HW 3

Math 5587
Elementary Partial Differential Equations I

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Contents

1 Problem 3.1.2

Find all separable eigensolutions to the heat equation $u_t = u_{xx}$ on $0 \leq x \leq \pi$ subject to (a) homogeneous boundary conditions $u(t,0) = 0, u(t,\pi) = 0$. (b) mixed boundary conditions $u(t,0) = 0, u_x(t,\pi) = 0$

solution

Using separation of variables, let $u(t,x) = T(t)X(x)$. Substituting this into $u_t = u_{xx}$ gives $T'X = TX''$. Dividing by $XT \neq 0$ results in

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Where λ is the seperation constant. The above gives the following ODE's to solve

$$X''(x) + \lambda X(x) = 0$$
$$T'(t) + \lambda T(t) = 0$$

The boundary and initial conditions are transfered from the PDE to the ODE as shown below.

1.1 Part (a)

Using $u(t,0) = 0, u(t,\pi) = 0$. Starting with the spatial ODE, and transferring the boundary conditions to the ODE results in

$$X''(x) + \lambda X(x) = 0$$
$$X(0) = 0$$
$$X(\pi) = 0$$

This is an eigenvalue boundary value ODE. The solution to the spatial ODE is

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

case $\lambda < 0$

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is positive. The above solution becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

Which can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

The solution now reduces to $X(x) = c_2 \sinh\left(\sqrt{\mu}x\right)$. At $x = \pi$ this gives

$$0 = c_2 \sinh\left(\sqrt{\mu}\pi\right)$$

But $\sinh$ is only zero when its argument is zero. Since $\mu \neq 0$, then the only choice is that $c_2 = 0$ also. But this gives trivial solution therefore $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case the solution is $X(x) = c_1 + c_2 x$. At $x = 0$ this gives $0 = c_1$. The solution becomes $X(x) = c_2 x$. At $x = \pi$, this gives $0 = c_2 \pi$. Therefore $c_2 = 0$ also. This also gives the trivial solution. Hence $\lambda = 0$ is not an eigenvalue.

case $\lambda > 0$

The solution in this case is

$$
\begin{aligned}
X(x) &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\
&= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x}
\end{aligned}
$$

Which can be rewritten as (the constants c_1, c_2 below will be different than the above c_1, c_2, but kept the same name for simplicity).

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

The solution now reduces to

$$X(x) = c_2 \sin\left(\sqrt{\lambda}x\right)$$

At $x = \pi$ this gives

$$0 = c_2 \sin\left(\sqrt{\lambda}\pi\right)$$

non-trivial solution requires that $\sin\left(\sqrt{\lambda}\pi\right) = 0$ which implies that $\sqrt{\lambda}\pi = n\pi, n = 1, 2, 3, \cdots$. Hence eigenvalues are

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

And corresponding eigenfunctions are

$$\boxed{X_n(x) = \sin(nx) \qquad n = 1, 2, 3, \cdots}$$

Now that the eigenvalues and eigenfunction are found, the time ODE can be solved. The time ODE now becomes

$$T'(t) + n^2 T(t) = 0$$

This is linear first order ode. The solution is $T_n(t) = C_n e^{-n^2 t}$. Therefore the fundamental solution is

$$
\begin{aligned}
u_n(t, x) &= C_n T_n(t) X_n(x) \\
&= C_n e^{-n^2 t} \sin(nx)
\end{aligned}
$$

Since this is a linear PDE, a linear combination of all fundamental solutions is a solution. Hence the general solution is

$$u(t, x) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx)$$

The constant C_n can be found if initial conditions are given.

1.2 Part (b)

Using $u(t, 0) = 0, u_x(t, \pi) = 0$. Starting with the spatial ODE, and transferring the boundary condition to X, it becomes

$$
\begin{aligned}
X''(x) + \lambda X(x) &= 0 \\
X(0) &= 0 \\
X'(\pi) &= 0
\end{aligned}
$$

This is an eigenvalue boundary value problem. The solution to the spatial ODE is

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

<u>case $\lambda < 0$</u>

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is positive. The solution becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

The above can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

Hence the solution now becomes

$$X(x) = c_2 \sinh\left(\sqrt{\mu}x\right)$$

Taking derivative gives

$$X'(x) = c_2\sqrt{\mu} \cosh\left(\sqrt{\mu}x\right)$$

And at $x = \pi$ the above gives

$$0 = c_2\sqrt{\mu} \cosh\left(\sqrt{\mu}\pi\right)$$

But $\mu \neq 0$ and $\cosh$ is never zero for any argument. Hence the only choice is that $c_2 = 0$. This gives the trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case the solution is $X(x) = c_1 + c_2 x$. At $x = 0$ this results in $0 = c_1$. The solution becomes $X(x) = c_2 x$. Hence $X'(x) = c_2$. At $x = \pi$, this implies $0 = c_2\pi$. Therefore $c_2 = 0$ also. This gives the trivial solution. Hence $\lambda = 0$ is not an eigenvalue.

case $\lambda > 0$

The solution in this case is

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\ &= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} \end{aligned}$$

Which can be rewritten as (the constants c_1, c_2 below will be different than the above c_1, c_2, but kept the same name for simplicity).

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

At $x = 0$ this gives

$$0 = c_1$$

The solution now reduces to

$$X(x) = c_2 \sin\left(\sqrt{\lambda}x\right)$$

Therefore

$$X'(x) = \sqrt{\lambda}c_2 \cos\left(\sqrt{\lambda}x\right)$$

At $x = \pi$

$$0 = \sqrt{\lambda}c_2 \cos\left(\sqrt{\lambda}\pi\right)$$

Non-trivial solution requires that $\cos\left(\sqrt{\lambda}\pi\right) = 0$, which implies $\sqrt{\lambda}\pi = \frac{n\pi}{2}, n = 1, 3, 5, \cdots$. or $\sqrt{\lambda} = \frac{n}{2}, n = 1, 3, 5, \cdots$. Therefore the eigenvalues are

$$\lambda_n = \left(\frac{n}{2}\right)^2 \qquad n = 1, 3, 5, \cdots$$

Or

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 \qquad n = 1, 2, 3, \cdots$$

Few eigenvalues are $\lambda = \left\{\frac{1}{4}, \frac{9}{4}, \frac{25}{4}, \cdots\right\}$. The corresponding eigenfunctions are

$$\boxed{X_n(x) = \sin\left(\tfrac{2n-1}{2}x\right) \qquad n = 1,2,3,\cdots}$$

Now that the eigenvalues and eigenfunction are found, the time ODE is solved. The time ODE now becomes

$$T'(t) + \left(\frac{2n-1}{2}\right)^2 T(t) = 0$$

This is linear first order ode. The solution is $T_n(t) = C_n e^{-\left(\frac{2n-1}{2}\right)^2 t}$. Therefore the fundamental solution is

$$\begin{aligned} u_n(t,x) &= C_n T_n(t) X_n(x) \\ &= C_n e^{-\left(\frac{2n-1}{2}\right)^2 t} \sin\left(\frac{2n-1}{2}x\right) \end{aligned}$$

A linear combination of all fundamental solution is a solution (due to linearity). Hence the general solution is

$$u(t,x) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{2n-1}{2}\right)^2 t} \sin\left(\frac{2n-1}{2}x\right)$$

2 Problem 3.1.5

(a) Find the real eigensolutions to the damped heat equation $u_t = u_{xx} - u$. (b) Which solutions satisfy the periodic boundary conditions $u(t,-\pi) = u(t,\pi), u_x(t,-\pi) = u_x(t,\pi)$?

solution

2.1 Part (a)

Using separation of variables, Let $u(t,x) = T(t)X(x)$. Substituting this into $u_t + u = u_{xx}$ gives $T'X + TX = TX''$. Dividing by $XT \neq 0$ gives

$$\frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda$$

Where λ is the separation constant. This gives the following ODE's to solve

$$X''(x) + \lambda X(x) = 0$$
$$T'(t) + (\lambda + 1)T(t) = 0$$

Eigenfunctions are solutions to the spatial ODE.

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

To determine the actual eigenfunctions and eigenvalues, boundary conditions are used. This is part b below.

2.2 Part (b)

Using $u(t,-\pi) = u(t,\pi), u_x(t,-\pi) = u_x(t,\pi)$. Starting with the spatial ODE above, and transferring the boundary condition to X gives

$$X''(x) + \lambda X(x) = 0$$
$$X(-\pi) = X(\pi)$$
$$X'(-\pi) = X'(\pi)$$

This is an eigenvalue boundary value problem. The solution is

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

case $\lambda < 0$

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is now positive. The solution (1) becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

The above can be written as

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) + c_2 \sinh\left(\sqrt{\mu}x\right) \tag{2}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using (2) gives

$$c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_1 \cosh\left(\sqrt{\mu}\pi\right) + c_2 \sinh\left(\sqrt{\mu}\pi\right)$$
$$c_2 \sinh\left(-\sqrt{\mu}\pi\right) = c_2 \sinh\left(\sqrt{\mu}\pi\right)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_2 = 0$ as only possibility to satisfy the above equation. The solution (2) now reduces to

$$X(x) = c_1 \cosh\left(\sqrt{\mu}x\right) \tag{3}$$

Taking derivative

$$X'(x) = c_1\sqrt{\mu} \sinh\left(\sqrt{\mu}x\right) \tag{4}$$

Applying the second BC $X'(-\pi) = X'(\pi)$ gives

$$c_1\sqrt{\mu} \sinh\left(-\sqrt{\mu}\pi\right) = c_1\sqrt{\mu} \sinh\left(\sqrt{\mu}x\right)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_1 = 0$. This means a trivial solution. Therefore $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case the solution is $X(x) = c_1 + c_2 x$. Applying first BC $X(-\pi) = X(\pi)$ gives

$$\begin{aligned} c_1 - c_2\pi &= c_1 + c_2\pi \\ -c_2\pi &= c_2\pi \end{aligned}$$

This gives $c_2 = 0$. The solution now becomes

$$X(x) = c_1$$

Therefore $X'(x) = 0$. Applying the second boundary conditions $X'(-\pi) = X'(\pi)$ is now satisfied for any c_1, since it gives $(0 = 0)$. Therefore $\underline{\lambda = 0}$ is an eigenvalue with eigenfunction $X_0(0) = 1$ (selecting $c_1 = 1$ since any arbitrary constant will work).

case $\lambda > 0$

The solution in this case is

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\ &= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} \end{aligned}$$

Which can be rewritten as (the constants c_1, c_2 below will be different than the above c_1, c_2, but kept the same name for simplicity).

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \tag{5}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using the above gives

$$\begin{aligned} c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(-\sqrt{\lambda}\pi\right) &= c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\ c_2 \sin\left(-\sqrt{\lambda}\pi\right) &= c_2 \sin\left(\sqrt{\lambda}\pi\right) \end{aligned}$$

There are two choices here. Either $c_2 = 0$ or $\sqrt{\lambda}\pi = n\pi, n = 1, 2, 3, \cdots$. Using the second choice for now, which implies that

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

And now we will now look to see what happens using the second BC with the above choice. The solution (5) now becomes

$$X(x) = c_1 \cos(nx) + c_2 \sin(nx) \qquad n = 1, 2, 3, \cdots$$

Therefore

$$X'(x) = -c_1 n \sin(nx) + c_2 n \cos(nx)$$

Applying the second BC $X'(-\pi) = X'(\pi)$ using the above gives

$$\begin{aligned} c_1 n \sin(n\pi) + c_2 n \cos(n\pi) &= -c_1 n \sin(n\pi) + c_2 n \cos(n\pi) \\ c_1 n \sin(n\pi) &= -c_1 n \sin(n\pi) \\ 0 &= 0 \end{aligned}$$

Since n is integer.

Therefore this means that using the choice $\lambda_n = n^2$ satisfied both boundary conditions with $c_2 \neq 0, c_1 \neq 0$. This means the solution (5) is

$$X_n(x) = A_n \cos(nx) + B_n \sin(nx) \qquad n = 1, 2, 3, \cdots$$

The above says that there are two eigenfunctions in this case. They are

$$X_n(x) = \begin{cases} \cos(nx) \\ \sin(nx) \end{cases}$$

Recalling that there is also a zero eigenvalue with constant as its eigenfunction, then the complete set of eigenfunctions is

$$X_n(x) = \begin{cases} 1 \\ \cos(nx) \\ \sin(nx) \end{cases}$$

Now that the eigenvalues are found, the solution to the time ODE can be found. The time ODE from above was found to be

$$T'(t) + (\lambda + 1) T(t) = 0$$

For the zero eigenvalue case, the above reduces to $T'(t) + T(t) = 0$ which has the solution $T_0(t) = C_0 e^{-t}$. For non zero eigenvalues $\lambda_n = n^2$, the ODE becomes $T'(t) + \left(n^2 + 1\right) T(t) = 0$, whose solution is $T_0(t) = C_n e^{-\left(n^2+1\right)t}$.

Putting all the above together, gives the fundamental solution as

$$u_n(t,x) = \begin{cases} C_0 e^{-t} & \\ C_n \cos(nx)\, e^{-\left(n^2+1\right)t} & n = 1,2,3,\cdots \\ B_n \sin(nx)\, e^{-\left(n^2+1\right)t} & n = 1,2,3,\cdots \end{cases}$$

The complete solution is the sum of the above solutions

$$u(t,x) = C_0 e^{-t} + \sum_{n=1}^{\infty} e^{-\left(n^2+1\right)t} \left(C_n \cos(nx) + B_n \sin(nx)\right)$$

The constants C_0, C_n, B_n can be found from initial conditions.

3 Problem 3.2.1

(d) Find the Fourier series of the following functions $f(x) = x^2$ (using $-\pi \leq x \leq \pi$)

solution

The Fourier series is given by

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + a_n \sin\left(\frac{2\pi}{T} nx\right)$$

Where T is the period of $f(x)$. Taking this period to be 2π, the above simplifies to

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

The function x^2 is even, hence all b_n are zero. The above becomes

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{1}$$

But

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 dx \\ &= \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi} \\ &= \frac{2}{3\pi} \pi^3 \\ &= \frac{2}{3} \pi^2 \end{aligned}$$

And

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx)\, dx \end{aligned} \tag{1A}$$

Let $I = \int_0^{\pi} x^2 \cos(nx)\, dx$. Using integration by parts $\int u dv = uv - \int v du$. Let $u = x^2, dv = \cos(nx)$. Then $du = 2x, v = \frac{\sin(nx)}{n}$. Hence

$$\begin{aligned} I &= \left[x^2 \frac{\sin(nx)}{n}\right]_0^{\pi} - 2\int_0^{\pi} x \frac{\sin(nx)}{n} dx \\ &= \overbrace{\frac{1}{n}\left[x^2 \sin(nx)\right]_0^{\pi}}^{0} - \frac{2}{n} \int_0^{\pi} x \sin(nx)\, dx \\ &= -\frac{2}{n} \int_0^{\pi} x \sin(nx)\, dx \end{aligned}$$

Integration by parts again. $u = x, dv = \sin(nx)$, then $du = 1, v = -\frac{\cos(nx)}{n}$. The above becomes

$$
\begin{aligned}
I &= -\frac{2}{n}\left(\left[-x\frac{\cos(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos(nx)}{n}dx\right)\\
&= -\frac{2}{n}\left(-\frac{1}{n}\left[x\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right)\\
&= \frac{2}{n^2}\left(\left[x\cos(nx)\right]_0^{\pi} - \int_0^{\pi}\cos(nx)\,dx\right)\\
&= \frac{2}{n^2}\left(\left[\pi\cos(n\pi)\right] - \left[\frac{\sin(nx)}{n}\right]_0^{\pi}\right)\\
&= \frac{2\pi}{n^2}\cos(n\pi)\\
&= \frac{2\pi}{n^2}(-1)^n
\end{aligned}
$$

The above is I. Substituting this result back in (1A) gives

$$
\begin{aligned}
a_n &= \frac{2}{\pi}I\\
&= \frac{2}{\pi}\frac{2\pi}{n^2}(-1)^n\\
&= \frac{4}{n^2}(-1)^n
\end{aligned}
$$

Therefore (1) becomes

$$
x^2 \sim \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)
$$

To verify this result, the Fourier series was compared to x^2 for an increasing number of terms to see if it converged to x^2. Here is the result. This shows the convergence is fast, after 6 terms only the approximation (in red color) is almost the same as the original function x^2.

