reappears unchanged and still moving to the right at the other end π. Similarly, when the left moving wave arrives at the left end, it reappears on the right end still moving left. The waves recombine into the original displacement after a time of $2\pi/c$, and then the process repeats periodically.

4.2.14c

(c) The initial displacement splits into two half sized replicas, initially moving off to the right and to the left with unit speed. When the left moving box collides with the origin, it reverses its direction, eventually following its right moving counterpart with the same unit speed at a fixed distance of 3 units. During the collision, the box temporarily increases its height before disengaging in its original upright form, but now moving to the right.

Plotted at times $t = 0, .25, .5, .75, 1., 1.25, 1.75, 2., 2.5, 3.5$:



4.2.22

The solution is periodic if and only if the initial velocity has mean zero: $\int_0^\ell g(x)\, dx = 0$. For generic solutions, the period is $2\ell/c$, although some special solutions oscillate more rapidly.

4.2.25

(*a*) The even, 2π periodic extension of the initial data is $f(x) = |\sin x|$. Thus, by d'Alembert's formula, $u(t,x) = \frac{1}{2}|\sin(x-2t)| + \frac{1}{2}|\sin(x+2t)|$.

(*b*) $u\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{1}{2}|\sin\frac{\pi}{2}| + \frac{1}{2}|\sin(\frac{3}{2}\pi)| = 1$. (*c*) $h(t) = |\cos 2t|$ is periodic of period $\frac{1}{2}\pi$.

(*d*) Yes. On the interval $0 \le x \le \pi$, discontinuities initially appear at $x = 0$ and $x = \pi$, and then propagate into the interval at speed 2, reflecting whenever they reach one of the ends, as sketched in the following figure: