

my study notes

Maths 4512  
Differential Equations with Applications

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## 1 Population models (Section 1.5 in book)

■ The most basic model is called Malthusian model given by  $\frac{dp}{dt} = ap(t)$  which says that rate of change of population is proportional to current population size.  $a$  is constant. The solution is  $p(t) = p_0 e^{a(t-t_0)}$ . Where  $p(t)$  is population at time  $t$  and  $p_0$  is initial population at time  $t_0$ . This model is OK when population is small. A better model is called logistic model given by

$$\begin{aligned}\frac{dp}{dt} &= ap(t) - bp^2(t) \\ p(t_0) &= p_0\end{aligned}$$

Where  $b$  is the competition factor. Also constant and positive. It is much smaller than  $a$ . The solution to the above is

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t-t_0)}} \quad (1)$$

In this model, we are normally given  $p_0$  and given  $(t - t_0)$  and given what is called the limiting value  $\frac{a}{b}$  which is  $\lim_{t \rightarrow \infty} p(t)$ . Then asked to find population  $p(t)$  after sometime. This will be  $(t - t_0)$ . We need to find  $a$ . Once we find  $a$ , then we find  $b$  from the limiting value. The trick is to find  $a$ . To do this, we first use (1) from the information given. The problem will always say that the population doubles every so many years, or the population increases at rate of some percentage per year and so on. Use this to find  $a$  from (1). Now we know  $b$ . Then use (1) again now to find the population at some future time as the problem says. See HW1, last problem for an example.

■ If a problem says substance decays exponentially, this means  $M(t) = M_0 e^{-Ct}$ . where  $C > 0$ . Need to find  $C$  from other problem information. Typically problem gives half life to do this. For example, see problem section 1.8, problem 3. It says:

substance  $x$  decays exponentially, and only half of the given quantity remains after 2 years. How long it takes for 5 lb decay to 1 lb? Solution is

$$M = M_0 e^{-Ct}$$

After 2 years,  $M = \frac{M_0}{2}$ , hence  $\frac{M_0}{2} = M_0 e^{-2C}$ . Hence  $\frac{1}{2} = e^{-2C}$  or  $\ln\left(\frac{1}{2}\right) = -2C$ , hence  $C =$

$-\frac{1}{2} \ln\left(\frac{1}{2}\right) = \frac{1}{2} \ln(2)$ . Now we know  $C$ , we can finish the solution.

$$\begin{aligned} M &= M_0 e^{-\frac{1}{2} \ln(2)t} \\ 1 &= 5e^{-\frac{1}{2} \ln(2)t} \\ \frac{1}{5} &= e^{-\frac{1}{2} \ln(2)t} \\ \ln\left(\frac{1}{5}\right) &= -\frac{1}{2} \ln\left(\frac{1}{5}\right) t \\ t &= -2 \frac{\ln\left(\frac{1}{5}\right)}{\ln(2)} \\ &= 2 \frac{\ln 5}{\ln 2} \\ &= 4.643 \text{ years} \end{aligned}$$

If it says it grows exponentially, then  $M = M_0 e^{Ct}$  instead.

## 2 Mixing problems (Section 1.8(b) in book)

The main idea is to set an ODE using  $\frac{dS(t)}{dt} = R_{in} - R_{out}$  where  $R_{in}$  is rate of mass of salt coming into the tank and  $R_{out}$  is rate of mass of salt leaving tank. This gives an ODE to solve for  $S(t)$  using initial conditions which is given. At end, divide by volume of tank to get concentration at time  $t$ . See book example at page 54.

## 3 Example 1, page 369

Book solution for example 1 is wrong. So I typed corrected solution.