Out[ ]=

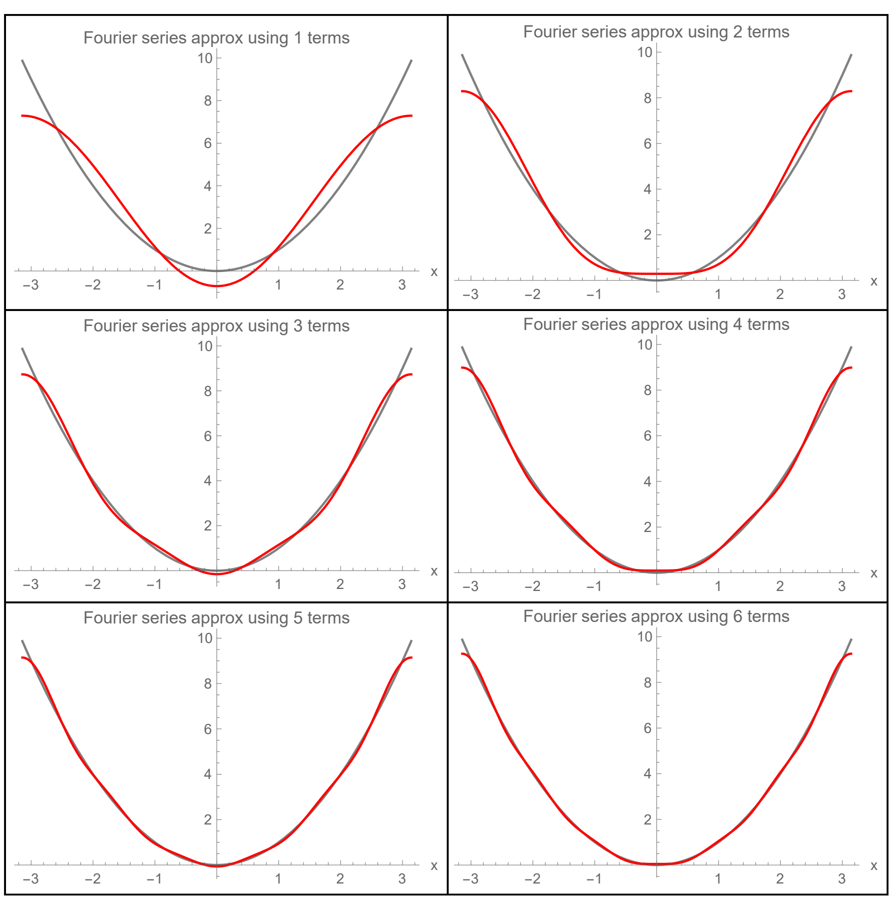

Figure 1: Fourier series of x^2

```
fs[x_, max_] := 1/3 π^2 + 4 Sum[(-1)^n/n^2 Cos[n x], {n, 1, max}]
makePlot[n_] := Plot[{x^2, fs[x, n]}, {x, -Pi, Pi},
         PlotStyle → {Gray, Red}, AxesLabel → {"x", None},
         PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
         ImageSize → 300
       ];
Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6}}], 2],
   Frame → All]
```

Figure 2: Code used for the above plot

the following plot shows how the Fourier series approximation to x^2 when it is periodically extended to outside $[-\pi, \pi]$. This uses the range $[-3\pi, 3\pi]$ by adding one period to left and one period to the right.

```
In[ ]:= fs[x_, max_] := 1/3 π^2 + 4 Sum[(-1)^n/n^2 Cos[n x], {n, 1, max}]
        fx[x_] := Piecewise[{
                  {(x + 2 Pi)^2, x < -Pi},
                  {x^2, -Pi < x < Pi},
                  {(x - 2 Pi)^2, x > Pi}}];
        makePlot[n_] := Plot[{fx[x], fs[x, n]}, {x, -3 Pi, 3 Pi},
                PlotStyle → {Gray, Red}, AxesLabel → {"x", None},
                PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
                ImageSize → 300
             ];
        Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6}}], 2],
           Frame → All]
```

Figure 3: Code used for the above plot

4 Problem 3.2.2

(d) Find the Fourier series of the following function $f(x) = \begin{cases} x & |x| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

<u>solution</u>

This is plot showing $f(x)$

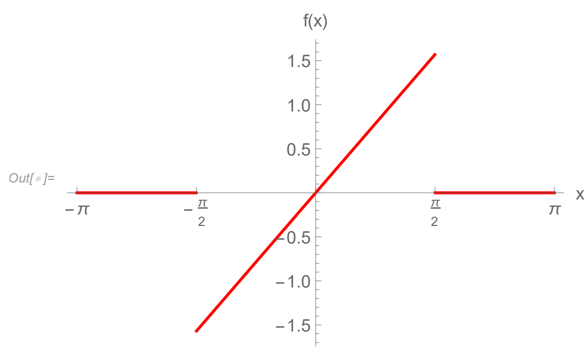

Figure 4: Plot of $f(x)$

The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + a_n \sin\left(\frac{2\pi}{T} nx\right)$$

Where T is the period of the function to be approximated. Taking this period to be 2π, the above simplifies to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

The function $f(x)$ is odd then all a_n will zero. The above simplifies to

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \\ &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin(nx)\, dx \end{aligned}$$

But x is odd and $\sin(x)$ is odd, hence the product is even. The above simplifies to

$$b_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} x \sin(nx)\, dx$$

Using integration by parts $\int u dv = uv - \int v du$. Let $x = u, du = 1, dv = \sin(nx), v = \frac{-\cos(nx)}{n}$, the above gives

$$\begin{aligned} b_n &= \frac{2}{\pi}\left(\frac{-1}{n}[x\cos(nx)]_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}} \cos(nx)\, dx\right) \\ &= \frac{2}{\pi n}\left(-[x\cos(nx)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(nx)\, dx\right) \\ &= \frac{2}{\pi n}\left(-\left[\frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right)\right] + \frac{1}{n}[\sin(nx)]_0^{\frac{\pi}{2}}\right) \\ &= \frac{2}{\pi n}\left(-\left[\frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right)\right] + \frac{1}{n}\left[\sin\left(n\frac{\pi}{2}\right)\right]\right) \\ &= \frac{2}{\pi n^2}\left(\sin\left(n\frac{\pi}{2}\right) - \frac{n\pi}{2}\cos\left(n\frac{\pi}{2}\right)\right) \end{aligned}$$

Therefore the Fourier series becomes

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(\sin\left(\frac{n\pi}{2}\right) - \frac{1}{2} n\pi \cos\left(\frac{n\pi}{2}\right)\right) \sin(nx)$$

To verify this result, the Fourier series was compared to $f(x)$ for increasing number of terms to see if it converges to x^2. Here is the result. This shows the convergence is fast, but not as fast as last problem due to jump discontinuity in $f(x)$. 10 terms are used below.

