HW 6

Math 5587
Elementary Partial Differential Equations I

Fall 2019
University of Minnesota, Twin Cities

Nasser M. Abbasi

January 24, 2020 Compiled on January 24, 2020 at 9:04pm

Contents

1 Problem 4.1.4

Find a series solution to the initial-boundary value problem for the heat equation $u_t = u_{xx}$ for $0 < x < 1$ when one the end of the bar is held at 0 degree and the other is insulated. Discuss the asymptotic behavior of the solution as $t \to \infty$

Solution

The problem did not say which end is insulated. So assuming the left end is at 0 degree and the right end is the one that is insulated.

Using L for the length to make the solution more general and at the end L is replaced by 1. Assuming the initial conditions is $u(x,0) = f(x)$. Therefore the problem to solve is to solve for $u(x,t)$ in

$$u_t = u_{xx} \qquad 0 < x < L, t > 0$$

With boundary conditions

$$u(0,t) = 0$$
$$u_x(L,t) = 0$$

And initial conditions

$$u(x,0) = f(x)$$

Let $u(x,t) = T(t)X(x)$, then the PDE becomes

$$T'X = X''T$$

Dividing by XT

$$\frac{T'}{T} = \frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is real.

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The two ODE's are

$$T' + \lambda T = 0 \tag{1}$$

And the eigenvalue ODE

$$X'' + \lambda X = 0 \tag{2}$$
$$X(0) = 0$$
$$X'(L) = 0$$

Now we solve (2) to find the eigenvalues and eigenfunctions.

Case $\lambda < 0$

Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$X(x) = C_1 \cosh(\omega x) + C_2 \sinh(\omega x)$$

At $x=0$ the above gives

$$0 = C_1$$

Hence the solution now becomes

$$X(x) = C_2 \sinh(\omega x)$$

Taking derivative gives

$$X'(x) = \omega C_2 \sinh(\omega x)$$

At $x = L$

$$0 = \omega C_2 \cosh(\omega L)$$

But $\cosh(\omega L)$ is never zero. Therefore $C_2 = 0$ which leads to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$

The space equation becomes $X'' = 0$ with the solution

$$X = Ax + B$$

At $x = 0$ the above gives $0 = B$. Therefore the solution is $X = Ax$. Taking derivative gives $X' = A$. At $x = L$ this gives $0 = A$. Which leads to trivial solutions. Therefore $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$

Starting with the space ODE, the solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Left B.C. gives

$$0 = A$$

The solution becomes

$$X(x) = B\sin\left(\sqrt{\lambda}x\right)$$

Taking derivative gives

$$X'(x) = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}x\right)$$

Applying right B.C. gives

$$0 = \sqrt{\lambda}B\cos\left(\sqrt{\lambda}L\right)$$

For non trivial solution we want $\cos\left(\sqrt{\lambda}L\right) = 0$ or

$$\sqrt{\lambda} = \frac{n\pi}{2L} \qquad n = 1, 3, 5, \cdots$$

Hence the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1, 3, 5, \cdots$$

Therefore the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{2L}x\right) \qquad n = 1, 3, 5, \cdots$$

Now that we found the eigenvalues, we can solve the time ODE (1).

$$\begin{aligned} T_n' + \lambda_n T &= 0 \\ T_n &= B_n e^{-\lambda_n t} \\ &= B_n e^{-\left(\frac{n\pi}{2L}\right)^2 t} \end{aligned}$$

Hence the fundamental solution is

$$\begin{aligned} u_n(x,t) &= X_n T_n \\ u(x,t) &= \sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\frac{n\pi}{2L}x\right) e^{-\left(\frac{n\pi}{2L}\right)^2 t} \end{aligned} \tag{3}$$

From initial conditions

$$f(x) = \sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\frac{n\pi}{2L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{2L}x\right)$ and integrating

$$\int_0^L f(x)\sin\left(\frac{m\pi}{2L}x\right)dx = \int_0^L \left(\sum_{n=1,3,5,\cdots}^{\infty} B_n \sin\left(\frac{m\pi}{2L}xx\right)\sin\left(\frac{n\pi}{2L}x\right)\right)dx$$

Interchanging order of summation and integration and applying orthogonality between $\cos$ functions results in

$$\begin{aligned} \int_0^L f(x)\sin\left(\frac{m\pi}{2L}x\right)dx &= \int_0^L B_m \sin^2\left(\frac{m\pi}{2L}x\right)dx \\ &= B_m \frac{L}{2} \end{aligned}$$

Therefore

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{2L}x\right)dx$$

Therefore the solution is (3) becomes

$$u(x,t) = \frac{2}{L}\sum_{n=1,3,5,\cdots}^{\infty}\left(\int_0^L f(x)\sin\left(\frac{n\pi}{2L}x\right)dx\right)\sin\left(\frac{n\pi}{2L}x\right)e^{-\left(\frac{n\pi}{2L}\right)^2 t}$$

For $L = 1$ the above becomes

$$u(x,t) = 2\sum_{n=1,3,5,\cdots}^{\infty}\left(\int_0^1 f(x)\sin\left(\frac{n\pi}{2}x\right)dx\right)\sin\left(\frac{n\pi}{2}x\right)e^{-\left(\frac{n\pi}{2}\right)^2 t}$$

The above can be rewritten as

$$u(x,t) = 2\sum_{n=0}^{\infty}\left(\int_0^1 f(x)\sin\left(\frac{(2n+1)\pi}{2}x\right)dx\right)\sin\left(\frac{(2n+1)\pi}{2}x\right)e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t}$$

As $t \to \infty$ and since $\left(\frac{(2n-1)\pi}{2}\right)^2$ is positive and assuming the integral is finite which is valid for well behaved $f(x)$ the solution then $\lim_{t\to\infty} e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t} \to 0$ and the solution above becomes

$$\lim_{t\to\infty} u(x,t) = 0$$

This makes sense, since the right side of the bar is insulated, meaning no heat will escape from that side, and the left side is at kept a zero temperature. Therefore after long time the initial temperature distribution given by $f(x)$ will reach equilibrium which is zero temperature due to the left side kept at zero and since there are no external heat sources or heat sinks.

2 Problem 4.1.7

A metal bar of length $L=1$ and thermal diffusivity $\gamma=1$ is fully insulated, including its ends. Suppose the initial temperature distribution is

$$u(x,0)=f(x)=\begin{cases} x & 0\leq x\leq\frac{1}{2}\\ 1-x & \frac{1}{2}\leq x\leq 1\end{cases}$$

(a) Use Fourier series to write down the temperature distribution at time $t>0$. (b) What is the equilibrium temperature distribution in the bar, i.e., for $t\gg 0$? (c) How fast does the solution go to equilibrium? (d) Just before the temperature distribution reaches equilibrium, what does it look like? Sketch a picture and discuss

Solution

2.1 Part (a)

Using L for the length to make the solution more general and at the end L is replaced 1.

$$\begin{aligned}\frac{\partial u}{\partial t}&=\gamma\frac{\partial^2 u}{\partial x^2}\\ u_x(0,t)&=0\\ u_x(L,t)&=0\\ u(x,0)&=f(x)\end{aligned}$$

Let $u(x,t)=T(t)X(x)$, then the PDE becomes

$$\frac{1}{\gamma}T'X=X''T$$

Dividing by $XT\neq 0$

$$\frac{1}{\gamma}\frac{T'}{T}=\frac{X''}{X}$$

Since each side depends on different independent variable and both are equal, they must be both equal to same constant, say $-\lambda$. Where λ is assumed real.

$$\frac{1}{\gamma}\frac{T'}{T}=\frac{X''}{X}=-\lambda$$

The two ODE's generated are

$$T'+\gamma\lambda T=0 \tag{1}$$

And the eigenvalue ODE

$$\begin{aligned}X''+\lambda X&=0 \qquad (2)\\ X'(0)&=0\\ X'(L)&=0\end{aligned}$$

Starting with the eigenvalue ODE equation (2). The following cases are considered.

case $\lambda<0$

In this case, $-\lambda$ is positive. Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$X(x) = C_1 \cosh(\omega x) + C_2 \sinh(\omega x)$$

Therefore

$$X' = C_1 \sinh(\omega x) + C_2 \cosh(\omega x)$$

Applying the left B.C. gives

$$0 = C_2$$

Therefore the solution becomes $X(x) = C_1 \cosh(\omega x)$ and $X'(x) = C_1 \sinh(\omega x)$. Applying the right B.C. gives

$$0 = C_1 \sinh(\omega L)$$

For non-trivial solution we want $\sinh(\omega L) = 0$. But this is not possible since $\sinh$ is zero when its argument is zero, which is not the case here. Hence only trivial solution results from this case. $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

The solution is

$$\begin{aligned} X(x) &= c_1 x + c_2 \\ X'(x) &= c_1 \end{aligned}$$

Applying left boundary conditions gives

$$0 = c_1$$

Hence the solution becomes $X(x) = c_2$. Therefore $\frac{dX}{dx} = 0$. Applying the right B.C. provides no information. Any c_2 will work. Therefore this case leads to the solution $X(x) = c_2$. Associated with this one eigenvalue, the time equation becomes $T_0'(t) = 0$ hence $T_0(t)$ is a constant. Hence the solution $u_0(x,t)$ associated with this $\lambda = 0$ is

$$\begin{aligned} u_0(x,t) &= X_0 T_0 \\ &= A_0 \end{aligned}$$

where constant $c_2 T_0$ was renamed to $\frac{A_0}{2}$ to indicate it is associated with $\lambda = 0$. $\lambda = 0$ is an eigenvalue with eigenfunction constant $\frac{A_0}{2}$.

case $\lambda > 0$

The solution is

$$\begin{aligned} X(x) &= c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \\ X'(x) &= -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right) + c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right) \end{aligned}$$

Applying the left B.C. gives

$$0 = c_2 \sqrt{\lambda}$$

Therefore $c_2 = 0$ as $\lambda > 0$. The solution becomes

$$X(x) = c_1 \cos\left(\sqrt{\lambda} x\right)$$

And $X'(x) = -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$. Applying the right B.C. gives

$$0 = -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)$$

$c_1 = 0$ gives a trivial solution. Selecting $\sin\left(\sqrt{\lambda}L\right) = 0$ gives

$$\sqrt{\lambda}L = n\pi \qquad n = 1, 2, 3, \cdots$$

Or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Therefore the eigenfunctions are

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right) \qquad n = 1, 2, 3, \cdots$$

The time solution is found by solving

$$T_n'(t) + \gamma\lambda_n T_n(t) = 0$$

This has the solution

$$\begin{aligned} T_n(t) &= A_n e^{-\gamma\lambda_n t} \\ &= A_n e^{-\gamma\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1, 2, 3, \cdots \end{aligned}$$

The solution to the PDE is

$$u_n(x,t) = T_n(t) X_n(x) \qquad n = 0, 1, 2, 3, \cdots$$

But for linear system sum of eigenfunctions is also a solution. Hence

$$\begin{aligned} u(x,t) &= u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\gamma\left(\frac{n\pi}{L}\right)^2 t} \end{aligned} \tag{1}$$

From the solution found above, setting $t = 0$ gives

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

Hence A_0, A_n are the Fourier $\cos$ coefficients for the function $f(x)$. Doing an even extension of $f(x)$ from $[-L, L]$, then $\frac{A_0}{2}$ is the average of the function $f(x)$ over $[-L, L]$. But this average is seen as $\frac{2\left(\frac{1}{2}\times\frac{1}{2}\right)}{2} = \frac{1}{4}$. The term $\frac{1}{2}\times\frac{1}{2}$ is the area of $f(x)$ from $[0, L]$.

$$\frac{A_0}{2} = \frac{1}{4}$$

For A_n

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

Replacing $L = 1$ and using the definition of $f(x)$ given above gives

$$A_n = \int_{-1}^{1} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

But $f(x)$ is even (after even extending) and $\cos$ is even, hence the above becomes

$$\begin{aligned}
A_n &= 2\int_0^1 f(x)\cos(n\pi x)\,dx \\
&= 2\left(\int_0^{\frac{1}{2}} x\cos(n\pi x)\,dx + \int_{\frac{1}{2}}^1 (1-x)\cos(n\pi x)\,dx\right) \\
&= 2\left(\int_0^{\frac{1}{2}} x\cos(n\pi x)\,dx + \int_{\frac{1}{2}}^1 \cos(n\pi x)\,dx - \int_{\frac{1}{2}}^1 x\cos(n\pi x)\,dx\right) \qquad (2)
\end{aligned}$$

But

$$\begin{aligned}
\int_a^b x\cos(n\pi x)\,dx &= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_a^b - \frac{1}{n\pi}\int_a^b \sin(n\pi x)\,dx \\
&= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_a^b + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_a^b \qquad (3)
\end{aligned}$$

When $a = 0, b = \frac{1}{2}$ the above gives

$$\begin{aligned}
\int_0^{\frac{1}{2}} x\cos(n\pi x)\,dx &= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_0^{\frac{1}{2}} + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^{\frac{1}{2}} \\
&= \frac{1}{n\pi}\left(\frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\right) + \frac{1}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) \\
&= \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) \\
&= \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2} \qquad (4)
\end{aligned}$$

And when $a = \frac{1}{2}, b = 1$ (3) gives

$$\begin{aligned}
\int_{\frac{1}{2}}^1 x\cos(n\pi x)\,dx &= \frac{1}{n\pi}\left[x\sin(n\pi x)\right]_{\frac{1}{2}}^1 + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_{\frac{1}{2}}^1 \\
&= \frac{1}{n\pi}\left[\sin(n\pi) - \frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\right] + \frac{1}{n^2\pi^2}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right] \\
&= -\frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos(n\pi) - \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) \qquad (5)
\end{aligned}$$

Substituting (4,5) into (2) gives

$$\begin{aligned}\frac{A_n}{2} &= \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\\ &+ \int_{\frac{1}{2}}^{1}\cos(n\pi x)\,dx\\ &- \left(-\frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos(n\pi) - \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right)\end{aligned}$$

Or

$$\begin{aligned}\frac{A_n}{2} &= \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\\ &+ \frac{1}{n\pi}\overbrace{\sin(n\pi)}^{0} - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\\ &+ \frac{1}{2n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\cos(n\pi) + \frac{1}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\end{aligned}$$

Or

$$\begin{aligned}\frac{A_n}{2} &= \left(\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\right) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\cos(n\pi)\\ &= \frac{2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n}{n^2\pi^2}\end{aligned}$$

Therefore the solution (1) becomes, replacing $L = 1$

$$\begin{aligned}u(x,t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(n\pi x)\,e^{-\gamma n^2\pi^2 t}\\ &= \frac{1}{4} + 2\sum_{n=1}^{\infty}\frac{2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n}{n^2\pi^2}\cos(n\pi x)\,e^{-\gamma n^2\pi^2 t} \qquad (6)\end{aligned}$$

2.2 Part b

From the solution (6) in part (a), since $\gamma > 0$ then $\lim_{t\to\infty} e^{-\gamma n^2\pi^2 t} = 0$ and the solution becomes

$$\lim_{t\to\infty} u(x,t) = \frac{1}{4}$$

This is the average of the initial temperature distribution. This makes sense, since there are no sources or sinks, and both ends are insulated. So all of the initial heat will remain in the bar but will average evenly over the bar length at the average which is $\frac{1}{4}$.

2.3 Part c

due to the exponential decay term $e^{-\gamma n^2\pi^2 t}$ and also having $\frac{1}{n^2}$ term, the decay of the sum is very fast. High frequency terms decay very fast since $e^{-\gamma n^2\pi^2 t} \lll 1$ for large n. Using $\gamma = 1$ only few terms are needed to show this. The solution goes to the average (the constant term in the Fourier series) at exponential rate.

This will be shown explicitly in the next part by plotting the solution using $\gamma = 1$ for illustration.

2.4 Part d

The following shows how fast the initial temperature reach equilibrium $\frac{1}{4}$ degree over the whole bar. Using only 4 terms in the Fourier series, and using $\gamma = 1$, it took only 0.1 seconds. Looking at the middle of the bar, where the initial temperature was highest at 0.5, we first see that initial temperature which was not smooth, become instantaneously smooth. Then it took 0.5 seconds for the temperature in the middle of the bar to go down to 0.3 degrees. And the next 0.5 second to go down to 0.25. This shows that the initial decay was rapid, then it slows down relatively until it reaches 0.25 degree which is the average then stops there.

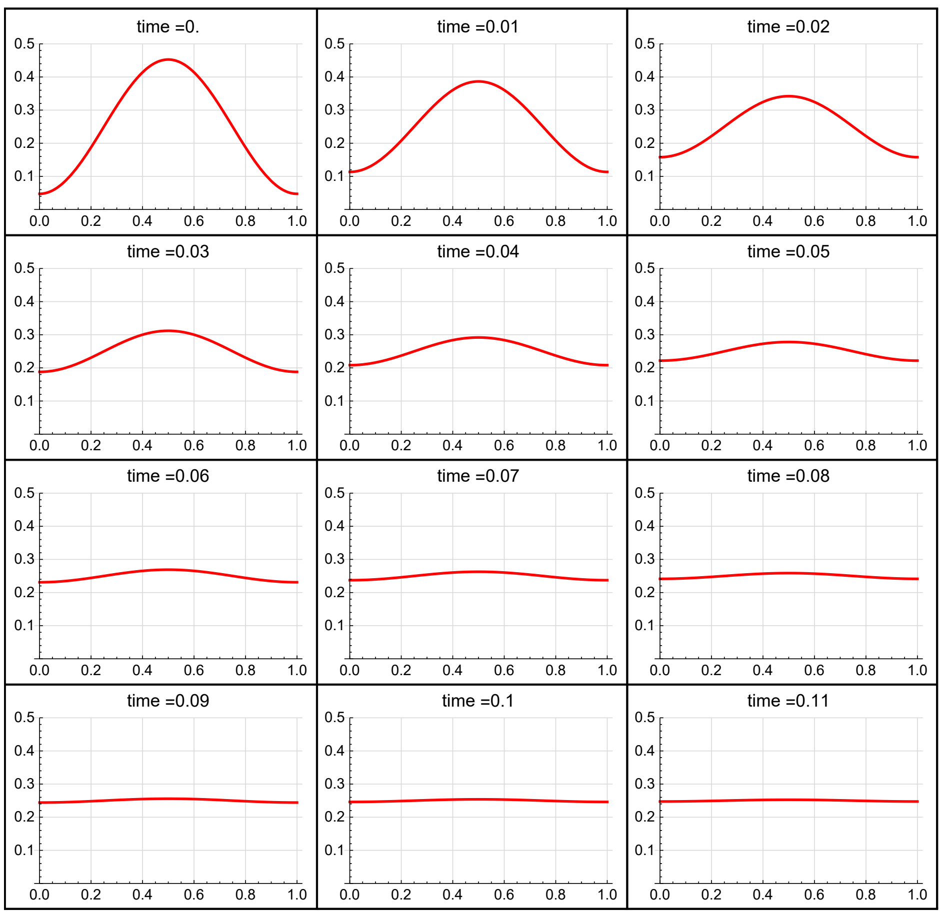

Figure 1: Plot showing solution in time

```
u[x_, t_, max_] :=
        1/4 + 2 Sum[ 1/(n^2 π^2) (2 Cos[n π/2] - 1 - (-1)^n) Cos[n π x] Exp[- n^2 π^2 t], {n, 1, max}];
p = Grid[Partition[Table[Quiet@Plot[u[x, t, 4], {x, 0, 1}, PlotRange → {Automatic, {0, 0.5}},
                         GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red,
                         PlotLabel → Row[{"time =", t}]], {t, 0, .11, 0.01}], 3], Frame → All];
```

Figure 2: Code used for the above plot

3 Problem 4.1.10c

For each of the following initial temperature distributions, (i) write out the Fourier series solution to the heated ring (4.30–32), and (ii) find the resulting equilibrium temperature as $t \to \infty$ (c) $u(x,0) = |x|$

Solution

3.1 Part I

The heated ring is given by 4.30-4.32 as solving for $u(x,t)$ in

$$u_t = u_{xx} \qquad -\pi < x < \pi, t > 0$$

With periodic BC

$$\begin{aligned} u(-\pi,t) &= u(\pi,t) \\ u_x(-\pi,t) &= u_x(\pi,t) \end{aligned}$$

And initial conditions $u(x,0) = f(x) = |x|$. As given in the text, the Fourier series solution is (4.35)

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) e^{-n^2 t}$$

Since $f(x)$ is even, then all $b_n = 0$.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{2}{\pi} \int_0^{\pi} x dx \\ &= \frac{2}{\pi} \frac{1}{2} \left[x^2 \right]_0^{\pi} \\ &= \pi \end{aligned}$$

And

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx \\
&= \frac{2}{\pi}\left(\overbrace{\left[\frac{x\sin nx}{n\pi}\right]_0^{\pi}}^{0} - \int_0^{\pi}\frac{\sin nx}{n\pi}dx\right) \\
&= \frac{2}{\pi}\left(\frac{1}{n\pi}\left[\cos nx\right]_0^{\pi}\right) \\
&= \frac{2}{n\pi^2}\left(\cos n\pi - 1\right) \\
&= \frac{2}{n\pi^2}\left((-1)^n - 1\right)
\end{aligned}$$

Hence the solution becomes

$$u(x,t) = \frac{\pi}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\cos(nx)\,e^{-n^2 t}$$

3.2 Part II

From the solution above, we see that

$$\lim_{t\to\infty} u(x,t) = \frac{\pi}{2}$$

Which is the average of the original temperature distribution.

4 Problem 4.1.16

The cable equation $v_t = \gamma v_{xx} - \alpha v$ with $\gamma, \alpha > 0$, also known as the lossy heat equation,was derived by the nineteenth-century Scottish physicist William Thomson to model propagation of signals in a transatlantic cable. Later, in honor of his work on thermodynamics, including determining the value of absolute zero temperature, he was named Lord Kelvin by Queen Victoria. The cable equation was later used to model the electrical activity of neurons. (a) Show that the general solution to the cable equation is given by $v(x,t) = e^{-\alpha t} u(x,t)$ where $u(x,t)$ solves the heat equation $u_t = \gamma u_{xx}$.

(b) Find a Fourier series solution to the Dirichlet initial-boundary value problem $v_t = \gamma v_{xx} - \alpha v$, with initial conditions $v(x,0) = f(x)$ and boundary conditions $v(0,t) = 0, v(1,t) = 0$ for $0 \leq x \leq 1, t > 0$. Does your solution approach an equilibrium value? If so, how fast? (c) Answer part (b) for the Neumann problem

$$v_t = \gamma v_{xx} - \alpha v \qquad 0 \leq x \leq 1, t > 0$$

With initial conditions

$$v(x,0) = f(x)$$

And B.C.

$$\begin{aligned} v_x(0,t) &= 0 \\ v_x(1,t) &= 0 \end{aligned}$$

Solution

4.1 Part c

Part (a,b) were solved in HW5 so we only need to solve part c here.

Using separation of variable, let $v = T(t)X(x)$ where $T(t)$ is function that depends on time only and $X(x)$ is a function that depends on x only. Using this substitution in (1) gives

$$T'X = \gamma X''T - \alpha XT$$

Dividing by $XT \neq 0$ gives

$$\frac{1}{\gamma}\frac{T'}{T} + \frac{\alpha}{\gamma} = \frac{X''}{X} = -\lambda$$

Where λ is the separation constant. The above gives two ODE's to solve

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X'(1) &= 0 \end{aligned} \tag{2}$$

And

$$
\begin{aligned}
\frac{1}{\gamma}\frac{T'}{T}+\frac{\alpha}{\gamma} &= -\lambda\\
T'+\alpha T &= -\lambda\gamma T\\
T'+\alpha T+\lambda\gamma T &= 0\\
T'+\left(\alpha+\lambda\gamma\right)T &= 0 \qquad (3)
\end{aligned}
$$

ODE (2) is the boundary value ODE which will generate the eigenvalues and eigenfunctions.

case $\lambda < 0$

Let $-\lambda = \omega^2$. The solution to (2) becomes

$$
\begin{aligned}
X &= c_1\cosh(\omega x)+c_2\sinh(\omega x)\\
X' &= \omega c_1\sinh(\omega x)+\omega c_2\cosh(\omega x)
\end{aligned}
$$

At $x=0$

$$0=\omega c_2$$

Therefore $c_2=0$. The solution becomes

$$
\begin{aligned}
X &= c_1\cosh(\omega x)\\
X' &= \omega c_1\sinh(\omega x)
\end{aligned}
$$

At $x=1$ this gives $0=\omega c_1\sinh(\omega)$. But $\sinh(\omega)=0$ only when $\omega=0$ which is not the case here. Hence $c_1=0$ leading to trivial solution. Therefore $\lambda<0$ is not eigenvalue.

case $\lambda = 0$

The solution is $X(x)=c_1x+c_2$ and $X'=c_1$. At $x=0$ this gives $0=c_1$. Hence solution is $X=c_2$ and $X'=0$. At $x=1$ this gives $0=0$. Therefore any c_2 will work. Taking $c_2=1$ the eigenfunction is $X_0(x)=1$ and $\lambda=0$ is eigenvalue.

case $\lambda > 0$

Solution is

$$
\begin{aligned}
X(x) &= c_1\cos\left(\sqrt{\lambda}x\right)+c_2\sin\left(\sqrt{\lambda}x\right)\\
X'(x) &= \sqrt{\lambda}c_1\sin\left(\sqrt{\lambda}x\right)+c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)
\end{aligned}
$$

At $x=0$ this results in $0=c_2\sqrt{\lambda}$. Hence $c_2=0$. The solution now becomes The above now becomes

$$
\begin{aligned}
X(x) &= c_1\cos\left(\sqrt{\lambda}x\right)\\
X'(x) &= -c_1\sin\left(\sqrt{\lambda}x\right)
\end{aligned}
$$

At $x=1$

$$0=-c_1\sin\left(\sqrt{\lambda}\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\lambda}\right) = 0$ or $\sqrt{\lambda} = n\pi, n = 1, 2, \cdots$. Hence

$$\lambda_n = n^2\pi^2 \qquad n = 1, 2, \cdots$$

And the corresponding eigenfunctions

$$X_n(x) = \cos(n\pi x) \tag{4}$$

Now we can solve the time ODE (3). For the zero eigenvalue, (3) becomes

$$T' + \alpha T = 0$$

With solution

$$T_0(t) = \frac{A_0}{2} e^{-\alpha t}$$

And for the non zero eigenvalues $\lambda_n = n^2\pi^2$ the ODE (3) becomes

$$T' + \left(\alpha + n^2\pi^2\gamma\right) T = 0$$

With solution

$$T_n(t) = A_n e^{-\left(\alpha + n^2\pi^2\gamma\right)t}$$

The general solution is linear combination of the above

$$v(x,t) = \frac{A_0}{2} e^{-\alpha t} + \sum_{n=1}^{\infty} A_n e^{-\left(\alpha + n^2\pi^2\gamma\right)t} \cos(n\pi x) \tag{6}$$

At $t = 0$ the above becomes

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

We see that A_n are the cosine Fourier coefficients of $f(x)$, after <u>even extending</u> $f(x)$ to $[-1, 1]$, the period of $f(x)$ becomes 2 giving

$$\begin{aligned} A_0 &= \int_{-1}^{1} f(x)\, dx \\ &= 2\int_{0}^{1} f(x)\, dx \end{aligned}$$

And

$$\begin{aligned} A_n &= \int_{-1}^{1} f(x) \cos(nx)\, dx \\ &= 2\int_{0}^{1} f(x) \cos(nx)\, dx \end{aligned}$$

Using the above in solution (6) gives

$$v(x,t) = \left(\int_0^1 f(x)\, dx\right) e^{-\alpha t} + 2\sum_{n=1}^{\infty} \left(\int_0^1 f(x) \cos(nx)\, dx\right) e^{-\left(\alpha + n^2\pi^2\gamma\right)t} \cos(n\pi x)$$

To find equilibrium, we let $t \to \infty$ then $e^{-\left(\alpha + n^2\pi^2\gamma\right)t} \to 0$ and also $e^{-\alpha t}$ because $\alpha, \gamma > 0$ and the above becomes

$$v_{eq}(x,t) = 0$$

The decay is fast due to $e^{-\left(\alpha+n^2\pi^2\gamma\right)t} \gg 1$ for large n. Hence it is exponential decay. Solution each equilibrium value of 0 where it remains there.

5 Problem 4.2.3d

Write down the solutions to the following initial-boundary value problems for the wave equation in the form of a Fourier series

$$u_{tt} = 4u_{xx} \tag{1}$$

With boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0$$

And initial conditions

$$u(x,0) = x$$
$$u_t(x,0) = -x$$

<u>Solution</u>

To make the solution more general and useful, the length is taken as L and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ and $c^2 = 4$, and then at the end these are replaced by the actual values given in this problem which are $L = 1, f(x) = x, g(x) = -x, c^2 = 4$.

Hence the PDE to solve is $u_{tt} = c^2 u_{xx}$ with BC $u(0,t) = 0, u(L,0) = 0$ and $u(x,0) = f(x), u_t(x,0) = g(x)$.

Using separation of variables, let $u = X(x)T(t)$. The PDE becomes

$$\frac{T''X}{c^2} = X''T$$
$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda$$

Where λ is separation constant. Hence the eigenvalue ODE is

$$X'' + \lambda X = 0 \tag{2}$$
$$X(0) = 0$$
$$X(L) = 0$$

And the time ODE is

$$T'' + c^2\lambda T = 0 \tag{3}$$

Starting by the eigenvalue ODE to determine the eigenvalues and eigenfunctions.

<u>Case $\lambda < 0$</u>

Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$X(x) = C_1 \cosh(\omega x) + C_2 \sinh(\omega x)$$

At $x = 0$ the above gives

$$0 = C_1$$

Hence the solution now becomes

$$X(x) = C_2 \sinh(\omega x)$$

At $x = L$ the above gives

$$0 = C_2 \sinh(\omega L)$$

But sinh is zero only when its argument is zero which is not the case here. Therefore $C_2 = 0$ which leads to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$

The space equation becomes $X''(x) = 0$ with the solution

$$X = Ax + B$$

At $x = 0$ the above gives $0 = B$. Therefore the solution is $X = Ax$. At $x = L$ this gives $0 = AL$. Hence $A = 0$, which leads to trivial solutions. Therefore $\lambda = 0$ is not an eigenvalue.

case $\lambda > 0$

The solution to the above ODE now is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Since $X(0) = 0$ then $A = 0$ and the solution becomes

$$X(x) = B\sin\left(\sqrt{\lambda}x\right)$$

Since $X(L) = 0$ then for non trivial solution we want $\sin\left(\sqrt{\lambda}L\right) = 0$ or $\sqrt{\lambda}L = n\pi$ or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Hence the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1, 2, 3, \cdots$$

The time ODE (3) now becomes

$$T'' + c^2\left(\frac{n\pi}{L}\right)^2 T = 0$$

Which has the solution

$$T(t) = B_n\cos\left(c\frac{n\pi}{L}t\right) + A_n\sin\left(c\frac{n\pi}{L}t\right)$$

Therefore the complete solution becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n\cos\left(c\frac{n\pi}{L}t\right) + A_n\sin\left(c\frac{n\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right) \tag{4}$$

Now we can replace the given values in the above solution which gives

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n\cos(2n\pi t) + A_n\sin(2n\pi t)\right)\sin(n\pi x) \tag{4A}$$

At $t = 0$ the above becomes

$$f(x) = \sum_{n=1}^{\infty} B_n\sin(n\pi x)\,dx$$

Hence B_n are the Fourier sine coefficients of $f(x) = x$. After odd extending $f(x)$ to $[-1,1]$ we obtain

$$
\begin{aligned}
B_n &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx \\
&= 2\int_0^1 f(x)\sin(n\pi x)\,dx \\
&= 2\int_0^1 x\sin(n\pi x)\,dx \\
&= 2\left(\frac{-1}{n\pi}\left[x\cos n\pi x\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x dx\right) \\
&= 2\left(\frac{-1}{n\pi}(\cos n\pi) + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1\right) \\
&= \frac{-2(-1)^n}{n\pi}
\end{aligned}
$$

To find A_n, taking time derivative of (4A) gives

$$
u_t(x,t) = \sum_{n=1}^{\infty} \left(-B_n 2n\pi\sin(2n\pi t) + 2n\pi A_n\cos(2n\pi t)\right)\sin(n\pi x)
$$

At $t=0$ the above becomes, using the initial conditions where $g(x) = -x$

$$
g(x) = \sum_{n=1}^{\infty} (2n\pi A_n)\sin(n\pi x)
$$

The above is the Fourier sine series for $g(x)$. By odd extending $-x$ to $[-1,1]$ then

$$
\begin{aligned}
2n\pi A_n &= \int_{-1}^{1} g(x)\sin(n\pi x)\,dx \\
&= 2\int_0^1 g(x)\sin(n\pi x)\,dx \\
&= -2\int_0^1 x\sin(n\pi x)\,dx \\
&= -2\left(-\frac{1}{n\pi}\left[x\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right) \\
&= -2\left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n^2\pi^2}\overbrace{\left[\sin(n\pi x)\right]_0^1}^{0}\right) \\
&= \frac{2}{n\pi}\left[\cos(n\pi)\right] \\
&= \frac{2(-1)^n}{n\pi}
\end{aligned}
$$

Therefore

$$
A_n = \frac{(-1)^n}{n^2\pi^2}
$$

Now that we found A_n, B_n, then the solution (4A) is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left(\frac{-2(-1)^n}{n\pi} \cos(2n\pi t) + \frac{(-1)^n}{n^2\pi^2} \sin(2n\pi t) \right) \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} \left(\sin(2n\pi t) - 2n\pi \cos(2n\pi t) \right) \sin(n\pi x) \end{aligned}$$

6 Problem 4.2.4b

Find all separable solutions to the wave equation $u_{tt} = u_{xx}$ on the interval $0 \leq x \leq \pi$ subject to (b) Neumann boundary conditions $u_x(0,t) = 0, u_x(\pi,t) = 0$.

Solution

Using separation of variables, let $u = X(x)T(t)$. The PDE becomes

$$\begin{aligned} T''X &= X''T \\ \frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where λ is separation constant. Hence the eigenvalue ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X'(\pi) &= 0 \end{aligned} \tag{2}$$

And the time ODE is

$$T'' + \lambda T = 0 \tag{3}$$

Starting by the eigenvalue ODE to determine the eigenvalues and eigenfunctions.

Case $\lambda < 0$

Let $-\lambda = \omega^2$. Hence the ODE is $X'' - \omega^2 X = 0$ and the solution becomes

$$\begin{aligned} X(x) &= C_1 \cosh(\omega x) + C_2 \sinh(\omega x) \\ X'(0) &= C_1 \omega \sinh(\omega x) + C_2 \omega \cosh(\omega x) \end{aligned}$$

At $x = 0$ the above gives

$$0 = C_2$$

Hence the solution now becomes

$$\begin{aligned} X(x) &= C_1 \cosh(\omega x) \\ X'(x) &= C_1 \omega \sinh(\omega x) \end{aligned}$$

At $x = \pi$ the above gives

$$0 = C_1 \omega \sinh(\omega \pi)$$

But $\sinh$ is zero only when its argument is zero which is not the case here. Therefore $C_1 = 0$ which leads to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$

The space equation becomes $X''(x) = 0$ with the solution

$$\begin{aligned} X &= Ax + B \\ X'(x) &= A \end{aligned}$$

At $x = 0$ the above gives $0 = A$. Therefore the solution is $X = B$. Therefore $X' = 0$. At $x = \pi$ this gives $0 = 0$. Therefore any value of B will work. Using the constant as 1, then the $\lambda = 0$

is an eigenvalue with corresponding eigenfunction $X_0 = 1$.

case $\lambda > 0$

The solution to the above ODE now is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

Since $X'(0) = 0$ then $B = 0$ and the solution becomes

$$X(x) = A\cos\left(\sqrt{\lambda}x\right)$$
$$X'(x) = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$$

Since $X'(\pi) = 0$ then for non trivial solution we want $\sin\left(\sqrt{\lambda}\pi\right) = 0$ or $\sqrt{\lambda}\pi = n\pi$ or

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

Hence the eigenfunctions are

$$X_n(x) = \cos(nx) \qquad n = 1, 2, 3, \cdots$$

The time ODE (3) is now solved. For $\lambda = 0$ it becomes $T'' = 0$. Hence the solution is $T_0(t) = \frac{B_0}{2}t + \frac{A_0}{2}$ and for $\lambda_n = n^2$ it becomes

$$T_n'' + n^2 T_n = 0$$

Which has the solution

$$T_n(t) = A_n\cos(nt) + B_n\sin(nt)$$

Therefore the complete solution becomes

$$\begin{aligned} u(x,t) &= \overbrace{X_0}^{1}\ \overbrace{T_0}^{\frac{B_0}{2}t+\frac{A_0}{2}} + \sum_{n=1}^{\infty} X_n T_n \\ &= \frac{B_0}{2}t + \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos(nt) + B_n\sin(nt)\right)\cos(nx) \end{aligned} \tag{4}$$

To find A_0, B_0, A_n, B_n we need initial conditions which are not given. I was not sure if we are supposed to assume such initial conditions or not in order to continue. If so, then assuming $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, then at $t = 0$ the above becomes

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(nx)$$

Hence A_n are the Fourier cosine coefficients of $f(x)$. After even extending $f(x)$ to $[-\pi, \pi]$ we obtain

$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \end{aligned}$$

And

$$
\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx
\end{aligned}
$$

To find B_n, taking time derivative of (4) gives

$$
u_t(x,t) = \frac{B_0}{2} + \sum_{n=1}^{\infty} \left(-nA_n\sin(nt) + nB_n\cos(nt)\right)\cos(nx)
$$

At $t = 0$ the above gives

$$
g(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} nB_n\cos(nx)
$$

Hence was done above for A_0, A_n we obtain

$$
B_0 = \frac{2}{\pi}\int_{0}^{\pi} g(x)\,dx
$$

And

$$
\begin{aligned}
nB_n &= \frac{2}{\pi}\int_{0}^{\pi} g(x)\cos(nx)\,dx \\
B_n &= \frac{2}{n\pi}\int_{0}^{\pi} g(x)\cos(nx)\,dx
\end{aligned}
$$

Now that we found A_n, B_n, then the solution (4) is

$$
\begin{aligned}
u(x,t) = t&\left(\frac{1}{\pi}\int_{0}^{\pi} g(x)\,dx\right) + \left(\frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx\right) \\
&+ \sum_{n=1}^{\infty}\left[\left(\frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx\right)\cos(nt) + \left(\frac{2}{n\pi}\int_{0}^{\pi} g(x)\cos(nx)\,dx\right)\sin(nt)\right]\cos(nx)
\end{aligned}
$$

Or

$$
\begin{aligned}
u(x,t) = t&\left(\frac{1}{\pi}\int_{0}^{\pi} g(x)\,dx\right) + \left(\frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx\right) \\
&+ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(n\cos(nt)\int_{0}^{\pi} f(x)\cos(nx)\,dx + \sin(nt)\int_{0}^{\pi} g(x)\cos(nx)\,dx\right)\cos(nx)
\end{aligned}
$$

7 Problem 4.2.6

(a) Formulate the periodic initial-boundary value problem for the wave equation on the interval $-\pi \le x \le \pi$, modeling the vibrations of a circular ring. (b) Write out a formula for the solution to your problem in the form of a Fourier series. (c) Is the solution a periodic function of t? If so, what is the period? (d) Suppose the initial displacement coincides with that in Figure 4.6, while the initial velocity is zero. Describe what happens to the solution as time evolves.

Solution

7.1 Part a

Solving for $u(x,t)$ in

$$u_{tt} = c^2 u_{xx} \tag{1}$$

With periodic boundary conditions

$$\begin{aligned} u(-\pi,t) &= u(\pi,t) \\ u_x(-\pi,t) &= u_x(\pi,t) \end{aligned}$$

And initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

7.2 Part b

Using separation of variables, let $u = X(x)T(t)$. Substituting in (1) gives

$$\begin{aligned} \frac{1}{c^2}T''X &= X''T \\ \frac{1}{c^2}\frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where λ is the separation variable. This gives two ODE's to solve. The time ODE

$$T'' + c^2\lambda T = 0 \tag{2}$$

And the eigenvalue ODE

$$X'' + \lambda X = 0 \tag{3}$$

case $\lambda < 0$

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is now positive. The solution to (3) becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

The above can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right) \tag{4}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using (4) gives

$$c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(\sqrt{\mu}\pi\right)$$
$$c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_2 \sinh\left(\sqrt{\mu}\pi\right)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_2 = 0$. The solution (4) now reduces to

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right)$$

Taking derivative gives

$$X'(x) = c_1\sqrt{\mu} \sinh\left(\sqrt{\mu}x\right)$$

Applying the second BC $X'(-\pi) = X'(\pi)$ the above gives

$$c_1\sqrt{\mu} \sinh\left(-\sqrt{\mu}\pi\right) = c_1\sqrt{\mu} \sinh\left(\sqrt{\mu}x\right)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_1 = 0$. This means a trivial solution. Therefore $\lambda < 0$ is not an eigenvalue.

<u>case $\lambda = 0$</u>

In this case the solution is $X(x) = c_1 + c_2 x$. Applying first BC $X(-\pi) = X(\pi)$ gives

$$c_1 - c_2\pi = c_1 + c_2\pi$$
$$-c_2\pi = c_2\pi$$

This gives $c_2 = 0$. The solution now becomes $X(x) = c_1$ and $X'(x) = 0$. Applying the second boundary conditions $X'(-\pi) = X'(\pi)$ is not satisfies $(0 = 0)$. Therefore <u>$\lambda = 0$</u> is an eigenvalue with eigenfunction $X_0(0) = 1$ (selected $c_1 = 1$ since an arbitrary constant).

<u>case $\lambda > 0$</u>

The solution in this case is

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \tag{5}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using the above gives

$$c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(-\sqrt{\lambda}\pi\right) = c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right)$$
$$c_2 \sin\left(-\sqrt{\lambda}\pi\right) = c_2 \sin\left(\sqrt{\lambda}\pi\right)$$

There are two choices here. If $\sin\left(-\sqrt{\lambda}\pi\right) \neq \sin\left(\sqrt{\lambda}\pi\right)$, then this implies that $c_2 = 0$. If $\sin\left(-\sqrt{\lambda}\pi\right) = \sin\left(\sqrt{\lambda}\pi\right)$ then $c_2 \neq 0$. Assuming for now that $\sin\left(-\sqrt{\lambda}\pi\right) = \sin\left(\sqrt{\lambda}\pi\right)$. This happens when $\sqrt{\lambda}\pi = n\pi, n = 1, 2, 3, \cdots$, or

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

Using this choice, we will now look to see what happens using the second BC. The solution (5) now becomes

$$X(x) = c_1 \cos(nx) + c_2 \sin(nx) \qquad n = 1, 2, 3, \cdots$$

Therefore

$$X'(x) = -c_1 n \sin(nx) + c_2 n \cos(nx)$$

Applying the second BC $X'(-\pi) = X'(\pi)$ using the above gives

$$\begin{aligned} c_1 n \sin(n\pi) + c_2 n \cos(n\pi) &= -c_1 n \sin(n\pi) + c_2 n \cos(n\pi) \\ c_1 n \sin(n\pi) &= -c_1 n \sin(n\pi) \\ 0 &= 0 \end{aligned}$$

Since n is integer.

Therefore this means that using $\lambda_n = n^2$ has satisfied both boundary conditions with $c_2 \neq 0, c_1 \neq 0$. This means the solution (5) becomes

$$X_n(x) = A_n \cos(nx) + B_n \sin(nx) \qquad n = 1, 2, 3, \cdots$$

The above says that there are two eigenfunctions in this case. They are

$$X_n(x) = \begin{cases} \cos(nx) \\ \sin(nx) \end{cases}$$

Since there is also zero eigenvalue, then the complete set of eigenfunctions become

$$X_n(x) = \begin{cases} 1 \\ \cos(nx) \\ \sin(nx) \end{cases}$$

Now that the eigenvalues are found, we go back and solve the time ODE. Recalling that the time ODE (2) from above was found to be

$$T'' + c^2 \lambda T = 0$$

When $\lambda = 0$ this becomes $T'' = 0$ with solution $T_0(t) = At + B$. When $\lambda_n = n^2$ the ODE becomes $T'' + c^2 n^2 T = 0$ with solution

$$T_n(t) = C_n \cos(cnt) + E_n \sin(cnt)$$

Adding all the above solutions using $u_n(x,t) = X_n(x) T_n(t)$ gives the final solution as

$$\begin{aligned} u(x,t) &= X_0(x) T_0(t) + \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= At + B + \sum_{n=1}^{\infty} (\cos(nx) + \sin(nx)) (C_n \cos(cnt) + E_n \sin(cnt)) \end{aligned}$$

Or

$$\begin{aligned} u(x,t) = At &+ B \\ &+ \sum_{n=1}^{\infty} (C_n \cos(cnt) + E_n \sin(cnt)) \cos(nx) \\ &+ \sum_{n=1}^{\infty} (C_n \cos(cnt) + E_n \sin(cnt)) \sin(nx) \end{aligned}$$

7.3 Part c

The solution is periodic in time. To find the period, solving $ct = \frac{2\pi}{T}t$ for T gives

$$T = \frac{2\pi}{c}$$

7.4 Part d

The solution will behave similar to the one on page 148 initially, where initial conditions splits in half, one half moving left and one moving right until each half reach the boundary conditions. But now, each half wave reflects off the boundary staying upside and starts moving back toward the middle again, until the two halves reunite again to reproduce the same initial conditions shape. This process then repeats again and again.