Solve  $\dot{x} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$  with  $x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Solution

$$\begin{aligned} (sI - A)X(s) &= F(s) + x(0) \\ \left[ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} \\ \frac{1}{s-1} \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} s-1 & -4 \\ -1 & s-1 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \begin{pmatrix} \frac{1}{s-1} + 2 \\ \frac{1}{s-1} + 1 \end{pmatrix} \end{aligned}$$

Multiplying the second row by  $(s-1)$  and adding the result to the first row to obtain Gaussian

elimination. First multiplying second row by  $(s - 1)$  gives

$$\begin{pmatrix} s-1 & -4 \\ -(s-1) & (s-1)^2 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s-1} + 2 \\ 1 + (s-1) \end{pmatrix}$$

$$\begin{pmatrix} s-1 & -4 \\ -(s-1) & (s-1)^2 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s-1} + 2 \\ s \end{pmatrix}$$

Now replacing row 2 by row 2 plus row 1 gives

$$\begin{pmatrix} s-1 & -4 \\ 0 & (s-1)^2 - 4 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s-1} + 2 \\ s + \left(\frac{1}{s-1} + 2\right) \end{pmatrix}$$

$$\begin{pmatrix} s-1 & -4 \\ 0 & s^2 - 2s - 3 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s-1} + 2 \\ \frac{1}{s-1}(s^2 + s - 1) \end{pmatrix} \quad (1)$$

Hence

$$x_2(s) = \frac{1}{s-1} \left( \frac{s^2 + s - 1}{s^2 - 2s - 3} \right)$$

$$= \frac{s^2 + s - 1}{(s-1)(s-3)(s+1)} \quad (2)$$

Partial fractions:

$$\frac{s^2 + s - 1}{(s-1)(s-3)(s+1)} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{s+1}$$

$$A = \left( \frac{s^2 + s - 1}{(s-3)(s+1)} \right)_{s=1} = \frac{1+1-1}{(1-3)(1+1)} = -\frac{1}{4}$$

And

$$B = \left( \frac{s^2 + s - 1}{(s-1)(s+1)} \right)_{s=3} = \frac{9+3-1}{(3-1)(3+1)} = \frac{11}{8}$$

And

$$C = \left( \frac{s^2 + s - 1}{(s-1)(s-3)} \right)_{s=-1} = \frac{1-1-1}{(-1-1)(-1-3)} = -\frac{1}{8}$$

Hence

$$x_2(s) = -\frac{1}{4} \frac{1}{s-1} + \frac{11}{8} \frac{1}{s-3} - \frac{1}{8} \frac{1}{s+1} \quad (3)$$

Therefore

$$x_2(t) = -\frac{1}{4}e^t + \frac{11}{8}e^{3t} - \frac{1}{8}e^{-t}$$

Now we go back to (1) and use the first row to find  $x_1(s)$  since we know  $x_2(s)$  which is given

in (2). This results in

$$\begin{aligned}
 (s-1)x_1(s) - 4x_2(s) &= \frac{1}{s-1} + 2 \\
 (s-1)x_1(s) &= \frac{1}{s-1} + 2 + 4x_2 \\
 x_1(s) &= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)}x_2 \\
 &= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)} \left( \frac{s^2 + s - 1}{(s-1)(s-3)(s+1)} \right) \\
 &= \frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{4}{(s-1)} \left( -\frac{1}{4} \frac{1}{(s-1)} + \frac{11}{8} \frac{1}{(s-3)} - \frac{1}{8} \frac{1}{(s+1)} \right) \\
 &= \frac{1}{(s-1)^2} + \frac{2}{s-1} - \frac{1}{(s-1)^2} + \frac{44}{8} \frac{1}{(s-1)(s-3)} - \frac{4}{8} \frac{1}{(s-1)(s+1)} \\
 &= \frac{2}{s-1} + \frac{44}{8} \frac{1}{(s-1)(s-3)} - \frac{4}{8} \frac{1}{(s-1)(s+1)} \\
 &= \frac{2}{s-1} + \frac{11}{2} \frac{1}{(s-1)(s-3)} - \frac{1}{2} \frac{1}{(s-1)(s+1)} \\
 &= \frac{2}{s-1} + \frac{11}{2} \left( \frac{1}{2s-3} - \frac{1}{2s-1} \right) - \frac{1}{2} \left( \frac{1}{2s-1} - \frac{1}{2s+1} \right) \\
 &= \frac{2}{s-1} + \frac{11}{4} \frac{1}{s-3} - \frac{11}{4} \frac{1}{s-1} - \frac{1}{4} \frac{1}{s-1} + \frac{1}{4} \frac{1}{s+1} \\
 &= \frac{-1}{s-1} + \frac{11}{4} \frac{1}{s-3} + \frac{1}{4} \frac{1}{s+1}
 \end{aligned}$$

Therefore

$$x_1(t) = -e^t + \frac{11}{4}e^{3t} + \frac{1}{4}e^{-t}$$

We see that book solution is wrong. It gives  $x_1(t) = 2e^{3t} + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ .

Solving the same problem, but using the variation of parameters method:

Since  $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$  then

$$\begin{aligned}
 \det(A - \lambda I) &= 0 \\
 \det \begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} &= 0 \\
 (1-\lambda)^2 - 4 &= 0
 \end{aligned}$$

Hence roots are  $\lambda = -1, \lambda = 3$

$$\underline{\lambda = -1}$$

$$\begin{pmatrix} 1 - \lambda & 4 \\ 1 & 1 - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first row,  $2v_1 + 4v_2 = 0$  or  $v_1 = -2v_2$ . Hence  $v^1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  and  $x^1(t) = e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\underline{\lambda = 3}$$

$$\begin{pmatrix} 1 - \lambda & 4 \\ 1 & 1 - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first row,  $-2v_1 + 4v_2 = 0$  or  $v_1 = 2v_2$ . Hence  $v^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $x^2(t) = e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ . Therefore

$$X(t) = \begin{pmatrix} -2e^{-t} & 2e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix}$$

$$X(0) = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$$

Therefore  $X^{-1}(0) = \frac{\text{adj}(X(0))}{\det(X(0))} = \frac{\begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix}^T}{-4} = -\frac{1}{4} \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$ . Hence

$$e^{At} = X(t) X^{-1}(0)$$

$$= \begin{pmatrix} -2e^{-t} & 2e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} & -e^{-t} + e^{3t} \\ -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix}$$

Using (since  $t_0 = 0$ )

$$x(t) = e^{At}x(0) + e^{At} \int_0^t e^{-As} f(s) ds$$

But

$$\begin{aligned}
 e^{-As}f(s) &= \begin{pmatrix} \frac{1}{2}e^s + \frac{1}{2}e^{-3s} & -e^s + e^{-3s} \\ -\frac{1}{4}e^s + \frac{1}{4}e^{-3s} & \frac{1}{2}e^s + \frac{1}{2}e^{-3s} \end{pmatrix} \begin{pmatrix} e^s \\ e^s \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2}e^{2s} + \frac{1}{2}e^{-2s} - e^{2s} + e^{-2s} \\ -\frac{1}{4}e^{2s} + \frac{1}{4}e^{-2s} + \frac{1}{2}e^{2s} + \frac{1}{2}e^{-2s} \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{2}e^{2s} + \frac{3}{2}e^{-2s} \\ \frac{1}{4}e^{2s} + \frac{3}{4}e^{-2s} \end{pmatrix}
 \end{aligned}$$

Integrating

$$\begin{aligned}
 \int_0^t e^{-As}f(s) ds &= \begin{pmatrix} \int_0^t -\frac{1}{2}e^{2s} + \frac{3}{2}e^{-2s} ds \\ \int_0^t \frac{1}{4}e^{2s} + \frac{3}{4}e^{-2s} ds \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{4}(e^{2s})_0^t - \frac{3}{4}(e^{-2s})_0^t \\ \frac{1}{8}(e^{2s})_0^t - \frac{3}{8}(e^{-2s})_0^t \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{4}(e^{2t} - 1) - \frac{3}{4}(e^{-2t} - 1) \\ \frac{1}{8}(e^{2t} - 1) - \frac{3}{8}(e^{-2t} - 1) \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{4}e^{2t} + \frac{1}{4} - \frac{3}{4}e^{-2t} + \frac{3}{4} \\ \frac{1}{8}e^{2t} - \frac{1}{8} - \frac{3}{8}e^{-2t} + \frac{3}{8} \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{4}e^{2t} - \frac{3}{4}e^{-2t} + 1 \\ \frac{1}{8}e^{2t} - \frac{3}{8}e^{-2t} + \frac{1}{4} \end{pmatrix}
 \end{aligned}$$

Hence

$$\begin{aligned}
 e^{At} \int_0^t e^{-As}f(s) ds &= \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} & -e^{-t} + e^{3t} \\ -\frac{1}{8}e^{-t} + \frac{1}{4}e^{3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{4}e^{2t} - \frac{3}{4}e^{-2t} + 1 \\ \frac{1}{8}e^{2t} - \frac{3}{8}e^{-2t} + \frac{1}{4} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{4}e^{-t} - e^t + \frac{3}{4}e^{3t} \\ -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{3}{8}e^{4t} \end{pmatrix}
 \end{aligned}$$

And

$$\begin{aligned}
 e^{At}\mathbf{x}(0) &= \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} & -e^{-t} + e^{3t} \\ -\frac{1}{8}e^{-t} + \frac{1}{4}e^{3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}
 \end{aligned}$$



Hence

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} + \begin{pmatrix} \frac{1}{4}e^{-t} - e^t + \frac{3}{4}e^{3t} \\ -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{3}{8}e^{4t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}e^{-t} - e^t + \frac{11}{4}e^{3t} \\ -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{11}{8}e^{4t} \end{pmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} x_1(t) &= \frac{1}{4}e^{-t} - e^t + \frac{11}{4}e^{3t} \\ x_2(t) &= -\frac{1}{8}e^{-t} - \frac{1}{4}e^{2t} + \frac{11}{8}e^{4t} \end{aligned}$$

Which agrees with result using Laplace transform method.

## 4 Orthogonal projections

Given  $F(x, y, c)$  we need to find the orthogonal projections. The first step is to find the slope of the orthogonal projection, which is given by (it is orthogonal to the given curve slope)

$$\frac{dy}{dx} = \frac{F_y}{F_x} \quad (1)$$

Next step, check if  $c$  still shows up in the above (i.e. did not cancel out), then solve for  $c$  from  $F(x, y, c) = 0$  and replace it in (1). Now (1) will not have  $c$  in it any more. Next, solve (1) for  $y$ . This gives the curve for the orthogonal projection. This solution will have new  $c$  in it (since we need to integrate to find  $y$ ). See HW 2 for example problem.

## 5 Existence-uniqueness for 1D ODE

Given by theorem 2 for existence and uniqueness: given  $\frac{dy}{dx} = f(x, y)$ , with initial value  $y(x_0) = y_0$ . Let  $f$  and  $\frac{\partial f}{\partial x}$  be continuous in the rectangle  $R : t \leq t \leq t_0 + a, |y - y_0| \leq b$ . Compute  $M = \max_{(x,y)} |f(x, y)|$  and set  $\alpha = \min\left(a, \frac{b}{M}\right)$  then ODE has at least one solution in interval  $t \leq t \leq t_0 + \alpha$  and this solution is unique. (I do not know why book split this into theorem 2 and 2').

Notice in the above, if  $f$  or  $\frac{\partial f}{\partial x}$  not continuous in the range (the range must include the initial point) then not unique solution exist. For example  $y' = \sin(2t)y^{\frac{1}{3}}$  with  $y(0) = 0$ . Here  $f'$  is not continuous at  $y = 0$ .

How to use the above The first step is to find  $M$ . This is done by finding maximum in  $R$ . This is normally done by inspection from looking at  $f(x, y)$ . Next, let  $g(y) = \frac{b}{M}$ . Find where this one is maximum. set its value in  $\alpha = \min\left(a, \frac{b}{M}\right)$  and this finds  $\alpha$ . Done. Example. Show

$y(t)$  solution to  $\frac{dy}{dt} = t^2 + e^{-y^2}, y(0) = 0$  exists in  $0 \leq t \leq \frac{1}{2}$  and  $|y(t)| \leq 1$ . Here it is clear  $M = \frac{5}{4}$  and hence  $\alpha = \min\left(\frac{1}{2}, \frac{b}{M}\right)$  but  $b = 1$ , hence  $\alpha = \min\left(\frac{1}{2}, \frac{1}{\frac{5}{4}}\right) = \alpha = \min\left(\frac{1}{2}, \frac{4}{5}\right) = \frac{1}{2}$ . Therefore solution exist for  $0 \leq t \leq 0 + \alpha$  or  $0 \leq t \leq \frac{1}{2}$ .