Out[∘]=

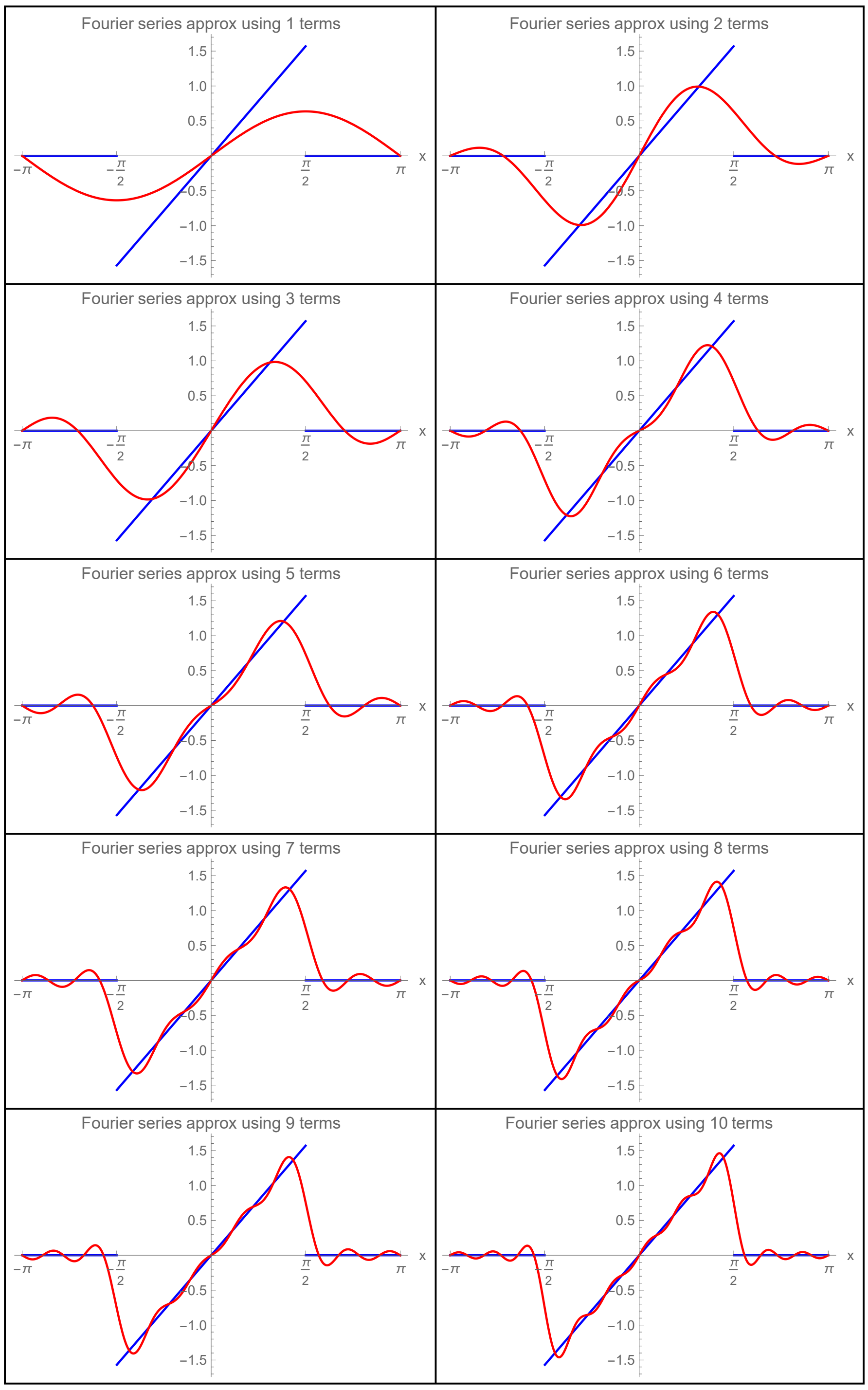

Figure 5: Fourier series approximation of $f(x)$

```
In[∘]:= fs[x_, max_] := Sum[2/(n^2 π) (Sin[n π/2] - 1/2 n π Cos[n π/2]) Sin[n x], {n, 1, max}];
f[x_] := Piecewise[{{x, Abs[x] < Pi/2}, {0, True}}];
makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -Pi, Pi},
        PlotStyle → {Blue, Red}, AxesLabel → {"x", None},
        PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
        ImageSize → 300,
        Ticks → {Range[-Pi, Pi, Pi/2], Automatic}
    ];
Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}}], 2],
  Frame → All]
```

Figure 6: Code used for the above plot

the following plot shows how the Fourier series approximate $f(x)$ when it is periodically extended to outside $[-\pi, \pi]$. This uses the range $[-3\pi, 3\pi]$ by adding one more period to left and to the right.

Out[ ]=

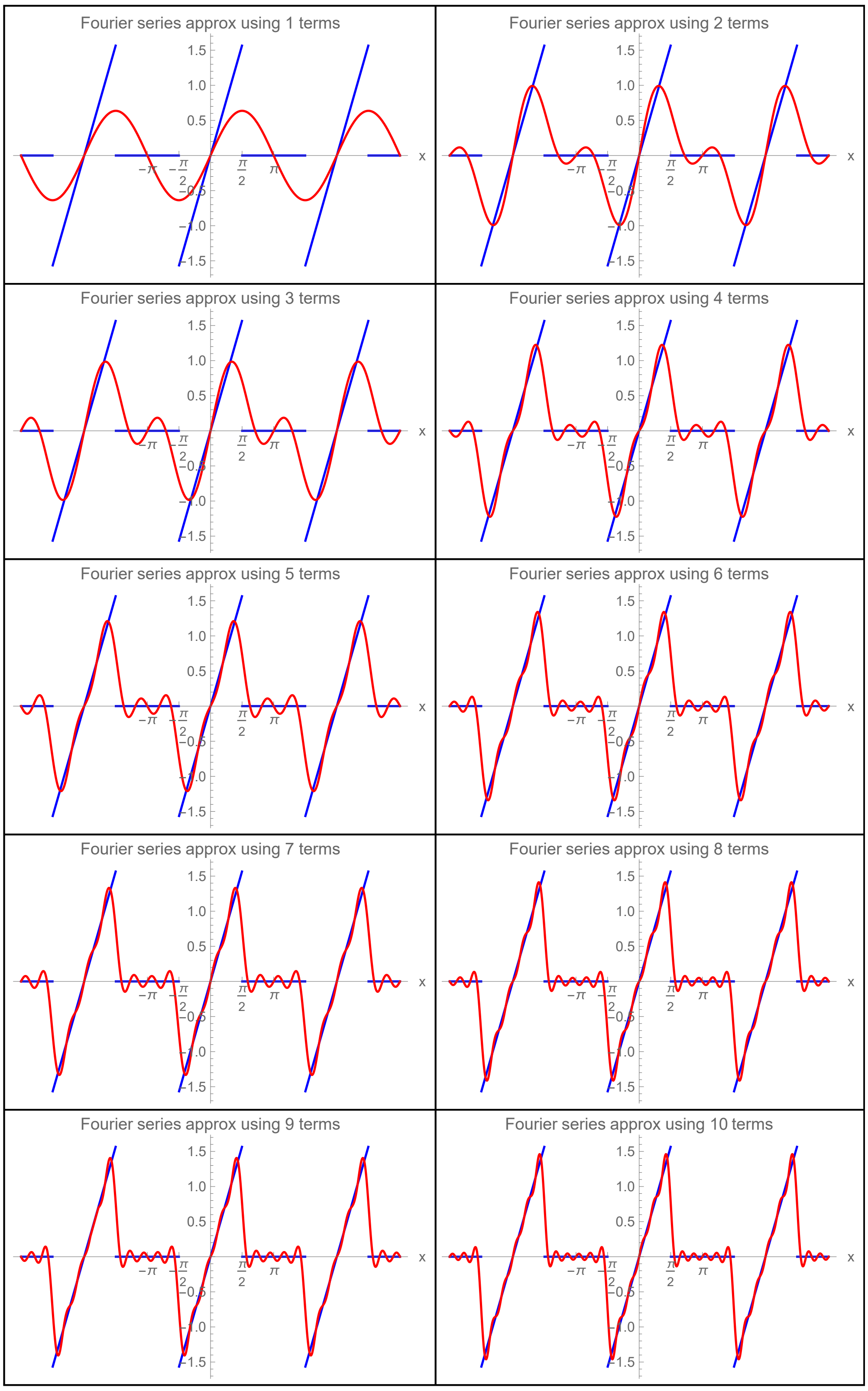

Figure 7: Fourier series of periodic extension $f(x)$

```
In[•]:= fs[x_, max_] := Sum[2/(n^2 π) (Sin[n π/2] - 1/2 n π Cos[n π/2]) Sin[n x], {n, 1, max}];
f[x_] := Piecewise[{
            {0, x < -5 / 2 Pi},
            {x + 2 Pi, -5 / 2 Pi < x < -3 / 2 Pi},
            {0, -3 / 2 Pi < x < -Pi / 2},
            {x, -Pi / 2 < x < Pi / 2},
            {0, Pi / 2 < x < 3 / 2 Pi},
            {x - 2 Pi, 2 / 3 Pi < x < 5 / 2 Pi},
            {0, 5 / 2 Pi < x < 3 Pi}}];
makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -3 Pi, 3 Pi},
         PlotStyle → { Blue, Red}, AxesLabel → {"x", None},
         PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
         ImageSize → 300,
         Ticks → {Range[-Pi, Pi, Pi / 2], Automatic}
      ];
Grid[Partition[Table[makePlot[n], {n, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}}], 2],
   Frame → All]
```

Figure 8: Code used for the above plot

5 Problem 3.2.3

Find the Fourier series of $\sin^2 x$ and $\cos^2 x$ without directly calculating the Fourier coefficients.

solution

Using the known trig identity

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) \tag{1}$$

And comparing the the above to the Fourier series expansion

$$\sin^2 x = \frac{a_0}{2} + (a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + \cdots) + (b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \cdots) \tag{A}$$

Shows that $\frac{a_0}{2} = \frac{1}{2}$ and $a_2 = \frac{-1}{2}$ and all other terms are zero. Because the Fourier series is unique for a function, then (1) is the Fourier series for $\sin^2 x$.