So the difference between periodic boundary conditions, and having ends fixed as the case in Figure 4.6, is that when ends are fixed, the two half waves reflect upside down at the boundaries, while here they do not not. The solution above was animated and plotted showing this. Initial conditions used is small triangle similar to one used in Figure 4.6 with zero initial conditions and using $c = 1$ for speed. The following is the result

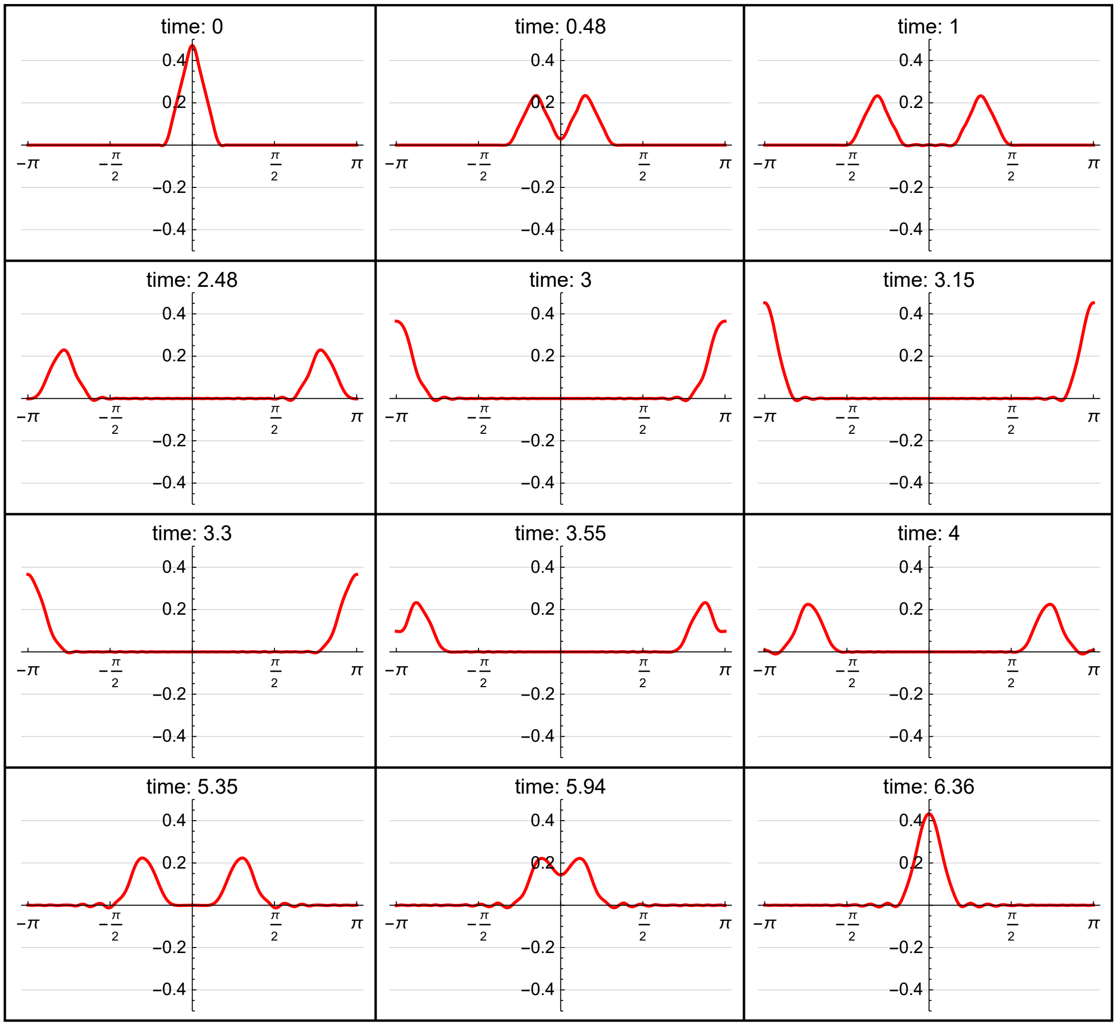

Figure 3: Plot showing solution in time, Periodic B.C.

In the above at $t = 3.15$ sec. is when each half wave reaches the boundary at $x = -\pi$ and $x = \pi$. At $t > 3.3$ the waves half reflects and are starting to moving back towards the center. At $t = 6.36$ the initial conditions shape is reconstructed again. For higher times, the above motion repeats.

8 Problem 4.2.14c

Sketch the solution of the wave equation $u_{tt} = u_{xx}$ and describe its behavior when the initial displacement is the box function $u(x,0) = \begin{cases} 1 & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$ while the initial velocity is 0 in each of the following scenarios (c) on the half-line $0 \le x < \infty$, with homogeneous Neumann boundary condition at the end.

Solution

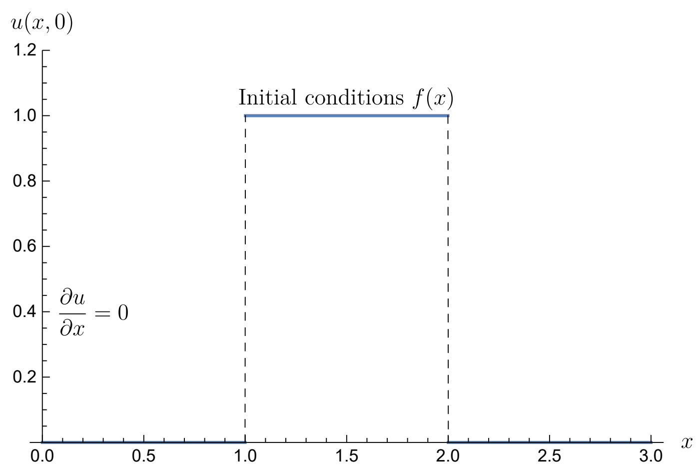

Figure 4: Initial conditions

Let $f(x) = u(x,0)$ and let $g(x) = u_t(x,0) = 0$. Since the boundary condition is homogeneous Neumann, then $f(x)$ is even extended to make it periodic with period 4. This is done so we can use d'Alembert solution which is valid for unbounded domain. Let $\tilde{f}(x)$ be the new periodic initial conditions as shown the in the following diagram.

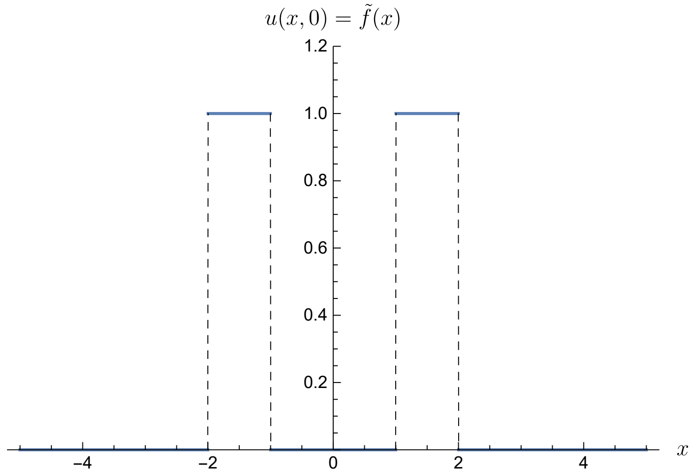

Figure 5: Initial conditions

With the new periodic initial conditions, we now can apply d'Alembert solution

$$\tilde{u}(x,t)=\frac{1}{2}\left(\tilde{f}(x-ct)+\tilde{f}(x+ct)\right)$$

Since $c=1$ then above becomes

$$\tilde{u}(x,t)=\frac{1}{2}\left(\tilde{f}(x-t)+\tilde{f}(x+t)\right)$$

We will use the solution from above only for $x>0$ since that is the physical domain.