## 5.1 practice problems

### 5.1.1 Problem 5, section 1.10

Show that the solution  $y(t)$  exists on  $y(0) = 0; 0 \leq t \leq \frac{1}{3}$

$$y' = 1 + y + y^2 \cos t$$

solution

Here  $a = \frac{1}{3}$ .

$$\begin{aligned} M &= \max(f(t, y)) \\ &= 1 + b + b^2 \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \min\left(\frac{1}{3}, \frac{b}{M}\right) \\ &= \min\left(\frac{1}{3}, \frac{b}{1 + b + b^2}\right) \end{aligned}$$

Let  $g(b) = \frac{b}{1+b+b^2}$  then  $\frac{dg}{db} = \frac{(1+b+b^2) - b(1+2b)}{(1+b+b^2)^2}$ . Setting this to zero and solving for  $b$

$$\begin{aligned} (1 + b + b^2) - b(1 + 2b) &= 0 \\ 1 + b + b^2 - b - 2b^2 &= 0 \\ 1 - b^2 &= 0 \end{aligned}$$

Hence  $b = 1$ . At  $b = 1$ , then  $g(b) = \frac{1}{1+1+1} = \frac{1}{3}$ . Therefore

$$\begin{aligned} \alpha &= \min\left(\frac{1}{3}, \frac{1}{3}\right) \\ &= \frac{1}{3} \end{aligned}$$

Therefore  $y(t)$  solution exist for  $0 \leq t \leq 0 + \alpha$  or  $0 \leq t \leq \frac{1}{3}$ .

### 5.1.2 Problem 16, section 1.10

Consider  $y' = t^2 + y^2, y(0) = 0$  and let  $R$  be rectangle  $0 \leq t \leq a, -b \leq y \leq b$ . (a) Show the solution exist for  $0 \leq t \leq \min\left(a, \frac{b}{a^2+b^2}\right)$  (b) Show the maximum value of  $\frac{b}{a^2+b^2}$ , for  $a$  fixed is

$\frac{1}{2a}$ . (c) Show that  $\alpha = \min\left(a, \frac{1}{2}a\right)$  is largest when  $a = \frac{1}{\sqrt{2}}$

$$y' = 1 + y + y^2 \cos t$$

solution

(a)

$$\begin{aligned} M &= \max(f(t, y)) \\ &= a^2 + b^2 \end{aligned}$$

Hence

$$\begin{aligned} \alpha &= \min\left(a, \frac{b}{M}\right) \\ &= \min\left(\frac{1}{3}, \frac{b}{a^2 + b^2}\right) \end{aligned}$$

Hence solution exist for  $0 \leq t \leq \min\left(a, \frac{b}{a^2 + b^2}\right)$ .

(b) Let  $g(b) = \frac{b}{a^2 + b^2}$  then  $\frac{dg}{db} = \frac{(a^2 + b^2) - b(2b)}{(1 + b + b^2)^2}$ . Setting this to zero and solving for  $b$

$$\begin{aligned} (a^2 + b^2) - b(2b) &= 0 \\ a^2 + b^2 - 2b^2 &= 0 \\ a^2 &= b^2 \end{aligned}$$

Hence  $b = \pm a$ . At  $b = a$ , then  $g(b) = \frac{a}{a^2 + a^2} = \frac{a}{2a^2} = \frac{1}{2a}$ .

(c)

$$\begin{aligned} \alpha &= \min\left(a, g_{\max}(b)\right) \\ &= \min\left(a, \frac{1}{2a}\right) \end{aligned}$$

Solving  $a = \frac{1}{2a}$  or  $a^2 = \frac{1}{2}$ . Hence  $a = \frac{1}{\sqrt{2}}$  gives largest value.