Similarly, Using the known trig identity

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) \tag{2}$$

And comparing the the above to the Fourier series expansion (A), shows that $\frac{a_0}{2} = \frac{1}{2}$ and $a_2 = \frac{1}{2}$ and all other terms are zero. Therefore (2) is the Fourier series expansion for $\cos^2 x$.

6 Problem 3.2.6

Graph the 2π periodic extension of each of the following functions (h) $f(x) = \frac{1}{x}$. Which extension are continuous? Differentiable?

solution

6.1 Part (h)

The original function $f(x) = \frac{1}{x}$ is always taken from $-\pi \leq x \leq \pi$ (before extending it periodically). At $x = 0$ the function is not defined.

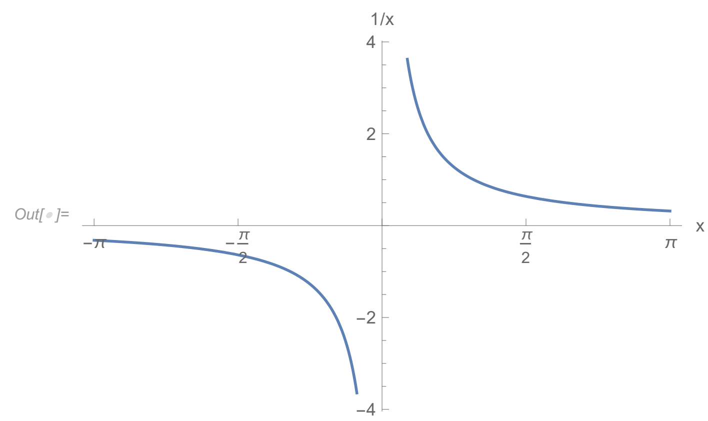

Figure 9: Plot of $f(x) = \frac{1}{x}$

Periodically extending it, it becomes (showing one extra period to the left and right) then following

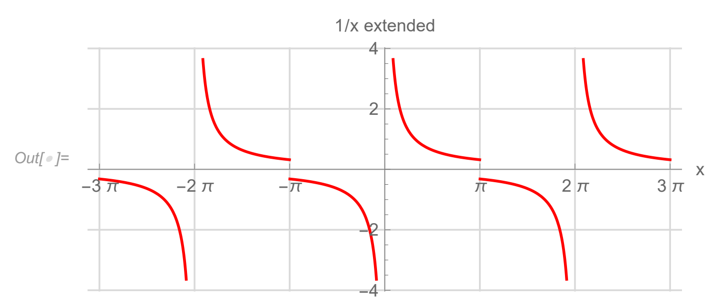

Figure 10: Plot of periodic extension of $f(x) = \frac{1}{x}$

```
In[ ]:= f[x_] := Piecewise[{
            {1 / (x + 2 Pi), x < -Pi},
            {1 / x, -Pi < x < Pi},
            {1 / (x - 2 Pi), Pi < x}
         }];
     Plot[f[x], {x, -3 Pi, 3 Pi}, Ticks → {Range[-3 Pi, 3 Pi, Pi], Automatic},
       AxesLabel → {"x", "1/x extended"},
       GridLines → {Range[-3 Pi, 3 Pi, Pi], Automatic},
       GridLinesStyle → LightGray, PlotStyle → Red, AspectRatio → Automatic]
```

Figure 11: Code for the above plot

Looking at the above plot shows the extension is not continuous and also not Differentiable due to jump discontinuities.

7 Problem 3.2.9

Suppose that $f(x)$ is periodic with period T (using T instead of l as in book as it is more clear). Prove that for any a (a) $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$. (b) $\int_0^T f(x+a)\,dx = \int_0^T f(x)\,dx$

solution

7.1 Part (a)

$$\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx = \overbrace{\left(\int_a^T f(x)\,dx + \int_T^{a+T} f(x)\,dx\right)}^{\int_a^{a+T} f(x)dx} - \overbrace{\left(\int_0^a f(x)\,dx + \int_a^T f(x)\,dx\right)}^{\int_0^T f(x)dx}$$

Simplifying the RHS above gives

$$\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx = \int_T^{a+T} f(x)\,dx - \int_0^a f(x)\,dx \tag{1}$$

But

$$\int_T^{a+T} f(x)\,dx = \int_0^a f(x+T)\,dx \tag{2}$$

To show how Eq(2) was derived: Let $u = x - T$. Then $du = dx$. When $x = T$ then $u = 0$. When $x = a + T$ then $u = a$. Hence $\int_T^{a+T} f(x)\,dx = \int_0^a f(u+T)\,du$. But u is arbitrary integral variable. Renaming it back to x gives that $\int_T^{a+T} f(x)\,dx = \int_0^a f(x+T)\,dx$.

Now, substituting (2) back into RHS of (1) gives

$$\begin{aligned}
\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx &= \int_0^a f(x+T)\,dx - \int_0^a f(x)\,dx \\
&= \int_0^a f(x+T) - f(x)\,dx
\end{aligned}$$

But since $f(x)$ is periodic, then $f(x+T) = f(x)$. Therefore the RHS above is zero.

$$\begin{aligned}
\int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx &= 0 \\
\int_a^{a+T} f(x)\,dx &= \int_0^T f(x)\,dx
\end{aligned}$$

Which is what the problem is asking to show.

7.2 Part (b)

Starting by rewriting $\int_0^T f(x+a)\,dx$ as the following. Let $u = x + a$. Hence $du = dx$. When $x = 0, u = a$ and when $x = T, u = a + T$. The integral becomes $\int_a^{a+T} f(u)\,du$. But now u is arbitrary integration variable. Renaming is back to x then we obtain that

$$\int_0^T f(x+a)\,dx = \int_a^{a+T} f(x)\,dx \tag{1}$$

Now, to show that main result, considering

$$\int_0^T f(x+a)\,dx - \int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx - \int_0^T f(x)\,dx$$

Where in the above, (1) was used to obtain RHS. The above can now be written as

$$\int_0^T f(x+a)\,dx - \int_0^T f(x)\,dx = \overbrace{\left(\int_a^T f(x)\,dx + \int_T^{T+a} f(x)\,dx\right)}^{\int_a^{a+T} f(x)dx} - \int_0^T f(x)\,dx$$

But $\int_T^{T+a} f(x)\,dx = \int_0^a f(x)\,dx$ since $f(x)$ is periodic with period T. The above now becomes

$$\begin{aligned}
\int_0^T f(x+a)\,dx - \int_0^T f(x)\,dx &= \left(\int_a^T f(x)\,dx + \int_0^a f(x)\,dx\right) - \int_0^T f(x)\,dx \\
&= \int_0^T f(x)\,dx - \int_0^T f(x)\,dx \\
&= 0
\end{aligned}$$

Therefore $\int_0^T f(x+a)\,dx = \int_0^T f(x)\,dx$ which is what the problem is asking to show.