The solution will starts by splitting each packet into 2 halves. One that move to the right and one that move to the left. When the half that moves to the left reach $x=0$, at that same time the half wave that was moving to the right from $x<0$ arrives. And they pass through each others. This appears as the wave half deflecting off $x=0$ turning around, remaining upright, and starts to move to the right behind the half that was moving to the right from the start. So we end up with 2 half waves moving to the right after that. This is sketch of what happens in time.

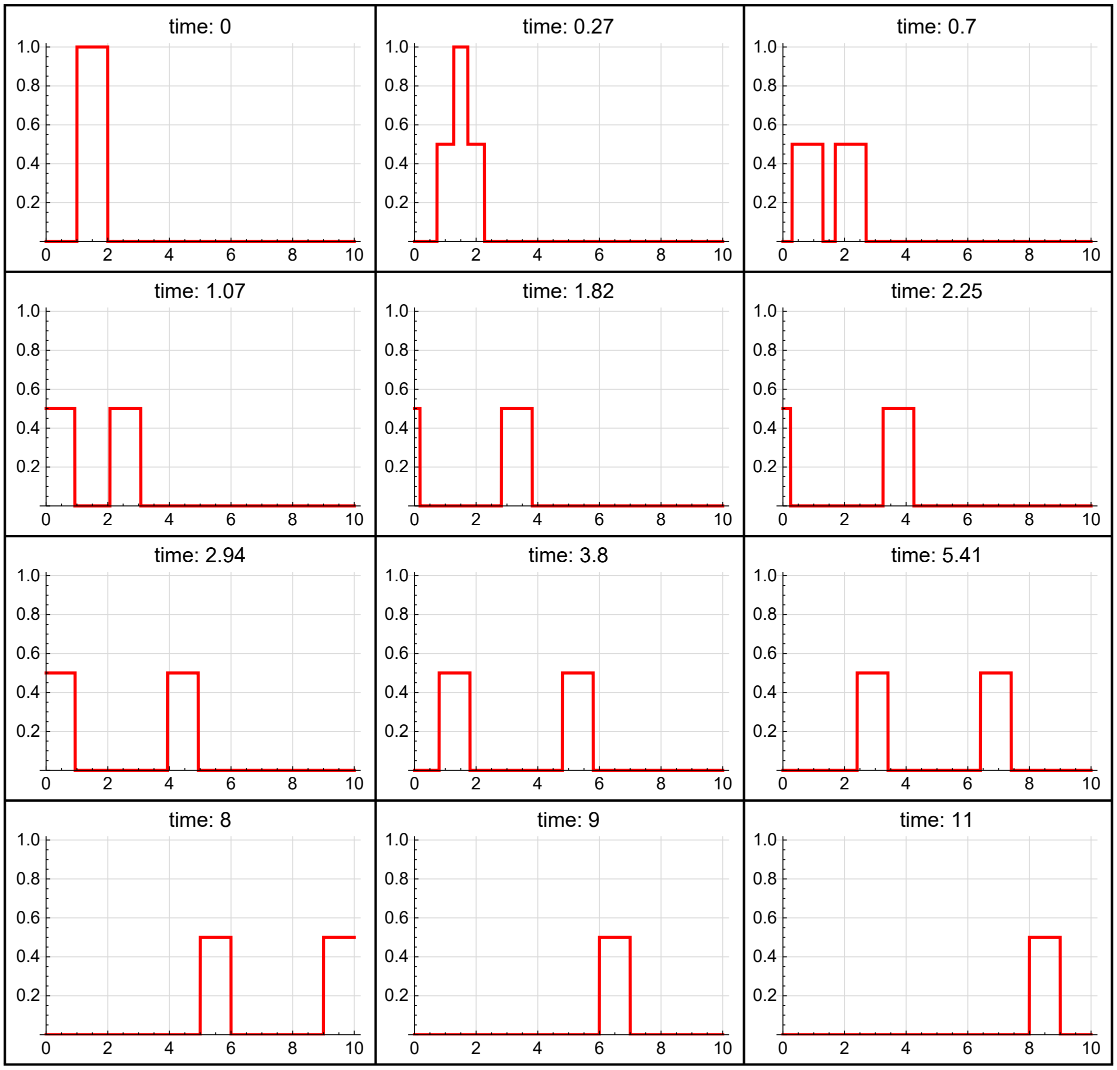

Figure 6: sketch of solution over time

```
pde = D[u[x, t], {t, 2}] == D[u[x, t], {x, 2}];
f[x_] := Piecewise[{{1, 1 < x < 2}, {0, True}}];
fbar[x_] := If[-3 < x < -1, fbar[x + 4], f[x]];
u[x_, t_] := 1/2 (fbar[x - t] + fbar[x + t]);
Table[Plot[u[x, t0], {x, 0, 10}, PlotRange → {Automatic, {0, 1.02}},
          GridLines → Automatic, GridLinesStyle → LightGray,
          PlotStyle → Red, PlotLabel → Row[{"time: ", t0}],
          PlotPoints → 40, Exclusions → None],
       {t0, {0, 0.27, 0.7, 1.07, 1.82, 2.25, 2.94, 3.8, 5.41, 8, 9, 11}}];
p = Grid[Partition[%, 3], Frame → All];
```

Figure 7: Code used for the above

9 Problem 4.2.22

Under what conditions is the solution to the Neumann boundary value problem for the wave equation on a bounded interval $[0,1]$ periodic in time? What is the period?

Solution

By even-extending the initial displacement and initial velocity to $[-1,1]$ and then repeating this again for the whole line $-\infty < x < \infty$, and then using the d'Alembert solution, then the resulting solution $u(x,t)$ will always be periodic since initial conditions are periodic. The period of the solution will $2L$ in x, where $L = 1$ here. Hence period is 2 in x.

10 Problem 4.2.25

Write down a formula for the solution $u(x,t)$ to the initial-boundary value problem $u_{tt} = 4u_{xx}$ with boundary conditions

$$u_x(0,t) = 0$$
$$u_x(\pi,t) = 0$$

And initial conditions

$$u(x,0) = \sin x$$
$$u_t(x,0) = 0$$

For $0 < x < \pi, t > 0$

Solution

Since boundary conditions are Neumann, then to use d'Alembert solution, we start by even extending both initial position $u(x,0) = \sin(x)$ and initial velocity (which is zero here) to be even over $[-\pi,\pi]$. Next we duplicate this over the whole line $-\infty < x < \infty$. Now we are able to use d'Alembert solution to solve the wave equation. The solution will be periodic with period 2π in x. Let $f(x) = \sin x$ and let $\tilde{f}(x)$ be its even periodic extension such that

$$\tilde{f}(-x) = f(x)$$
$$\tilde{f}(x+2\pi) = f(x)$$
$$\tilde{f}(x-2\pi) = f(x)$$

Hence the solution is

$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{f}(x-ct) + \tilde{f}(x+ct)\right)$$

But $c = 2$ therefore the above becomes

$$\tilde{u}(x,t) = \frac{1}{2}(\sin(x-2t) + \sin(x+ct))$$

The actual solution we want is over $[0,\pi]$ from the above since that is the physical domain of the original problem.