### 5.1.3 Problem 17, section 1.10

Prove that  $y(t) = -1$  is only solution for  $y' = t(1 + y), y(0) = -1$

solution

Since  $f = t(1 + y)$  is continuous for all  $t, y$  and  $f_y = t$  is continuous for all  $y$ , then if we find a solution, it will be unique solution by theorem 2'. But  $y(t) = -1$  is a solution since we can show easily it satisfies the ODE. Hence it is the only solution over all  $t$  by theorem 2'

### 5.1.4 Problem 19, section 1.10

Find solution of  $y' = t\sqrt{1-y^2}$ ,  $y(0) = 1$  other than  $y(t) = 1$ . Does this violate theorem 2'?

solution

$$\begin{aligned}\frac{dy}{dt} &= t\sqrt{1-y^2} \\ \int \frac{dy}{\sqrt{1-y^2}} &= \int t dt \\ \arcsin(y) &= \frac{t^2}{2} + C\end{aligned}$$

At  $t = 0$

$$\arcsin(1) = C$$

Hence solution is  $\arcsin(y) = \frac{t^2}{2} + \arcsin(1)$  or

$$\begin{aligned}y(t) &= \sin\left(\frac{t^2}{2} + \arcsin(1)\right) \\ &= \sin\left(\frac{t^2}{2} + \frac{\pi}{2}\right) \\ &= \sin\left(\frac{1}{2}(t^2 + \pi)\right)\end{aligned}$$

This does not violate theorem 2' because  $f(t, y) = t\sqrt{1-y^2}$ , hence  $f_y = \frac{-ty}{\sqrt{1-y^2}}$  which is not continuous at  $y = \pm 1$ . But  $y = -1$  is the initial conditions. Hence theorem 2' do not apply. Theorem 2' applies in the region where both  $f, f_y$  are continuous.

## 6 Stability of system

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**Algorithm 1** Determining stability of system  $\dot{x} = Ax + g(x)$

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```

1: if system is linear, i.e.  $\dot{x} = Ax$  then
2:   determine eigenvalues  $\lambda_i$  of  $A$  by solving  $|A - \lambda I| = 0$ 
3:   if all eigenvalues have real part smaller than zero then
4:     return stable
5:   else
6:     if at least one eigenvalue have positive real part then
7:       return not stable
8:     else ▷ we get here if at least one  $\lambda$  has zero real part
9:       for all  $\lambda_i$  with zero real part do
10:         $M =$  multiplicity of  $\lambda_i$ 
11:         $N =$  number of linearly independent eigenvectors that  $\lambda_i$  can generate
12:        if  $N < M$  then
13:          return not stable
14:        end if
15:      end for
16:      return stable
17:    end if
18:  end if
19: else ▷ system not linear
20:   will only consider case when origin is equilibrium point
21:   determine the Jacobian matrix  $J$ 
22:   evaluate  $J$  at origin  $x = 0$ 
23:   determine eigenvalues  $\lambda_i$  of  $J$  by solving  $|J - \lambda I| = 0$ 
24:   if all eigenvalues have real part smaller than zero then
25:     return stable
26:   else
27:     if at least one eigenvalue have positive real part then
28:       return not stable
29:     else ▷ we get here if at least one  $\lambda$  has zero real part
30:       return unable to decide
31:     end if
32:   end if
33: end if

```

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## 7 Laplace

If  $Y(s)$  has form  $\frac{s}{s^2+as+b}$  where roots of quadratic are complex, then complete the square.

Write  $s^2 + as + b = (s + A)^2 + B$  and find  $A, B$ . Then

$$\begin{aligned} \frac{s}{s^2 + as + b} &= \frac{s}{(s + A)^2 + B} \\ &= \frac{s + A - A}{(s + A)^2 + B} \\ &= \frac{s + A}{(s + A)^2 + B} - A \frac{1}{(s + A)^2 + B} \\ &= \frac{\tilde{s}}{\tilde{s}^2 + B} - A \frac{1}{\tilde{s}^2 + B} \\ &= \frac{\tilde{s}}{\tilde{s}^2 + B} - \frac{A}{\sqrt{B}} \frac{\sqrt{B}}{\tilde{s}^2 + B} \end{aligned}$$

And now use tables. Due to shifting, multiply result by  $e^{-Bt}$ . So inverse Laplace of the above is  $e^{-Bt} \left( \cos \sqrt{B}t - \frac{A}{\sqrt{B}} \sin \sqrt{B}t \right)$