8 Problem 3.2.25

(a) Sketch the 2π periodic half-wave $f(x) = \begin{cases} \sin x & 0 < x \leq \pi \\ 0 & -\pi \leq x < 0 \end{cases}$. (b) Find its Fourier series. (c) Graph the first five Fourier sums and compare the function. (d) Discuss convergence of the Fourier series.

solution

8.1 Part (a)

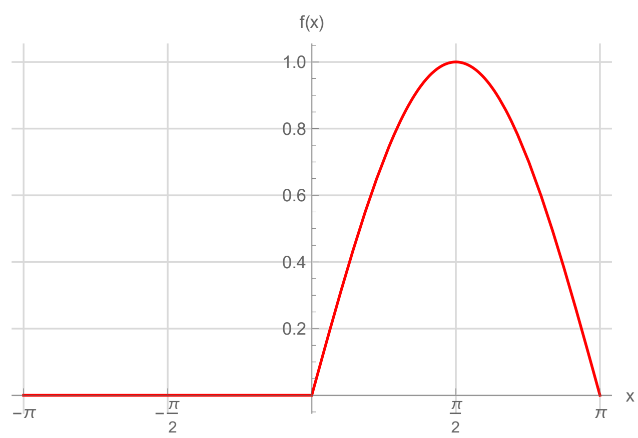

Figure 12: Plot of $f(x)$

```
f[x_] := Piecewise[{{Sin[x], 0 < x ≤ Pi}, {0, -Pi ≤ x < 0}}];
Plot[f[x], {x, -Pi, Pi}, Ticks → {Range[-Pi, Pi, Pi/2], Automatic},
  AxesLabel → {"x", "f(x)"},
  GridLines → {Range[-Pi, Pi, Pi/2], Automatic},
  GridLinesStyle → LightGray, PlotStyle → Red]
```

Figure 13: Code for the above plot

8.2 Part (b)

The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + a_n \sin\left(\frac{2\pi}{T} nx\right)$$

Where T is the period of the function to be approximated. Taking this period to be 2π, the above simplifies to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Hence

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\,dx\\
&= \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi}\\
&= \frac{-1}{\pi}\left[\cos(x)\right]_0^{\pi}\\
&= \frac{-1}{\pi}\left[\cos(\pi)-1\right]\\
&= \frac{-1}{\pi}\left[-1-1\right]\\
&= \frac{2}{\pi}
\end{aligned}
$$

And

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx\\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(nx)\,dx
\end{aligned}
$$

For $n = 1$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(x)\,dx\\
&= 0
\end{aligned}
$$

And for $n > 1$

$$
a_n = \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(nx)\,dx
$$

Using $\sin A\cos B = \frac{1}{2}\left(\sin(A-B)+\sin(A+b)\right)$, then $\sin(x)\cos(nx) = \frac{1}{2}\left(\sin(x-nx)+\sin(x+nx)\right)$. The above becomes

$$
\begin{aligned}
a_n &= \frac{1}{2\pi}\int_{0}^{\pi} \sin(x-nx)+\sin(x+nx)\,dx\\
&= \frac{1}{2\pi}\left(\int_{0}^{\pi}\sin(x-nx)\,dx + \int_{0}^{\pi}\sin(x+nx)\,dx\right)\\
&= \frac{1}{2\pi}\left(-\frac{1}{1-n}\left[\cos(x-nx)\right]_0^{\pi} - \frac{1}{1+n}\left[\cos(x+nx)\right]_0^{\pi}\right)\\
&= \frac{-1}{2\pi}\left(\frac{1}{1-n}\left[\cos(\pi-n\pi)-1\right] + \frac{1}{1+n}\left[\cos(\pi-n\pi)-1\right]\right)
\end{aligned}
$$

But $\cos(\pi-n\pi) = -\cos(n\pi)$. The above becomes

$$
\begin{aligned}
a_n &= \frac{-1}{2\pi}\left(\frac{1}{1-n}\left[-\cos(n\pi)-1\right] + \frac{1}{1+n}\left[-\cos(n\pi)-1\right]\right)\\
&= \frac{1}{2\pi}\left(\frac{\cos(n\pi)+1}{1-n} + \frac{\cos(n\pi)+1}{1+n}\right)\\
&= \frac{1}{2\pi}\left(\frac{(1+n)(\cos(n\pi)+1)+(1-n)(\cos(n\pi)+1)}{(1-n)(1+n)}\right)\\
&= \frac{1}{2\pi}\left(\frac{(1+n)(\cos(n\pi)+1)+(1-n)(\cos(n\pi)+1)}{\left(1-n^2\right)}\right)\\
&= \frac{1}{2\pi\left(1-n^2\right)}\left((1+n)(\cos(n\pi)+1)+(1-n)(\cos(n\pi)+1)\right)\\
&= \frac{1}{2\pi\left(1-n^2\right)}\left(2\cos(\pi n)+2\right)\\
&= \frac{1}{\pi\left(1-n^2\right)}\left(\cos(\pi n)+1\right)\\
&= \frac{1+(-1)^n}{\pi\left(1-n^2\right)}
\end{aligned}
$$

For odd $n=3,5,\cdots$ then $a_n=0$. For even n the above can be written as

$$a_n=\frac{2}{\pi\left(1-n^2\right)}\qquad n=2,4,6,\cdots$$

Now b_n is found

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\\
&= \frac{1}{\pi}\int_{0}^{\pi}\sin(x)\sin(nx)\,dx
\end{aligned}$$

Consider case $n=1$ first. The above gives

$$\begin{aligned}
b_1 &= \frac{1}{\pi}\int_0^{\pi}\sin^2(x)\,dx\\
&= \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}-\frac{1}{2}\cos(2x)\,dx\\
&= \frac{1}{\pi}\left(\int_0^{\pi}\frac{1}{2}dx-\frac{1}{2}\int_0^{\pi}\cos(2x)\,dx\right)\\
&= \frac{1}{\pi}\left(\frac{1}{2}\pi-\frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^{\pi}\right)\\
&= \frac{1}{2}
\end{aligned}$$

For $n>1$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi}\sin x\sin(nx)\,dx\\
&= \frac{1}{\pi}\frac{\sin(n\pi)}{n^2-1}\\
&= 0
\end{aligned}$$

Therefore the Fourier series is

$$\begin{aligned}
f(x) &\sim \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+b_n\sin(nx)\\
&= \frac{1}{\pi}+\frac{1}{2}\sin(x)+\frac{2}{\pi}\sum_{n=2,4,6,\cdots}^{\infty}\frac{1}{1-n^2}\cos(nx)\\
&= \frac{1}{\pi}+\frac{1}{2}\sin(x)+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{1-(2n)^2}\cos(2nx)
\end{aligned}$$

8.3 Part (c)

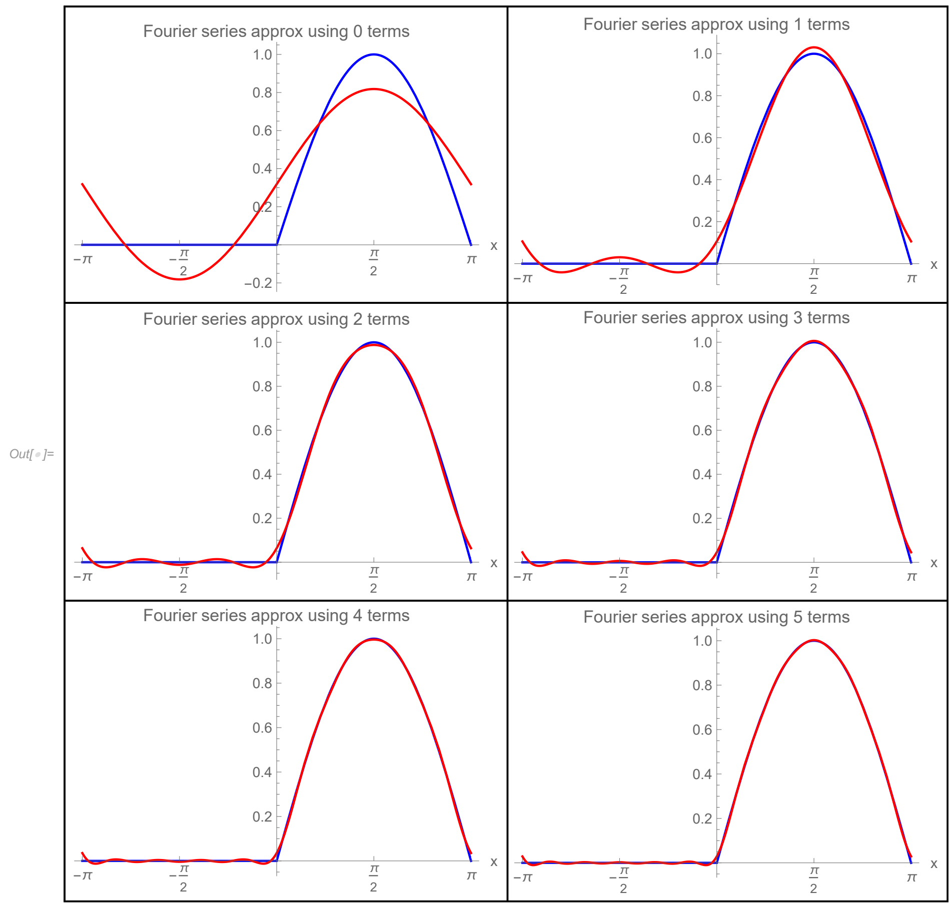

Figure 14: Plot of Fourier series approximation and $f(x)$

```
In[ ]:= fs[x_, max_] := 1/π + 1/2 Sin[x] + 2/π Sum[1/(1 - (2 n)^2) Cos[2 n x], {n, 1, max}];
f[x_] := Piecewise[{{Sin[x], 0 < x ≤ Pi}, {0, -Pi ≤ x < 0}}];
makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -Pi, Pi},
          PlotStyle → { Blue, Red}, AxesLabel → {"x", None},
          PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
          ImageSize → 300,
          Ticks → {Range[-Pi, Pi, Pi / 2], Automatic}
        ];
Grid[Partition[Table[makePlot[n], {n, 0, 5}], 2],
   Frame → All]
```

Figure 15: Code for the above plot

8.4 Part (d)

The function $f(x)$ is piecewise C^1 continuous over $-\pi \leq x \leq \pi$. Therefore the 2π periodic extension is also piecewise C^1 continuous over all x. There are no jump discontinues (only corner points). The Fourier series converges to $f(x)$ at each $x \in \Re$. (If there was a jump discontinuity at some x, then the Fourier series will converge to the average of $f(x)$ at that x, but this is not the case here).

9 Problem 3.2.27

(a) Find the Fourier series of $f(x) = e^x$. (b) For which values of x does the Fourier series converges? Is the convergence uniform? (c) Graph the function it converges to.

solution

9.1 Part (a)

For generality, the Fourier series for e^{ax} is found, then at the end a is set to be one. It is assumed the period is 2π.

$$e^{ax} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right)$$

But $T = 2\pi$ and the above becomes

$$e^{ax} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx \\
&= \frac{1}{\pi} \left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi a} \left(e^{a\pi} - e^{-a\pi}\right)
\end{aligned}$$

But $\frac{e^{a\pi} - e^{-a\pi}}{2} = \sinh(a\pi)$ hence the above simplifies to

$$a_0 = \frac{2}{\pi a} \sinh(a\pi)$$

And for $n > 0$

$$\begin{aligned}
a_n &= \frac{1}{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left(\frac{2\pi}{T} nx\right) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos(nx)\, dx \qquad (1)
\end{aligned}$$

Let $I = \int_{-\pi}^{\pi} e^{ax} \cos(nx)\, dx$. Using integration by parts, $\int u dv = uv - \int v du$. Let $u = \cos nx, dv = e^{ax}$ then $v = \frac{e^{ax}}{a}, du = -n \sin(nx)$. Hence

$$\begin{aligned}
I &= uv - \int v du \\
&= \left[\cos(nx) \frac{e^{ax}}{a}\right]_{-\pi}^{\pi} + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\, dx \\
&= \left[\cos(n\pi) \frac{e^{a\pi}}{a} - \cos(n\pi) \frac{e^{-a\pi}}{a}\right] + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\, dx \\
&= (-1)^n \left[\frac{e^{a\pi} - e^{-a\pi}}{a}\right] + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\, dx \\
&= \frac{2(-1)^n}{a} \left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right] + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\, dx \\
&= \frac{2(-1)^n}{a} \sinh(a\pi) + \frac{n}{a} \int_{-\pi}^{\pi} e^{ax} \sin(nx)\, dx
\end{aligned}$$

Applying integration by parts again on the integral above. Let $u=\sin nx, dv=e^{ax}$ then $v=\frac{e^{ax}}{a}, du=n\cos(nx)$ and the above becomes

$$\begin{aligned}
I &= \frac{2(-1)^n}{a}\sinh(a\pi)+\frac{n}{a}\left(\left(\sin nx\frac{e^{ax}}{a}\right)_{-\pi}^{\pi}-\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx\right)\\
&= \frac{2(-1)^n}{a}\sinh(a\pi)+\frac{n}{a}\left(\frac{1}{a}\overbrace{(\sin(n\pi)e^{a\pi}+\sin(n\pi)e^{-a\pi})}^{0}-\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx\right)\\
&= \frac{2(-1)^n}{a}\sinh(a\pi)-\frac{n^2}{a^2}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx
\end{aligned}$$

But $\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx=I$, the original integral we are solving for. Hence solving for I from the above gives gives

$$\begin{aligned}
I &= \frac{2(-1)^n}{a}\sinh(a\pi)-\frac{n^2}{a^2}I\\
I+\frac{n^2}{a^2}I &= \frac{2(-1)^n}{a}\sinh(a\pi)\\
I\left(1+\frac{n^2}{a^2}\right) &= \frac{2(-1)^n}{a}\sinh(a\pi)\\
I &= \frac{\frac{2(-1)^n}{a}\sinh(a\pi)}{1+\frac{n^2}{a^2}}\\
&= \frac{2a(-1)^n\sinh(a\pi)}{a^2+n^2} \qquad (2)
\end{aligned}$$

Using (2) in (1) gives

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx\\
&= \frac{a}{\pi}\frac{2(-1)^n\sinh(a\pi)}{a^2+n^2} \qquad (3)
\end{aligned}$$

Now we will do the same to find b_n

$$\begin{aligned}
b_n &= \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\sin\left(\frac{2\pi}{T}nx\right)dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx \qquad (4)
\end{aligned}$$

Let $I=\int_{-\pi}^{\pi}e^{ax}\sin(nx)\,dx$. Using integration by parts, $\int udv=uv-\int vdu$. Let $u=\sin(nx), dv=e^{ax}$ then $v=\frac{e^{ax}}{a}, du=n\cos(nx)$. Hence

$$\begin{aligned}
I &= uv-\int vdu\\
&= \left[\sin(nx)\frac{e^{ax}}{a}\right]_{-\pi}^{\pi}-\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx\\
&= \overbrace{\left[\sin(n\pi)\frac{e^{a\pi}}{a}-\sin(n\pi)\frac{e^{-a\pi}}{a}\right]}^{0}-\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx\\
&= -\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos(nx)\,dx
\end{aligned}$$

Now we apply integration by parts again on the integral above. Let $u = \cos nx, dv = e^{ax}$ then $v = \frac{e^{ax}}{a}, du = -n\sin(nx)$ and the above becomes

$$
\begin{aligned}
I &= -\frac{n}{a}\left(\left(\cos(nx)\frac{e^{ax}}{a}\right)_{-\pi}^{\pi} + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{1}{a}\left(\cos(n\pi)e^{a\pi} - \cos(n\pi)e^{-a\pi}\right) + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{1}{a}\cos(n\pi)\left(e^{a\pi} - e^{-a\pi}\right) + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{2}{a}\cos(n\pi)\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right) + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{n}{a}\left(\frac{2}{a}\cos(n\pi)\sinh(a\pi) + \frac{n}{a}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx\right)\\
&= -\frac{2n}{a^2}(-1)^n\sinh(a\pi) - \frac{n^2}{a^2}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx
\end{aligned}
$$

But $\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx = I$. Hence solving for I gives

$$
\begin{aligned}
I &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi) - \frac{n^2}{a^2}I\\
I + \frac{n^2}{a^2}I &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)\\
I\left(1 + \frac{n^2}{a^2}\right) &= -\frac{2n}{a^2}(-1)^n\sinh(a\pi)\\
I &= -\frac{\frac{2n}{a^2}(-1)^n\sinh(a\pi)}{1 + \frac{n^2}{a^2}}\\
I &= -\frac{2n(-1)^n}{a^2 + n^2}\sinh(a\pi) \qquad (5)
\end{aligned}
$$

Using (5) in (4) gives

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx\\
&= -\frac{1}{\pi}\frac{2n(-1)^n}{a^2 + n^2}\sinh(a\pi)
\end{aligned}
$$

Now that we found a_0, a_n, b_n then the Fourier series is

$$
\begin{aligned}
e^{ax} &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)\\
&\sim \frac{\frac{2}{\pi a}\sinh(a\pi)}{2} + \sum_{n=1}^{\infty}\frac{a}{\pi}\frac{2(-1)^n\sinh(a\pi)}{a^2 + n^2}\cos(nx) - \frac{1}{\pi}\frac{2n(-1)^n}{a^2 + n^2}\sinh(a\pi)\sin(nx)\\
&\sim \frac{\sinh(a\pi)}{\pi a} + \frac{1}{\pi}\sinh(a\pi)\sum_{n=1}^{\infty}\frac{2(-1)^n}{a^2 + n^2}\left(a\cos(nx) - n\sin(nx)\right)\\
&\sim \sinh(a\pi)\left(\frac{1}{\pi a} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n}{a^2 + n^2}\left(a\cos(nx) - n\sin(nx)\right)\right)\\
&\sim \frac{2\sinh(a\pi)}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2 + n^2}\left(a\cos(nx) - n\sin(nx)\right)\right)
\end{aligned}
$$

When $a = 1$ the above becomes

$$
e^x \sim \frac{2\sinh(\pi)}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1 + n^2}\left(\cos(nx) - n\sin(nx)\right)\right)
$$

9.2 Part (b)

The 2π periodic extended function shows it piecewise C^1 over all points except at the points $x = \cdots, -5\pi, -3\pi, \pi, 3\pi, 5\pi, \cdots$. These are points at the ends of the original domain. At these points, there is a jump discontinuity. Therefore the Fourier series at these points will converge to the average of the 2π periodic extended e^x. Due to the jump discontinuity Gibbs phenomena shows up at these points. This also means that the convergence is not uniform.

9.3 Part (c)

The following is a plot showing the convergence using different number of terms in the above sum. This shows the Fourier series converges to e^x at all points inside the interval, except at the end points $x = -\pi, \pi$ where it converges to the average of $f(x)$.

Out[ ]=

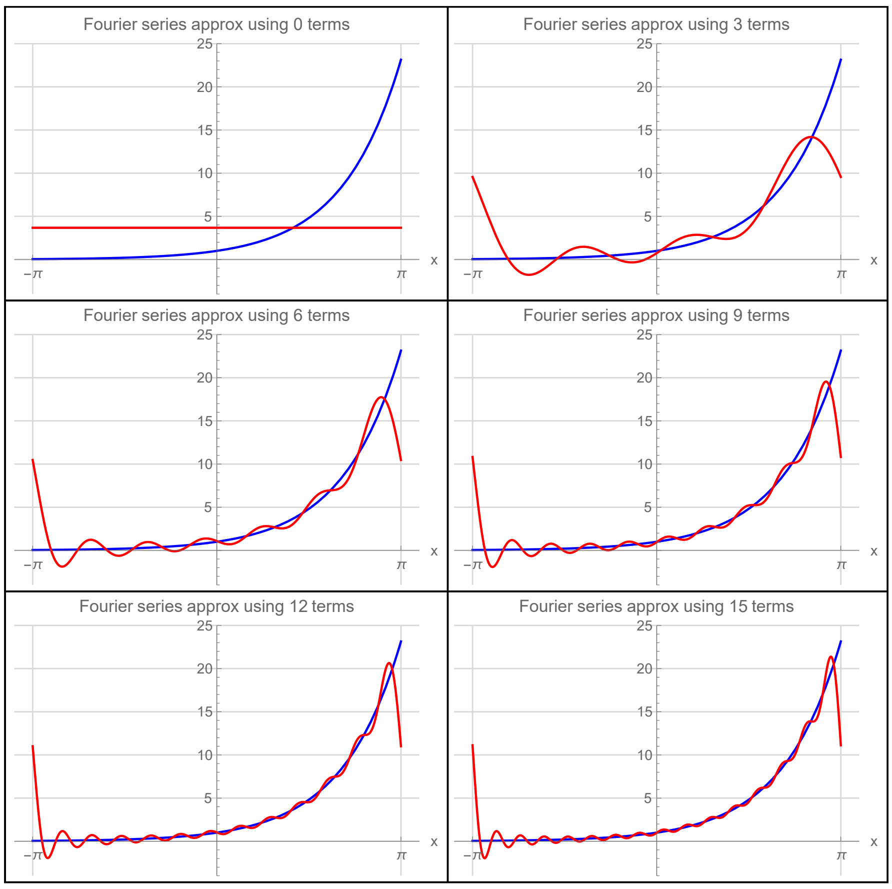

Figure 16: Plot of Fourier series approximation and $f(x)$

```
In[ ]:= padIt2[v_, f_List] := AccountingForm[v, f, NumberSigns → {"", ""},
            NumberPadding → {" ", " "}, SignPadding → True];
        fs[x_, max_] := (2 Sinh[Pi]/Pi) (1/2 + Sum[((-1)^n/(1 + n^2)) (Cos[n x] - n Sin[n x]), {n, 1, max}]);
        f[x_] := Exp[x];
        fp[x_] := Piecewise[{{f[x + 2 Pi], x ≤ -Pi}, {f[x], -Pi < x < Pi}, {f[x - 2 Pi], x > Pi}}];
        makePlot[n_] := Plot[{f[x], fs[x, n]}, {x, -Pi, Pi},
            PlotStyle → { Blue, Red}, AxesLabel → {"x", None},
            PlotLabel → Row[{"Fourier series approx using ", n, " terms"}],
            ImageSize → 300,
            Ticks → {Range[-Pi, Pi, Pi], Automatic},
            PlotRange → {{-1.1 Pi, 1.1 Pi}, {-4, 25}},
            GridLines → {Range[-Pi, Pi, Pi], Automatic}, GridLinesStyle → LightGray
          ];
        Grid[Partition[Table[makePlot[n], {n, {0, 3, 6, 9, 12, 15}}], 2],
          Frame → All]
```

Figure 17: Code for the above plot

10 Problem 3.2.30

Suppose a_k, b_k are the Fourier coefficients of the function $f(x)$. (a) To which function does the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx) + b_k \sin(2kx)$$

Converge? (b) Test your answer with the Fourier series (3.37) for $f(x) = x$.

$$x \sim 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right) \tag{3.37}$$

solution

10.1 Part (a)

Let

$$\begin{aligned} g(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx) + b_k \sin(2kx) \\ f(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \end{aligned}$$

Then $g(x)$ has as its period half the period of $f(x)$. This is because when $2kx = \frac{2\pi}{T}kx$ then $T = \pi$ and when $kx = \frac{2\pi}{T}kx$ then $T = 2\pi$.

Therefore, if $f(x)$ has fundamental period as $-\pi < x < \pi$, then $g(x)$ has a fundamental period as $-\frac{\pi}{2} < x < \frac{\pi}{2}$. And since $f(x), g(x)$ have the same Fourier series coefficients, then $g(x)$ converges to $2f(x)$ but only over $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

10.2 Part (b)

Let $f(x) = x$ whose we are given that its Fourier series is

$$\begin{aligned} f(x) &\sim 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right) \\ &= 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \cdots \end{aligned}$$

The above says that $a_k = 0$ and $b_k = \frac{2(-1)^{k+1}}{k}$. Hence

$$f(x) \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx)$$

Therefore $g(x)$ will converge to

$$\begin{aligned} g(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx) + b_k \sin(2kx) \\ &= \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(2kx) \\ &= 2(+1)\sin(2x) + \frac{-2}{2}\sin(4x) + \frac{2(+1)}{3}\sin(6x) + \frac{-2}{4}\sin(8x) + \cdots \\ &= 2\sin(2x) - \sin(4x) + \frac{2}{3}\sin(6x) - \frac{1}{2}\sin(8x) + \cdots \end{aligned}$$

Over $-\frac{\pi}{2} < x < \frac{\pi}{2}$. To verify the above, we will now find a_k, b_k directly for x but using $T = \pi$ and not $T = 2\pi$ to see if the above Fourier series is obtained.

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx \\ &= 0 \end{aligned}$$

And

$$a_k = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos(2kx)\, dx$$

Since x is odd function and $\cos$ is even, the product is odd. Hence $a_k = 0$.

$$\begin{aligned}
b_k &= \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin(2kx)\, dx \\
&= \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} x \sin(2kx)\, dx \\
&= \frac{4}{\pi} \left(\frac{-k\pi \cos(k\pi) + \sin(k\pi)}{4k^2} \right) \\
&= \frac{1}{\pi k^2} \left(-k\pi \cos(k\pi) \right) \\
&= \frac{-1}{k} \cos(k\pi) \\
&= \frac{-1}{k} (-1)^k \\
&= \frac{(-1)^{k+1}}{k}
\end{aligned}$$

Therefore

$$\begin{aligned}
g(x) &\sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx) + b_k \sin(2kx) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(2kx)
\end{aligned}$$